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Optimal ordering policies in response to a disposable coupon

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Abstract

This study investigates a retailer's ordering strategy under the framework of the economic order quantity (EOQ) model. A supplier offers a retailer a disposable coupon and allows it to place a special order at any time in a promotion period. The promotion period is not necessary short and shortages are allowed throughout the time horizon. In addition to the special order time and the special order quantity, the retailer needs to decide whether to place some regular orders in the promotion period before placing the special order for the purpose of making better use of this coupon. We show that the coupon should be used to the retailer's first order in the promotion period regardless of the duration of the promotion period. Moreover, the retailer's maximum inventory level is higher than that in the classical EOQ model. We find that a longer promotion period can benefit the retailer by endowing it with more flexibility in its decision-making. Therefore, the supplier can improve the cash flow and reduce the overstock by integrating a disposable coupon with a longer promotion period. Numerous managerial insights are obtained from sensitivity analysis and numerical experiments.

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1 Introduction

“Small profits and quick returns” has been widely adopted by functional managers in real industrial practices. To accelerate cash flow and reduce overstock, it is common for a supplier to encourage a retailer to place a larger order by temporarily charging a lower wholesale price. In this context, the retailer can improve its inventory system by placing one or more special orders. Despite a reduced wholesale price, the retailer's order decision is driven by a trade-off between the benefit from the special order (e.g., reduced purchasing cost and unit inventory holding cost) and the loss from it (e.g., increased ordering cost and inventory level). As such, the retailer is generally prudent to make its order decision, which depends on the discount rate and the promotion period simultaneously [1].

Given the diversity of retailers, the motivation effect of a short promotion period may be marginally pronounced because it undermines the flexibility of retailers in their decision-making. Nevertheless, if the length of the promotion period is long, retailers may repeatedly place small special orders for the purpose of cutting the purchasing cost and the inventory holding cost simultaneously, which works to the disadvantage of the supplier. To resolve the problem, the supplier can set a longer promotion period and offer the retailer a disposable coupon whereby the latter can order a special quantity only once during the promotion period [2]. We are interested in the following research questions:

RQ 1: Will the retailer place some regular orders in the promotion period before the special order?

RQ 2: How does the discount rate influence the retailer's optimal order decision?

RQ 3: How does the duration of the promotion period affect the retailer's total cost?

To address the above questions, we develop an inventory model in which a supplier offers a retailer a disposable coupon whereby the retailer can order a special quantity at a reduced wholesale price in a promotion period. The promotion period is not necessary short. Thus, in addition to the special order time and the special order quantity, the retailer needs to decide whether to place some regular orders in the promotion period before the special order. Shortages are allowed and all shortages are backordered. We derive the retailer's optimal order decision by first constructing its total cost function and then minimizing it. It is worthy mentioning that the retailer's total cost function is continuous with respect to the special order time and the special order quantity, while discrete with respect to the number of regular orders.

Our analytical results generate numerous managerial insights. Specifically, the coupon should be used to the first order in the promotion period regardless of the duration of the promotion period. Even if the supplier sets a long promotion period, the retailer has no incentive to deliberately postpone its special order, which improves the supplier's cash flow. Moreover, the maximum inventory level in our model is always higher than that in the classical economic order quantity (EOQ) model, indicating that the disposable coupon can reduce the supplier's overstock by passing on its excess stock to the downstream retailer. We investigate the effect of the discount rate on the retailer's optimal order decision. Overall speaking, a higher (lower) discount rate results in the retailer placing its special order earlier (later) and ordering less (more) special quantity simultaneously. In this sense, the supplier needs to make a trade-off between a earlier special order and a larger special order when setting the discount rate. Interestingly, the retailer's

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total cost decreases with the length of the promotion period, which implies that the supplier can attract more retailers by extending the promotion period in addition to reducing the wholesale price. The intuition is that a longer promotion period can make the retailer better off by endowing it with more flexibility in its ordering decision-making.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 describes the model. The retailer's optimal order decision and the corresponding managerial insights are presented in Section 4. In Section 5, we conduct numerical experiments to validate the proposed model. Section 6 concludes this study. All proofs are presented in Appendix.

2 Literature Review

The literature is reviewed from two perspectives: price discounts at a future time and price discounts over a short period.

For instantaneous price discounts, Lev and Weiss [3] investigated how a retailer adjusts its order quantity according to fluctuations of various operational costs. Tersine and Barman [4] developed a composite EOQ model which can be disaggregated into a family of hybrid models to deal with specific conditions. Wee and Yu [5] determined the optimal order quantity for deteriorating products. Cárdenas-Barrón et al. [6] generalized Tersine and Barman's model by allowing for shortages and two backorder costs. Yang et al. [7] examined an inventory setting in which the leading time hinges on the retailer's order quantity. Chang and Lin [8] generalized Lev and Weiss's model by incorporating perishable items. Yang et al. [9] formulated an inventory model with limited warehouse capacity. Taleizadeh [10] further extended Tersine and Barman's model by considering partial backordering shortages. Shaposhnik [11] developed an inventory model with a stochastic price discount. Inventory models with instantaneous price discounts include, among many others, [12,13,14,15,16,17,18,19].

For price discounts over a short period, Ardalan [20] suggested that the special order should be placed at a time when the inventory level reaches the minimum (i.e., the minimum inventory principle). Aull-Hyde [21] extended Ardalan's model by incorporating some supplier-restricted purchasing options. Ardalan [22] examined the retailer's replenishment and pricing strategy in a three-echelon supply chain. Aull-Hyde [23] investigated the retailer's ordering strategy under allowable shortages and restricted promotion period. Chu et al. [24] showed that the minimum inventory principle is still valid for Aull-Hyde's model. Abad [25] examined the retailer optimal order decision under a price-dependent demand. Sarker and Kindi [26] extended the time horizon from the special replenishment cycle to the whole year. Cárdenas-Barrón [27,28] generalized Sarker and Kindi's model by considering some practical extensions. Kindi and Sarker [29] further generalized Sarker and Kindi's model by allowing for shortages. Sari et al. [30] formulated an inventory model with time-based price discounts. Karimi-Nasab and Konstantaras [31] investigated the retailer's order strategy with stochastic replenishment cycles. Cárdenas-Barrón et al. [32] revised Kindi and Sarker's model and derived the closed-form optimal total gain costs. Wang et al. [33] developed an inventory model with a stochastic short-term price discount. Gao et al. [34] generalized Wang et al.'s model by allowing for partial backorders.

The literature referred to above generally assumes that the promotion period is too short to tolerate more than one order, which undermines the practicality of the price discount. Although Kindi and Sarker [26,29] examined the retailer's ordering strategy under a long promotion period, the start time of the promotion period is required to be exactly coincident with a regular replenishment point of the classical EOQ model. In this sense, the supplier's promotion policy is actually exclusive to a particular retailer and, thus, is hardly appropriate to various retailers. This paper complements the above literature by relaxing the assumption on the start time of the promotion period. To our best knowledge, this study is the first to allow the retailer to place some regular orders in the promotion period to better prepare for the subsequent special order, which endows the retailer with more flexibility in its decision-making and, thus, is suitable for a variety of retailers.

Our study is also related to Arcelus et al. [35], who investigated the retailer's special order time and special order quantity under a promotion period of unknown length. This study also examines the retailer's ordering strategy with a promotion period of arbitrary length, but differs from their model in two aspects. First, they allow the retailer to repeatedly place special orders throughout the promotion period, which enables the retailer to fully take advantage of the price discount. In contrast, we restrict the number of special orders to hurry the retailer into placing a larger special order for the purpose of improving the supplier's cash flow. Second, shortages are prohibited in their study, but they are allowed in this study, which endow the retailer with more flexibility in its decision-making.

