Spatio-temporal pattern detection in spatio-temporal graphs

Use case of invasive team sports and urban road traffic

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ABSTRACT. Spatio-temporal (ST) graphs have been used in many application domains to model evolving ST phenomenon. Such models represent the underlying structure of the phenomenon in terms of its entities and different types of spatial interactions between them. The reason behind using graph-based models to represent ST phenomenon is due to the existing well-established graph analysis tools and algorithms which can be directly applied to analyze the phenomenon under consideration. In this paper, considering the use case of two distinct, highly dynamic phenomena - invasive team sports, with a focus on handball and urban road traffic, we propose a spatio-temporal graph model applicable to both these phenomena. Different types of entities and spatial relations which make up these phenomena are highlighted to formalize the graph. Furthermore, the idea of graph-based pattern detection in both these phenomena is explored. Different types of ST patterns for both ST phenomena are discussed and the problem of pattern detection is formalized as the problem of subgraph isomorphism for dynamic graphs. Finally, the results of our algorithm to detect random ST patterns in random ST graphs are presented. The ideas discussed in this paper are applicable to other ST phenomena as well.

RÉSUMÉ. Les graphes spatio-temporels sont utilisés dans de nombreux domaines d'application pour modéliser les phénomènes spatio-temporels. Ces modèles représentent la structure sous-jacente du phénomène en termes d'entités et de différents types d'interactions spatiales entre elles. La motivation d'utiliser des modèles à base de graphes pour représenter le phénomène ST est due à l'existence d'outils et d'algorithmes d'analyse de graphes bien établis qui peuvent être directement appliqués pour analyser le phénomène considéré. Dans cet article, en considérant le cas d'usage de deux phénomènes distincts et très dynamiques - les sports collectifs, avec un regard particulier sur le handball et le trafic routier urbain, nous proposons un modèle

de graphe spatio-temporel applicable à ces deux phénomènes. Les différents types d'entités et les relations spatiales qui composent ces phénomènes sont soulignés pour formaliser ce graphe. En outre, nous explorons l'ideé de la détection de motifs basés sur les graphes dans le handball et le trafic routier urbain. Différents types de motifs spatio-temporels pour les deux phénomènes spatio-temporels sont discutés et le problème de la détection des motifs est formalisé comme le problème de l'isomorphisme des sous-graphes pour les graphes dynamiques. Enfin, les résultats de notre algorithme de détection de motifs spatio-temporels aléatoires dans les graphes aléatoires sont présentés. Les idées discutées dans cet article sont également applicables à d'autres phénomènes spatio-temporels.

1. Introduction

Graphs have proven to be a significant modeling tool in different kinds of applications such as bioinformatics, telecommunications, social networks etc. They represent the underlying structure of the phenomenon being modeled in terms of the entities which make up that phenomenon as well as the interactions/relations between those entities. For example, in bioinformatics, graphs represent molecules or protein interaction networks (Bonnici *et al*., 2013), in social network analysis, graphs represent relationships between people (such as drug traffickers in (Fan, 2012)), in video analysis for object tracking, graphs represent the adjacency between different object planes present in the video (Diot *et al*., 2012) etc. When both spatial and temporal dimensions are taken into account, the resultant graphs are called spatio-temporal (ST) graphs. Such graphs have been used to model the dynamics of street networks (Costes *et al*., 2015), for underlining spatial relations between places and events (Maduako, Wachowicz, 2019) etc.

Representing a ST phenomenon using a graph helps to explain the evolution of the phenomenon in terms of the evolution of the graph, i.e., by modeling change in the attributes of nodes and edges of the graph and by modeling change in its overall structure (addition/removal of nodes and edges). The evolution of the graph makes the temporal dimension of the model explicit. The spatial dimension of the model, on the other hand, can be incorporated directly using spatial relations between the entities (nodes) of the phenomenon (graph). Furthermore, with this modeling approach, well developed graph algorithms can be applied in order