3 Model Setup

Consider a supply chain setting in which a supplier sells a product to a retailer and charges a wholesale price w for each unit of its product. To promote sales, the supplier offers the retailer a disposable coupon whereby the latter can place a special order in a promotion period $[t_s, t_e]$ at a reduced wholesale price γw, $0 < \gamma < 1$. The promotion period may include one or more regular replenishment points. If that is the case, the retailer needs to decide whether to continue placing some regular orders after the coupon is available (at time t_s) to prepare for the special order. Shortages are allowed and fully backordered throughout the time horizon. The length of the time horizon is exogenously given and long enough to include the special order period. In addition to the special order time t_r, the special order quantity q_r, the retailer needs to determine the number of regular orders n placed in $[t_s, t_r]$. For convenience, we denote simply by a triple (q_r, t_r, n) the retailer's ordering strategy with a disposable coupon. To highlight the retailer's inventory mechanism, we adopt a constant demand rate under the framework of the classical EOQ model. To better illustrate our analytical model, we consider a two-echelon supply chain in which Coca-Cola and Costo act as the supplier and the retailer, respectively. The product is cola, which is produced by Coca-Cola and sold to Costco. It is worthy noting that the local demand for cola has tended to be steady [36]. A summary of the model notation is listed in the following table.

Table. 1 Model notation.

Notation	Description
D	Annual market demand
w	Wholesale price
γ	Discount rate, $0 < \gamma < 1$
A	Fixed ordering cost
h	Unit inventory holding cost
c_f	Fixed backorder cost
c_l	Unit backorder cost

Q^*	Regular economic order quantity (EOQ) order quantity
S^*	Regular economic order quantity (EOQ) backorder level
k	The length of the time horizon
t_s	The start time of the promotion period
t_e	The end time of the promotion period
m	The number of regular orders placed in $[0, t_s)$
n	The number of regular orders placed in $[t_s, t_r]$
q_r	Special order quantity
t_r	Special order time

3.1 Inventory Level

In this subsection, we characterize the retailer's inventory level with respect to its ordering strategy. To this end, we first examine the fixed number of regular EOQ orders placed before the promotion period. Let $m = \lceil t_s D/Q^* \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Thus, m denotes the number of regular orders placed in $[0, t_s)$, which satisfies $(m-1)Q^*/D < t_s \leq mQ^*/D$.

We then investigate the number of regular orders placed in $[t_s, t_r]$, wherein $[t_s, t_r]$ denotes the period from the coupon being available (at time t_s) to the special order being placed (at time t_r). Note that in the classic EOQ model, the i-th, $i \geq 1$, regular order is placed at time $(i-1)Q^*/D$. If the retailer decides to place n, $n \geq 0$, regular orders in $[t_s, t_r]$, it will place a total of $m+n$ regular orders in $[0, t_r]$. In particular, the $(m+n)$-th regular order is to be placed at time $(m+n-1)Q^*/D$. To measure the retailer's inventory level at the special order time t_r, let us define $f_n(t) = (m+n)Q^* - S^* - tD$ for $n \geq 0$ and $t \in [0, +\infty)$. In this light, $f_n(t_r)$ is exactly the inventory level at the special order time t_r.

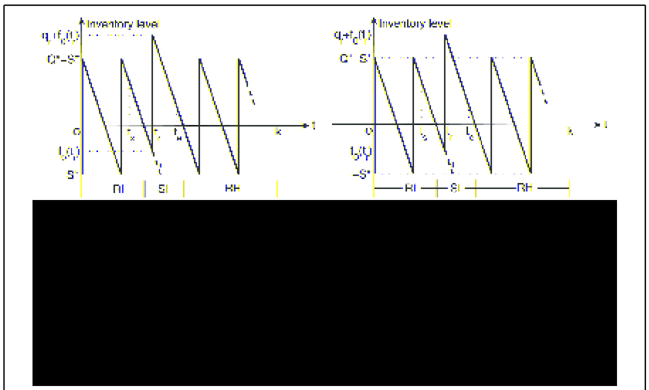

Figure 1: Retailer's Ordering Strategy with $n=0$.

Given that all items purchased through the last regular order before the promotion period will be sold out at time $((m+n)Q^* - S^*)/D$, we refer to $[0, ((m+n)Q^* - S^*)/D]$ as the regular order interval. Since all items purchased through the special order will be sold out at time $((m+n)Q^* - S^* + q_r)/D$, we refer to $(((m+n)Q^* - S^*)/D,\ ((m+n)Q^* - S^* + q_r)/D]$ as the the special order interval. The retailer's inventory level is illustrated in Figure 1, where “RI”, “SI”, and “RH” denote the regular order interval, the special order interval, and the remaining time horizon, respectively.

3.2 Total Cost

In this subsection, we examine the retailer's total cost with respect to the ordering strategy (q_r, t_r, n) by adding up its total costs in the regular order interval, the special order interval, and the remaining time horizon.

We first consider the retailer's total cost in the regular order interval. According to [37], the average cost caused by the regular order is $C_{ar}^* = wD + h(Q^* - S^*)$, where $Q^* = \sqrt{2AD/h}$ and $S^* = 0$ if $c_f \geq \sqrt{2Ah/D}$; otherwise, $Q^* = (2AD(h+c_l) - c_f^2 D^2)^{\frac{1}{2}}/(hc_l)^{\frac{1}{2}}$ and $S^* = (hQ^* - c_f D)/(h + c_l)$. Thus, the total cost in the regular order interval is $(m+n)Q^* C_{ar}^*/D - (c_l S^* + 2c_f D)S^*/2D$, where $(c_l S^* + 2c_f D)S^*/2D$ is the backorder cost caused by the last regular order before the promotion period, which actually occurs in the subsequent special order interval.

Next, we investigate the retailer's total cost in the special order interval. It is worthy noting that when the wholesale price is reduced from w to γw, the unit inventory holding cost falls from h to γh, while two backorder costs c_f and c_l remain unchanged [29,6].

Lemma 1: When $q_r + f_n(t_r) < 0$, the retailer can never achieve the minimum total cost.

Lemma 1 indicates that the inventory level should be non-negative after the retailer places a special order (i.e., $q_r + f_n(t_r) \geq 0$). Thus, we additionally assume that $q_r + f_n(t_r) \geq 0$ to simplify our discussion, which occurs if and only if $t_r \leq ((m+n)Q^* - S^* + q_r)/D$. If $f_n(t_r) \geq 0$, the retailer will bear the ordering cost, the purchasing cost, and the inventory holding cost simultaneously, in which case, the total cost in the special order interval is given by

$$C_1(q_r, t_r, n) = A + \gamma w q_r + \frac{\gamma h((q_r + f_n(t_r))^2 - f_n^2(t_r))}{2D};$$

if $f_n(t_r) < 0$, the retailer will additionally pay the fixed backorder cost and the linear backorder cost for its special order, in which case, the corresponding cost is

$$C_2(q_r, t_r, n) = A + \gamma w q_r + \frac{\gamma h(q_r + f_n(t_r))^2}{2D} - c_f f_n(t_r) + \frac{c_l f_n^2(t_r)}{2D}.$$

Then, we examine the retailer's total cost in the remaining time horizon $(((m+n)Q^* - S^* + q_r)/D,\ k]$. Given that our inventory model converts to the classical EOQ model after time $((m+n)Q^* - S^* + q_r)/D$, we adopt the average cost C_{ar}^* of the regular EOQ ordering strategy in the remaining time horizon, where k is large enough to contain the special order interval.[1]

Combining the above analysis, we can derive the retailer's total cost

$$f(q_r, t_r, n) = \frac{(m+n)Q^* C_{ar}^*}{D} - \frac{(c_l S^* + 2c_f D)S^*}{2D} + C_i(q_r, t_r, n) + \left(k - \frac{(m+n)Q^* - S^* + q_r}{D}\right)C_{ar}^*,$$

where $i = 1$ if $f_n(t_r) \geq 0$ and $i = 2$ otherwise. One can check that $f(q_r, t_r, n)$ is continuous with respect to q_r and t_r while discrete with respect to n. The retailer can determine the optimal order decision by minimizing $f(q_r, t_r, n)$ subject to the constraints: $q_r \geq 0$, $t_s \leq t_r \leq \min\{t_e, ((m+n)Q^* - S^* + q_r)/D\}$, and $n \geq 0$.

([1]) The retailer's optimal order decision is actually independent of the length of the time horizon k; see Proposition 1.

4 Analysis

Thus far, we have established the retailer's total cost function $f(q_r, t_r, n)$. In this subsection, we further examine the retailer's optimal order decision by minimizing $f(q_r, t_r, n)$. To this end, we first derive the minimizer, denoted by (q_n, t_n), of $f(q_r, t_r, n)$ for a fixed n and then determine the optimal number n^*. For ease of exposition, let us define $f_1(q_r, t_r, n) = f(q_r, t_r, n)|_{f_n(t_r) \geq 0}$ and $f_2(q_r, t_r, n) = f(q_r, t_r, n)|_{f_n(t_r) \leq 0}$; then $f(q_r, t_r, n)$ can be seen as a piecewise-defined function consisting of two sub-functions $f_1(q_r, t_r, n)$ and $f_2(q_r, t_r, n)$.

Lemma 2: For a given n, (i) $f_1(q_r, t_r, n)$ is strictly decreasing in t_r and convex in q_r; (ii) $f_2(q_r, t_r, n)$ is strictly convex in t_r and q_r.

Lemma 2 reveals the structural property of the sub-function $f_i(q_r, t_r, n)$ for $i = 1, 2$. Solving $\partial f_2/\partial t_r = 0$ yields $t_r = \overline{t}_n$, where $\overline{t}_n = ((m+n)Q^* - S^*)/D + (C_{ar}^* - c_f D - \gamma w D)/c_l D$. It is evident that $f_2(q_r, t_r, n)$ strictly decreases in t_r when $t_r \leq \overline{t}_n$ and increases in t_r when $t_r \geq \overline{t}_n$. Next, we derive the minimizer, denoted by $(q_{i,n}, t_{i,n})$, of $f_i(q_r, t_r, n)$ for a given n.

Lemma 3: For a given n, (i) the minimum of the sub-function $f_1(q_r, t_r, n)$ occurs at $(q_{1,n}, t_{1,n})$, where $q_{1,n} = (C_{ar}^* - \gamma w D)/\gamma h - f_n(t_{1,n})$ and $t_{1,n} = \min\{t_e, ((m+n)Q^* - S^*)/D\}$; (ii) the minimum of the sub-function $f_2(q_r, t_r, n)$ occurs at $(q_{2,n}, t_{2,n})$, where $q_{2,n} = (C_{ar}^* - \gamma w D)/\gamma h - f_n(t_{2,n})$ and $t_{2,n}$ such that if $c_f \leq \sqrt{2Ah/D} + (1-\gamma)w/D$, then $t_{2,n} = (m+n)Q^*/D$, otherwise,

$$t_{2,n} = \begin{cases} t_s, & \text{if} t_s > \overline{t}_n, \\ \overline{t}_n, & \text{if} t_s \leq \overline{t}_n < t_e, \\ t_e, & \text{if} t_e \leq \overline{t}_n. \end{cases}$$

Building upon the minimizers of the sub-functions $f_1(q_r, t_r, n)$ and $f_2(q_r, t_r, n)$ for a given n, we then examine the minimum of the piecewise function $f(q_r, t_r, n)$.

Lemma 4: For a given n, the minimum of $f(q_r, t_r, n)$ occurs at (q_n, t_n), where $(q_n, t_n) = (q_{1,n}, t_{1,n})$ if $t_e < ((m+n)Q^* - S^*)/D$; otherwise $(q_n, t_n) = (q_{2,n}, t_{2,n})$.

Although the retailer can determine the optimal special order time t_n and the optimal special order quantity q_n given the number of regular orders n, it is not clear whether the retailer should place some regular orders in the promotion period to prepare for the special order. In the following, we derive the retailer's optimal order decision, (q_{n^*}, t_{n^*}, n^*), by substituting $q_r = q_n$ and $t_r = t_n$ into $f(q_r, t_r, n)$ and solving the optimization problem $\min_n\{f(q_n, t_n, n)\}$ for $n \geq 0$.

Proposition 1: With a disposable coupon, the retailer's optimal order decision is $(q_0, t_0, 0)$.

Proposition 1 indicates that the coupon should be applied to the retailer's first order in the promotion period (i.e., $n^* = 0$). Even if the promotion period lasts for a long time, the retailer will quickly place a special order after the coupon is available, which improves the supplier's cash flow and mitigates its overstock simultaneously. In particular, when $t_e \leq (mQ^* - S^*)/D$, the retailer always places the special order at the end time of the promotion period (i.e., $t_0 = t_e$). The following proposition demonstrates how the discount rate γ affects the maximum inventory level of the retailer.

Proposition 2: The maximum inventory level is always higher than that in the classical EOQ model and is strictly decreasing in the discount rate γ.

From Proposition 2, it would be better for the retailer to check on the capacity of its own warehouse before placing a special order, especially when the forecasted discount rate is highly seductive.

Proposition 3: When $\sqrt{2Ah/D} \leq c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$ and $t_e > m\sqrt{2A/hD}$, shortages cannot benefit the retailer unless the promotion period sets in.

Proposition 3 shows that if the fixed backorder cost is in an intermediate range and the promotion period ends late, the retailer should take shortages into account in the promotion period, even if shortages are futile in the previous regular orders (see Figure 4 for a visual illustration). This result emphasizes the importance of flexibly utilizing shortages.

Proposition 4: When the supplier raises (reduces) the discount rate, (i) the retailer will bring forward (postpone) its special order if the inventory level is negative at the original special order time; otherwise, the retailer will keep the special order time unchanged; (ii) the retailer will reduce (increase) its special order quantity regardless of the current inventory level.

Proposition 4 demonstrates how the retailer adjusts its order decision with respect to the discount rare. In particular, when the inventory level is non-negative throughout the promotion period (i.e., $t_e \leq (mQ^* - S^*)D$), the retailer always places the special order at a time when the inventory level reaches the minimum (i.e., $t_0 = t_e$), regardless of the discount rate; see Proposition 1. This result coincides with the minimum inventory principle in [20]. Differently, our result complements the minimum inventory principle by allowing for shortages and extending the duration of the promotion period.

Proposition 5: The longer the promotion period is, the more attractive the coupon will be to the retailer.

While a lower discount rate can help the supplier sell its products to more retailers, it may hurt the supplier by cutting its sales revenue. Proposition 5 indicates that the supplier can attract more retailers by properly extending the promotion period in addition to reducing the wholesale price. The intuition is that a longer promotion period endows the retailer more flexibility in ordering decision-making, which benefits the retailer and, thus, renders the supplier better off. This result enlightens the supplier on the promotion strategy.

5 Numerical Experiments

In this section, some numerical experiments are performed to illustrate the validity of the model.

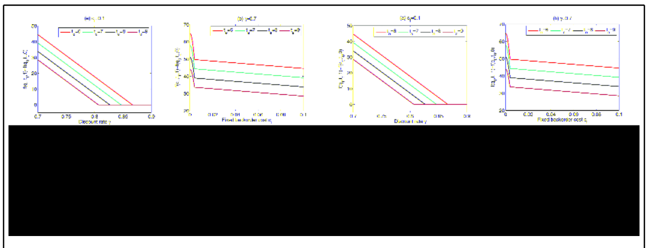

Figure 2: Comparison of Retailer's Ordering Strategies with n =0 and n =1 (D =1000, A =0.1, h =0.07, c_l =0.1, w =5, t_s =0.25, k =7).

When the promotion period contains a regular replenishment point, the retailer needs to decide whether to place a regular order at this point. If the retailer does so (i.e., n = 1), it incurs a total cost $f(q_1, t_1, 1)$; otherwise (i.e., n = 0), the corresponding total cost is $f(q_0, t_0, 0)$. Given that the retailer must make a trade-off between ``$n = 0$ *and* ``$n = 1$, the loss caused by the retailer adopting the ordering strategy with $n = 1$ can be measured by $f(q_1, t_1, 1) - f(q_0, t_0, 0)$, whose graphical illustration is shown in Figure 2. We observe that the ordering strategy with $n = 1$ always incurs a higher total cost than that with $n = 0$. Therefore, the retailer should promptly place the special order after the coupon is available, which is consistent with Proposition 1.

As depicted in Figure 2(a), the higher the discount rate is (or the later the promotion period ends), the lower the retailer's loss will be. In particular, when the discount rate is relatively high (e.g., $\gamma = 0.9$), the retailer may postpone placing its special order because there is no difference between the ordering strategies with $n = 0$ and $n = 1$. As such, it would be better for the supplier to reduce the wholesale price and shorten the promotion period simultaneously to facilitate the retailer to place the special order earlier. Figure 2(b) shows that the loss of the retailer increases as the fixed backorder cost c_f decreases. In particular, when c_f is reduced to below a certain threshold (i.e., $c_f < 0.00374$), there is a rapid jump in the loss of retailer due to the change of the regular order quantity Q^* and backorder level S^*. This emphasizes the importance of utilizing the coupon in time for a retailer who confronts a low fixed backorder cost.

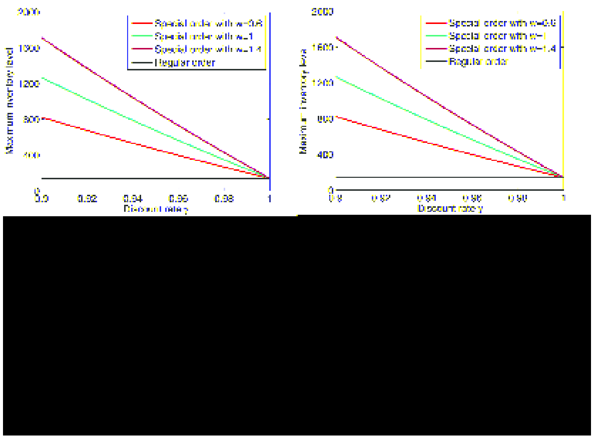

Figure 3: Maximum Inventory Level for γ and w (D=1000, A=1, h=0.1, c_f=0.2, c_l=0.1, t_s= 0.25, t_e=1, k=7).

The graphical illustration of Proposition 2 is depicted in Figure 3. It implies that the maximum inventory level caused by the special order is always higher than that caused by the regular order. Moreover, the curves gradually decrease and ultimately intersect at $\gamma = 1$. This displays how the maximum inventory level varies with the discount rate γ. In particular, when $\gamma = 1$, the maximum inventory level is an invariant constant regardless of the wholesale price and whether the retailer makes use of the coupon, because the benefit of the coupon vanishes. Another feature of Figure 3 is that the curve corresponding to a higher wholesale price (e.g., $w = 1.4$) is higher than that corresponding to a lower wholesale price (e.g., $w = 1$), indicating that a retailer who is charged a higher wholesale price should, if necessary, prepare a larger warehouse for the forthcoming promotion season.

See Figure 4 for a graphical illustration of Proposition 3. Note that shortages are currently not attractive to the retailer in the regular EOQ ordering strategy [37]. There are two noteworthy observations. First, the curve corresponding to "Shortages" is lower than that corresponding to "No shortages", indicating that although shortages cannot render the retailer better off in its regular orders, they benefits the retailer in the promotion season. Second, the gap between the two curves becomes wider as the discount rate decreases. This implies that making use of shortages in due time can help the retailer cut back on more spending from a lower discount rate.

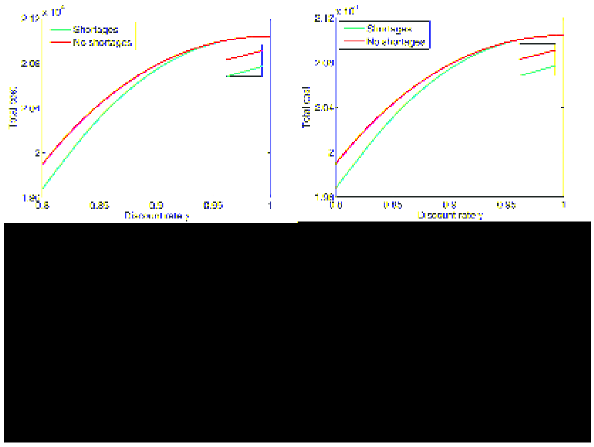

Figure 4: Comparison of Retailer's Total Costs under Allowable and Prohibitive Shortages (D=1000, A=0.1, h=0.2, c_f=0.1, c_l=0.06, w=3, t_s=0.25, t_e=0.7, k=7).

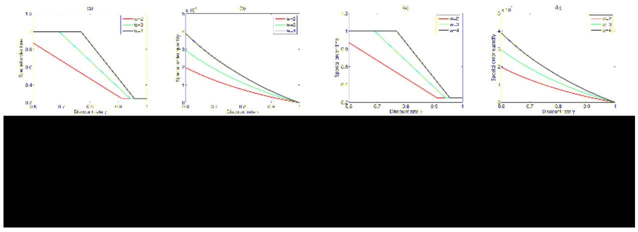

Figure 5: Retailer's Order Decision for γ and w (D=1000, A=0.1, h=0.07, c_f=0.2, c_l=1, t_s= 0.25, t_e=1, k=7).

Figure 5 illustrates the effect of the discount rate elaborated in Proposition 4. As shown in plot (a), the lower the discount rate is, the later the retailer will be to place a special order. Namely, a lower discount rate postpones the retailer's special order. From plot (b), a lower discount rate always facilitates the retailer to place a larger special order. An interesting observation is that the curves are smoother with a lower wholesale price (e.g., w = 2), but steeper as w increases. This implies that a retailer who suffers from a higher wholesale price is more sensitive to the discount rate.

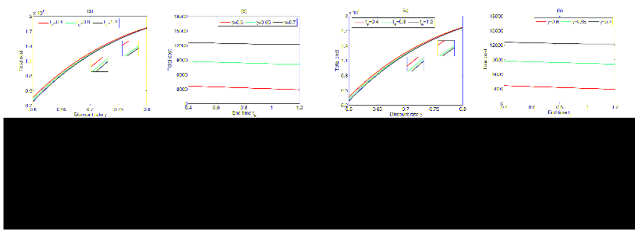

Figure 6: Retailer's Total Cost for γ and t_e (D=1000, A=0.1, h=0.07, c_f=0.2, c_l=0.1, w=3, t_s=0.25, k=7).

We illustrate Proposition 5 in Figure 6. It is evident that for any fixed end time t_e (e.g., t_e = 0.4), the retailer's total cost strictly increases with the discount rate γ; see plot (a). This result is intuitive because a higher discount rate increases the retailer's purchasing cost. In contrast, given the discount rate γ (e.g., γ = 0.7), the retailer's total cost slightly decreases with the end time t_e; see plot (b). This indicates that the later the coupon expires, the better off the retailer will be. Thus, the supplier can promote sales by extending the promotion period in addition to setting a lower wholesale price.

6 Conclusions

Many suppliers charge lower wholesale prices at times with an intent to attract more retailers and, thus, promote sales. To accelerate cash flow, the suppliers usually encourage their retailers to place one large order in the promotion period instead of many small orders. This paper focuses on an inventory system with allowable shortages under the framework of the EOQ model. The supplier offers the retailer a coupon, which can be utilized only once in the promotion season. The distinguishing feature of the model is that the duration of the promotion period is not necessary temporary, which makes the model more practical. In this sense, the retailer needs to decide the number of regular orders placed in the promotion period before making use of the coupon, in addition to the special order time and the special order quantity.

We derive the retailer's optimal order decision on the disposable coupon. With it, numerous managerial insights are obtained. First, the coupon should be applied to the first order in the promotion period regardless of the length of the promotion period. Second, we show that the maximum inventory level in our model is always higher than that in the classic EOQ model, which highlights the importance of the retailer checking its storage capacity before placing a special order. Third, we find that if the fixed backorder cost is in an intermediate range and the promotion period ends later, shortages can make the retailer better off even if they are not attractive to the retailer before the promotion period. Fourth, when the discount rate becomes lower, the retailer should place a larger special order while postponing the special order to a certain extent. Finally, in addition to reducing the discount rate, the supplier can promote sales by extending the promotion period, which benefits the retailer by endowing it with more flexibility in decision-making.

This paper has some limitations. First, the analysis in our model is constructed on the assumption that the market demand is common knowledge between the retailer and the supplier. The model could be generalized by considering a robust model with uncertain parameters (e.g., unpredictable demands and changeable lead times) [38,39]. Second, for analytical tractability, we assume that the demand rate is constant while normalize the leading time to zero. It could be interesting to consider (s,S) inventory systems with random leading times and multi-period resupply [40,41]. Third, in this paper, the supplier sells the product only through the retailer. In addition to the resell channel, the supplier can directly sell to end consumers by establishing a direct selling channel. Future research would be conducted to incorporate supplier encroachment [42,43].

BIBLIOGRAPHY

[1] Andriolo, A. and Battini, D. and Grubbström, R. W. and Persona, A and Sgarbossa, P. (2014) "A century of evolution from Harris's basic lot size model: Survey and research agenda", Volume 155. International Journal of Production Economics 16-38

[2] Ben-Zion, U. and Hibshoosh, A. and Spiegel, U. (1999) "The Optimal Face Value of a Discount Coupon.", Volume 51. Journal of Economics & Business 2 159-174

[3] Lev, B. and Weiss, H. J. (1990) "Inventory Models with Cost Changes.", Volume 38. Operations Research 1 53-63

[4] Tersine, R. J. and Barman, S. (1995) "Economic purchasing strategies for temporary price discounts.", Volume 80. European Journal of Operational Research 2 328-343

[5] Wee, H. and Yu, J. (1997) "A deteriorating inventory model with a temporary price discount.", Volume 53. International Journal of Production Economics 1 81–90

[6] Cárdenas-Barrón, L. E. and Smith, N. R. and Goyal, S. K. (2010) "Optimal order size to take advantage of a one-time discount offer with allowed backorders.", Volume 34. Applied Mathematical Modelling 6 1642–1652

[7] Yang, C. and Ouyang, L. Y. and Wu, K. S. and Yen, H. (2010) "An inventory model with temporary price discount when lead time links to order quantity", Volume 69. Journal of Scientific & Industrial Research 3 180-187

[8] Chang, H. J. and Lin, W. F. (2011) "A SIMPLE SOLUTION METHOD FOR THE FINITE HORIZON EOQ MODEL FOR DETERIORATING ITEMS WITH COST CHANGES", Volume 28. Asia-Pacific Journal of Operational Research 6 689-704

[9] Yang, C. T. and Ouyang, L. Y. and Wu, K. S. and Yen, H. F. (2012) "Optimal ordering policy in response to a temporary sale price when retailer's warehouse capacity is limited", Volume 6. European Journal of Industrial Engineering 1 26–49

[10] Taleizadeh, A. A. and Pentico, D. W. and Aryanezhad, M. and Ghoreyshi, S. M. (2012) "An economic order quantity model with partial backordering and a special sale price.", Volume 221. European Journal of Operational Research 3 571-583

[11] Shaposhnik, Y. and Herer, Y. T. and Naseraldin, H. (2015) "Optimal ordering for a probabilistic one-time discount.", Volume 244. European Journal of Operational Research 3 803–814

[12] Arcelus, F. J. and Shah, N. H. and Srinivasan, G. (2001) "Retailer's response to special sales: price discount vs. trade credit.", Volume 29. Omega 5 417–428

[13] Gaither, N. and Park, M. S. (1991) "Analysis of Constrained, One-Time, Multi-Item Discount Offers.", Volume 23. AIIE Transactions 3 228–235

[14] Goyal, S. K. and Srinivasan, G. and Arcelus, F. J. (1991) "One time only incentives and inventory policies.", Volume 54. European Journal of Operational Research 1 1-6

[15] Chang, H. J. and Lin, W. F. and Ho, J. F. (2011) "Closed-form solutions for Wee's and Martin's EOQ models with a temporary price discount.", Volume 131. International Journal of Production Economics 2 528-534

[16] Arcelus, F. J. and Shah, N. H. and Srinivasan, G. (2003) "Retailer's pricing, credit and inventory policies for deteriorating items in response to temporary price/credit incentives.", Volume 81-82. International Journal of Production Economics 153-162

[17] Abad, P. L. (2007) "Buyer's response to a temporary price reduction incorporating freight costs.", Volume 182. European Journal of Operational Research 3 1073-1083

[18] Hsu, W. K. K. and Yu, H. F. (2009) "EOQ model for imperfective items under a one-time-only discount.", Volume 37. Omega 5 1018–1026

[19] Tulika Chakraborty and Satyaveer S. Chauhan and Anjali Awasthi and Tatiana Bouzdine-Chameeva. (2019) "Two-period pricing and ordering policy with price-sensitive uncertain demand", Volume 70. Journal of the Operational Research Society 3 377-394

[20] Ardalan, A. (1988) "Optimal Ordering Policies in Response to a Sale.", Volume 20. IIE Transactions 3 292–294

[21] Aull-Hyde, R. L. (1992) "Evaluation of Supplier-Restricted Purchasing Options Under Temporary Price Discounts.", Volume 24. AIIE Transactions 2 184–186

[22] Ardalan, A. (1994) "Optimal prices and order quantities when temporary price discounts result in increase in demand.", Volume 72. European Journal of Operational Research 1 52–61

[23] Aull-Hyde, R. L. (1996) "A Backlog Inventory Model During Restricted Sale Periods.", Volume 47. Journal of the Operational Research Society 9 1192–1200

[24] Chu, P. and Chen, P. S. and Niu, T. (2003) "Note on supplier-restricted order quantity under temporary price discounts.", Volume 58. Mathematical Methods of Operations Research 1 141–147

[25] Abad, P. L. (2003) "Optimal price and lot size when the supplier offers a temporary price reduction over an interval.", Volume 30. Computers and Operations Research 1 63–74

[26] Sarker, B. R. and Kindi, M. A. (2006) "Optimal ordering policies in response to a discount offer.", Volume 100. International Journal of Production Economics 2 195–211

[27] Cárdenas-Barrón, L. E. (2009) "Optimal ordering policies in response to a discount offer: Corrections.", Volume 122. International Journal of Production Economics 2 783-789

[28] Cárdenas-Barrón, L. E. (2009) "Optimal ordering policies in response to a discount offer: Extensions.", Volume 122. International Journal of Production Economics 2 774-782

[29] Kindi, M. A. and Sarker, B. R. (2011) "Optimal inventory system with two backlog costs in response to a discount offer.", Volume 22. Production Planning and Control 3 325–333

[30] Sari, D. P. and Rusdiansyah, A. and Huang, L. (2012) "Models of Joint Economic Lot-sizing Problem with Time-based Temporary Price Discounts.", Volume 139. International Journal of Production Economics 1 145-154

[31] Karimi-Nasab, M. and Konstantaras, I. (2013) "An inventory control model with stochastic review interval and special sale offer", Volume 227. European Journal of Operational Research 1 81-87

[32] Cárdenas-Barrón, L. E. and Chung, K. J. and Kazemi, N. and Shekarian, E. (2018) "Optimal inventory system with two backlog costs in response to a discount offer: corrections and complements.", Volume 18. Operational Research 97–104

[33] Wang, Y. and Gao, H. and Wei, X. (2018) "Optimal replenishment and stocking strategies for inventory mechanism with a dynamically stochastic short-term price discount.", Volume 70. Journal of Global Optimization 1 27–53

[34] Gao, H. and Wang, D. and Santibanez Gonzalez, E. D. R. and Ju, Y. (2019) "Optimal stocking strategies for inventory mechanism with a stochastic short-term price discount and partial backordering.", Volume 57. International Journal of Production Research 2 1–30

[35] Arcelus, F. J. and Pakkala, T. P. M. and Srinivasan, G. (2009) "A retailer's decision process when anticipating a vendor's temporary discount offer", Volume 57. Computers & Industrial Engineering 1 253-260

[36] Zhang, Z. and Zhang, S. T. and Yue, M. S. (2022) "Joint ordering policy for a conditional trade credit model with two retailers", Volume 16. European Journal of Industrial Engineering 4 398-417

[37] Sphicas, G. P. (2006) "EOQ and EPQ with linear and fixed backorder costs: Two cases identified and models analyzed without calculus.", Volume 98. International Journal of Production Economics 1 59–64

[38] Shang, C. and You, F. Q. (2018) "Distributionally robust optimization for planning and scheduling under uncertainty", Volume 110. Computers & Chemical Engineering 53-68

[39] Wang, Y. D. and Shi, Q. and Li, L. and Li, F. and Xia, W. (2021) "Spare parts supply network optimization with uncertain distributed lead times and demands", Volume 37. Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería 1 1

[40] Bashyam, S. and Fu, M. C. (1998) "Optimization of (s, S) Inventory Systems with Random Lead Times and a Service Level Constraint", Volume 44. Management Science S243-S256

[41] Deng, H. Y. and Shi, Q. and Wang, Y. D. (2021) "Joint optimization of condition-based maintenance and inventory ordering based on status monitoring for multi-unit system", Volume 37. Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería 4 45

[42] Huang, S. and Guan, X. and Chen, Y. J. (2018) "Retailer Information Sharing with Supplier Encroachment", Volume 27. Production & Operations Management 6 1133-1147

[43] Yue, M. S. and Zhang, S. T. and Zhang, Z. (2022) "Deterrence effect of risk aversion information sharing on supplier encroachment", Volume 169. Computers & Industrial Engineering 108246

Appendix

Proof of Lemma 1: We denote by π the retailer's ordering strategy (q_r, t_r, n) satisfying $q_r + f_n(t_r) < 0$ and define $\pi' = (q_r, t'_r, n)$, where $t'_r = t_r +_r + f_n(n(t_r))/D$. Note that although π and π' involve the same special order quantity q_r, the special order time t'_r in π' is earlier than the special order time t_r in π (i.e., $t'_r < t_r$). We then prove that the total cost of π is higher than that of π'.

Since $q_r + f_n(t_r) < 0$, $q_r + f_n(t'_r) = 0$, and $t'_r < t_r$, the two ordering strategies (i.e., π and π') lead to the same inventory level in the time horizon $[0, k]$ except for the interval $[t'_r, t_r]$. Thus, we need only to compare the total costs of π and π' in $[t'_r, t_r]$. Given that the inventory level of π' is always non-positive and higher than that of π in $[t'_r, t_r]$, π and π' lead to the same fixed backorder cost and π' incurs a lower linear backorder cost than π. Therefore, the ordering strategy (q_r, t_r, n) with $q_r + f_n(t_r) < 0$ cannot help the retailer reach the minimum total cost

Proof of Lemma 2: Taking the partial derivatives of f_1 and f_2 with respect to q_r and t_r yields $\partial f_1/\partial t_r = -\gamma h q_r$, $\partial f_2/\partial t_r = -(\gamma h + c_l) f_n(t_r) - \gamma h q_r + c_f D$, $\partial f_1/\partial q_r = \partial f_2/\partial q_r = \gamma w + (\gamma h f_n(t_r) + \gamma h q_r - C^*_{ar})/D$. The corresponding second partial derivatives are $\partial^2 f_1/\partial t_r^2 = 0$, $\partial^2 f_1/\partial q_r^2 = \gamma h/D$, $\partial^2 f_2/\partial t_r^2 = (\gamma h + c_l)D$, $\partial^2 f_2/\partial t_r \partial q_r = -\gamma h$, and $\partial^2 f_2/\partial q_r^2 = \gamma h/D$.

Since $\partial f_1/\partial t_r < 0$ and $\partial^2 f_1/\partial q_r^2 > 0$, f_1 is decreasing t_r and convex in q_r. Constructing the Hessian matrix H of f_2 and calculating the determinant of it yields $|H| = \gamma h c_l > 0$. Hence, f_2 is convex with respect to q_r and t_r.

Proof of Lemma 3: (i) For a given n, because $f_n(t_r) \geq 0$ if and only if $t_r \leq ((m+n)Q^* - S^*)/D$, we need only to minimize $f_1(q_r, t_r, n)$ subject to the constrains: $q_r \geq 0$ and $t_s \leq t_r \leq \min\{t_e, ((m+n)Q^* - S^*)/D\}$. The result directly follows from Lemma 2(i) and the first-order optimality condition (i.e., $\partial f_1/\partial q_r = 0$).

(ii) By the same token, we minimize $f_2(q_r, t_r, n)$ for a fixed n subject to $q_r \geq 0$ and $\max\{t_s, ((m+n)Q^* - S^*)/D\} \leq t_r \leq \min\{t_e, ((m+n)Q^* - S^* + q_r)/D\}$. The stable point, $(\bar{q}_n, \bar{t}_n)$, of $f_2(q_r, t_r, n)$ for a fixed n satisfies $\bar{q}_n = ((1-\gamma)wD + h(Q^* - S^*))/\gamma h - f_n(\bar{t}_n)$ and $\bar{t}_n = ((m+n)Q^* - S^*)/D + (C^*_{ar} - c_f D - \gamma w D)/c_l D$. We then discuss whether the stable point $(\bar{q}_n, \bar{t}_n)$ locates in the feasible domain of f_2 mentioned above. It is worthy noting that $\bar{q}_n \geq 0$ always holds under the condition of $f_n(\bar{t}_n) \leq 0$, which occurs if and only if $\bar{t}_n \geq ((m+n)Q^* - S^*)/D$. Following $\bar{q}_n + f_n(\bar{t}_n) = ((1-\gamma)wD + h(Q^* - S^*))/\gamma h \geq 0$, we have $\bar{t}_n \leq ((m+n)Q^* - S^* + \bar{q}_n)/D$. Given that $\bar{t}_n \geq ((m+n)Q^* - S^*)/D$ if and only if $C^*_{ar} - c_f D - \gamma w D \geq 0$, the discussion is divided into the following two cases based on the values of Q^* and S^* [37].

Case 1 $c_f < \sqrt{2Ah/D}$. Following $C^*_{ar} - c_f D - \gamma w D = (1-\gamma)wD + c_l S^* \geq 0$, we have $\bar{t}_n \geq ((m+n)Q^* - S^*)/D$ (or equivalently, $f_n(\bar{t}_n) \leq 0$) and thus $\bar{q}_n \geq 0$. In this case, $(\bar{q}_n, \bar{t}_n)$ locates in the feasible domain of f_2 if and only if $t_s \leq \bar{t}_n \leq t_e$. Thus, $t_{2,n} = t_s$ if $\bar{t}_n < t_s$; $t_{2,n} = t_e$ if $\bar{t}_n > t_e$; and $t_{2,n} = \bar{t}_n$ if $t_s \leq \bar{t}_n \leq t_e$. And $q_{2,n}$ follows from the first-order optimality condition.

Case 2 $c_f \geq \sqrt{2Ah/D}$. In this case, we have $C^*_{ar} - c_f D - \gamma w D = (1-\gamma)wD + \sqrt{2ADh} - c_f D$. Since $S^* = 0$ and $t_s \leq mQ^*/D$, the feasible domain of f_2 is reduced to $q_r \geq 0$ and $(m+n)Q^*/D \leq t_r \leq \min\{t_e, ((m+n)Q^* + q_r)/D\}$. If $c_f \leq (1-\gamma)w + \sqrt{2Ah/D}$, then $\bar{t}_n \geq (m+n)Q^*/D$ and $\bar{q}_n \geq 0$. In this context, $(\bar{q}_n, \bar{t}_n)$ locates in the feasible domain of f_2 if and only if $t_s \leq \bar{t}_n \leq t_e$. Alternatively, if $c_f > \sqrt{2Ah/D} + (1-\gamma)w/D$, then $\bar{t}_n < (m+n)Q^*/D$. In this context, $(\bar{q}_n, \bar{t}_n)$ does not locate in the feasible domain of f_2. Thus, $t_{2,n} = (m+n)Q^*/D$ and $q_{2,n} = ((1-\gamma)wD + h(Q^* - S^*))/\gamma h \geq 0$.

Proof of Lemma 4: The discussion is divided into the following two cases.

Case 1 $t_e < ((m+n)Q^* - S^*)/D$. Given that $t_r < ((m+n)Q^* - S^*)/D$ for any $t_r \in [t_s, t_e]$, we always have $f_n(t_r) > 0$. In this context, the feasible domain of $f_2(q_r, t_r, n)$ is empty; thus, the minimizer of $f(q_r, t_r, n)$ is coincident with that of $f_1(q_r, t_r, n)$.

Case 2 $t_e \geq ((m+n)Q^* - S^*)/D$. If $t_s \leq ((m+n)Q^* - S^*)/D$, then $f_1(q_r, t_r, n)$ reaches the minimum at $(q_{1,n}, t_{1,n})$, where $q_{1,n} = (C^*_{ar} - \gamma w D)/\gamma h$ and $t_{1,n} = ((m+n)Q^* - S^*)/D$; see Lemma 3. Since $f_n(t_{1,n}) = 0$, the minimizer of f_1 also locates in the feasible domain of f_2. Hence, the minimizer of f can be regarded as that of f_2. Alternatively, if $t_s > ((m+n)Q^* - S^*)/D$, then $f_n(t_r) < 0$ always holds for any $t_r \in [t_s, t_e]$. In this context, the domain of $f_1(q_r, t_r, n)$ is empty. Thus, the minimizer of $f(q_r, t_r, n)$ is exactly that of $f_2(q_r, t_r, n)$.

Proof of Proposition 1: We first show that the optimal number $n^* \in \{0, 1\}$. To this end, we need to find some $n' \in \{0, 1\}$ for any ordering strategy (q_r, t_r, n) with $n \geq 2$ such that $f(q_{n'}, t_{n'}, n') \leq f(q_r, t_r, n)$, where $(q_{n'}, t_{n'})$ is the minimizer of $f(q_r, t_r, n')$. Specifically, when $t_r < t_s + nQ^*/D$, let $t'_r = t_r - (n-1)Q^*/D$; then $t'_r \in [t_s, t_e]$ and $f_1(t'_r) = f_n(t_r)$. Following $f_1(t'_r) = f_n(t_r)$, we have $f(q_r, t_r, n) = f(q_r, t'_r, 1) \geq f(q_1, t_1, 1)$, where (q_1, t_1) is the minimizer of $f(q_r, t_r, 1)$. Similarly, when $t_r \geq t_s + nQ^*/D$, let $t'_r = t_r - nQ/D$; then $f(q_r, t_r, n) = f(q_r, t'_r, 0) \geq f(q_0, t_0, 0)$. Therefore, it is not necessary for the retailer to place more than one regular orders in $[t_s, t_r]$; that is, $0 \leq n^* \leq 1$.

Next, we examine when the retailer places a regular order in $[t_s, t_r]$. For ease of exposition, we denote by t_f the first regular replenishment point after time t_s, i.e., $t_f = mQ^*/D$. Specifically, if $t_e < t_f$, there is no regular replenishment point in the promotion period $[t_s, t_e]$; thus, the retailer never places regular orders in $[t_s, t_r]$ (i.e., $n^* = 0$). If $t_e \geq t_f$, the discussion is divided into two cases based on the relationship between t_e and $((m+1)Q^* - S^*)/D$. Note that all items will be sold out at time $((m+1)Q^* - S^*)/D$ if the retailer places a regular order at time t_f.

Case 1 $t_f \leq t_e < ((m+1)Q^* - S^*)/D$. Given that t_f is the unique regular replenishment point in the promotion period $[t_s, t_e]$, the retailer needs to decide whether to place a regular order at time t_f. If he does so (i.e., $n = 1$), by $t_e < ((m+1)Q^* - S^*)/D$, Lemmas 3(i), and Lemma 4, $f(q_r, t_r, 1)$ reaches the minimum at $(q_{1,1}, t_{1,1})$; that is $f(q_r, t_r, 1) \geq f(q_{1,1}, t_{1,1}, 1) = f_1(q_{1,1}, t_{1,1}, 1)$ for any $q_r \geq 0$ and $t_r \in [t_s, t_e]$. If he does not so (i.e., $n = 0$), given that $t_e \geq (mQ^* - S^*)/D$, the minimum of $f(q_r, t_r, 0)$ occurs at $(q_{2,0}, t_{2,0})$; that is, $f(q_r, t_r, 1) \geq f(q_{2,0}, t_{2,0}, 0) = f_2(q_{2,0}, t_{2,0}, 0)$ for any $q_r \geq 0$ and $t_r \in [t_s, t_e]$. In this sense, the retailer places a regular order at time t_s (i.e., $n^* = 1$) if and only if $f_1(q_{1,1}, t_{1,1}, 1) < f_2(q_{2,0}, t_{2,0}, 0)$. Let $t'_r = (mQ^* - S^*)/D$, the discussion is further divided into the following two subcases based on the relationship between t_s and t'_r.

Case 1.1 $t'_r \geq t_s$. From Lemma 3 (ii), we have $f_2(q_{1,1}, t'_r, 0) \geq f_2(q_{2,0}, t_{2,0}, 0)$, where $(q_{2,0}, t_{2,0})$ is the minimizer of $f_2(q_r, t_r, 0)$. We then prove $f_1(q_{1,1}, t_{1,1}, 1) > f_2(q_{1,1}, t'_r, 0)$ by mildly extending the promotion period from $[t_s, t_e]$ to $[t_s, t'_e]$, where $t'_e = ((m+1)Q^* - S^*)/D$. It is straightforward that Lemmas 1-3 hold for the alternative promotion period $[t_s, t'_e]$. Following Lemma 2(i) and $f_0(t'_r) = f_1(t'_e) = 0$, we have $f_1(q_{1,1}, t_{1,1}, 1) > f_1(q_{1,1}, t'_e, 1) = f_2(q_{1,1}, t'_r, 0)$. Combining the above analysis, we have $f_1(q_{1,1}, t_{1,1}, 1) > f_2(q_{2,0}, t_{2,0}, 0)$. Thus, $n^* = 0$.

Case 1.2 $t_r < t_s$. Suppose that $c_f \geq \sqrt{2Ah/D}$, following $S^* = 0$ and

$t_s \square Q^*/D$, we have $t_s \square t'_r$, which contradicts with $t'_r < t_s$. Thus, $c_f < \sqrt{2Ah/D}$. It is evident that when $c_f < \sqrt{2Ah/D}$, $\bar{t}_0 > mQ^*/D$ if and only if $(1-\gamma)wD > 0$, which always holds because $0 < \gamma < 1$. Following $\bar{t}_0 > mQ^*/D$ and $mQ^*/D \square t_s$, we have $\bar{t}_0 > t_s$. We then prove the result by extending the promotion period from $[t_s, t_e]$ to $[t't_e]_e]$, where $t'_s = t_r$ and $t'_e = ((m+1)Q^* - S^*)/D$. Note that Lemmas 1-3 still hold for the alternative promotion period $[t't_e]_e]$. Following Lemma 2(i) and $f_0(t'_r) = f_1(t'_e) = 0$, we have $f_1(q_{1,1}, t_{1,1}, 1) > f_1(q_{1,1}, t'_e, 1 = f_2(q_{1,1}, t'_r, 0)$. Because $\bar{t}_0 > t'_r$, $f_2(q_{1,1}, t_r, 0)$ is strictly decreasing in t_r when $t_r \in [t'_r, t''_r]$, where $t''_r = \min\{\bar{t}_0, t_e\}$ satisfying $t''_r \in [t_s, t_e]$; see Lemma 2. Thus, $f_2(q_{1,1}, t'_r, 0)_> f_2(q_{1,1}, t''_r, 0)$. Given that $t''_r \in [t_s, t_e]$ and that $(q_{2,0}, t_{2,0})$ is the minimizer of $f_2(q_r, t_r, 0)$ under the condition of the promotion period being $[t_s, t_e]$, we have $f_2(q_{1,1}, t''_r, 0) \square f_2(q_{2,0}, t_{2,0}, 0)$. Based on the above, we conclude that $f_1(q_{1,1}, t_{1,1}, 1) > f_2(q_{2,0}, t_{2,0}, 0)$; that is, $n^* = 0$.

Case 2 $t_e \square ((m+1)Q^* - S^*)/D$. By Lemmas 3 and 4, the retailer's minimum total cost is $f_2(q_{2,1}, t_{2,1}, 1)$ if the retailer places a regular order at time t_f; otherwise, its minimum total cost is $f_2(q_{2,0}, t_{2,0}, 0)$. If $c_f \square \sqrt{2Ah/D} + (1-\gamma)wD$, then $t_{2,1} = (m+1)Q^*/D$; see Lemma 3(ii). Otherwise, following $\bar{t}_1 > (m+1)Q^*/D > t_s$, $t_e \square ((m+1)Q^* - S^*)/D$, and Lemma 3(ii), we have $t_{2,1} \square ((m+1)Q^* - S^*)/D$. Let $t'_{2,1} = t_{2,1} - Q^*/D$, then $t'_{2,1} \in [(mQ^* - S^*)/D, t_e]$. Using $f_0(t'_{2,1}) = f_1(t_{2,1})$, we have $f_2(q_{2,1}, t_{2,1}, 1) = f_2(q_{2,1}, t'_{2,1}, 0)$. In this sense, the retailer places a regular order at time t_f (i.e., $n^* = 1$) if and only if $f_2(q_{2,1}, t'_{2,1}, 0) \square f_2(q_{2,0}, t_{2,0}, 0)$. Specifically, if $t'_{2,1} \square t_s$, using $t'_{21} \in [t_s, t_e]$, we have $f_2(q_{2,0}, t_{2,0}, 0) \square f_2(q_{2,1}, t'_{2,1}, 0)$, where $(q_{2,0}, t_{2,0})$ is the minimizer of $f_2(q_r, t_r, 0)$. Thus, $n^* = 0$. Alternatively, if $t'_{2,1} < t_s$, the discussion is further divided into the following two subcases based on the relationship between $\bar{t}_0$ and t_e. Note that $t'_{2,1} < t_s \square t_{2,0}$.

Case 2.1 $\bar{t}_0 < t_e$. Suppose that $c_f \square \sqrt{2Ah/D}$, then $S^* = 0$. Using $S^* = 0$, $t_s \square mQ^*/D$, and $t'_{2,1} \square (mQ^* - S^*)/D$, we have $t_s \square t'_{2,1}$, which contradicts with $t'_{2,1} < t_s$. Hence, $c_f < \sqrt{2Ah/D}$. Following $c_f < \sqrt{2Ah/D}$ and $0 < \gamma < 1$, we have $\bar{t}_0 \square mQ^*/D$ and thus $\bar{t}_0 \square t_s$. The result will be proven by replacing $[t_s, t_e]$ with $[t'_{2,1}, t_{2,0}]$. Lemmas 1-3 still hold for the promotion period $[t'_{2,1}, t_{2,0}]$. By $c_f < \sqrt{2Ah/D}$, $t_s \square \bar{t}_0 < t_e$, and Lemma 3 (ii), we have $t_{2,0} = \bar{t}_0$. Thus, $f_2(q_r, t_r, 0)$ is strictly decreasing in t_r when $t_r \in [t'_{2,0}, t_{2,0}]$; see Lemma 2. In this sense, we have that $f_2(q_{2,1}, t'_{2,1}, 0) > f_2(q_{2,1}, t_{2,0}, 0) \square f_2(q_{2,0}, t_{2,0}, 0)$, where $(q_{2,0}, t_{2,0})$ is the minimizer of $f_2(q_r, t_r, 0)$ under the condition of the promotion period being $[t_s, t_e]$. Thus, $n^* = 0$.

Case 2.2 $\bar{t}_0 \square t_e$. By Lemma 3 (ii), we have $t_{2,0} = t_e$. We prove by replacing $[t_s, t_e]$ with $[t'_{2,1}, t_{2,0}]$. Lemmas 1-3 hold for $[t'_{2,1}, t_{2,0}]$. Following $\bar{t}_0 \square t_e$ and $t_e = t_{2,0}$, we have that $f_2(q_r, t_r, 0)$ is strictly decreasing in t_r when $t_r \in [t'_{2,1}, t_{2,0}]$. Thus, $f_2(q_{2,1}, t'_{2,1}, 0) > f_2(q_{2,1}, t_{2,0}, 0) \square f_2(q_{2,0}, t_{2,0}, 0)$ (i.e., $n^* = 0$), where $(q_{2,0}, t_{2,0})$ is the minimizer of $f_2(q_r, t_r, 0)$ under the condition of the promotion period being $[t_s, t_e]$.

Proof of Proposition 2: Let $g(\gamma) = q_0 + f_0(t_0) - (Q^* - S^*)$, the result directly follows from $g(\gamma) = ((1-\gamma)wD + h(Q^* - S^*))/\gamma h > 0$ and $\mathrm{d}g(\gamma)/\mathrm{d}\gamma = -(wD + h(Q^* - S^*))/\gamma^2 h < 0$.

Proof of Proposition 3: Following $c_f \square \sqrt{2Ah/D}$, we have $Q^* = \sqrt{2AD/h}$ and $S^* = 0$, indicating that shortages cannot make the retailer better off through regular EOQ orders. We then prove by showing that the minimum inventory level in the promotion period is negative (i.e., $f_0(t_0) < 0$). Using $t_e > mQ^*/D$ and Lemma 4, we have $t_0 = t_{2,0}$. From $c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$, we have $\bar{t}_0 > mQ^*/D$ and thus $\bar{t}_0 > t_s$. According to $c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$ and $\bar{t}_0 > t_s$, we have $t_{2,0} = \min\{t_e, \bar{t}_0\}$; see Lemma 3(ii). Given that $t_e > mQ^*/D$ and $\bar{t}_0 > mQ^*/D$, $t_{2,0} > mQ^*/D$. Thus, $f_0(t_0) = f_0(t_{2,0}) < 0$.

Proof of Proposition 4: (i) If $f_0(t_0) < 0$, then $t_0 > (mQ^* - S^*)/D$. Suppose that $t_0 = t_{1,0}$, we have $f_0(t_0) \square 0$, which yields a contradiction. Thus, $t_0 = t_{2,0}$. Suppose that $c_f \square \sqrt{2Ah/D} + (1-\gamma)w/D$, then $t_{2,0} = (mQ^* - S^*)/D$, wherein $S^* = 0$. This contradicts with $t_{2,0} > (mQ^* - S^*)/D$. Following $t_0 = t_{2,0}$, $c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$, and Lemma 3(ii), we have $dt_0/d\gamma = dt_{2,0}/d\gamma = d\bar{t}_0/d\gamma = -w/c_l < 0$ if $\bar{t}_0 \in [t_s, t_e]$; otherwise $dt_0/d\gamma = dt_{2,0}/d\gamma = 0$. Alternatively, if $f_0(t_0) \square 0$, then $t_0 \square (mQ^* - S^*)/D$. Recall that $\bar{t}_0 > mQ^*/D$ holds for all $c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$. Suppose that $t_0 = t_{2,0}$ and $c_f < \sqrt{2Ah/D} + (1-\gamma)w/D$ hold simultaneously, using $\bar{t}_0 > mQ^*/D$, we have $t_0 = t_{2,0} > mQ^*/D$; see Lemma 3(ii). This contradicts with $t_0 \square (mQ^* - S^*)/D$. Thus, we can conclude that either $t_0 = t_{1,0}$ or $t_0 = mQ^*/D$ holds, which leads to $dt_0/d\gamma = 0$.

(ii) The result directly follows from $dq_0/d\gamma = -C^*_{ar}/\gamma^2 h < 0$.

Proof of Proposition 5: If the supplier extends the promotion period, the retailer will always have an option to place the same special order as before. As a consequence, a longer promotion can only benefit the retailer instead of making it worse off.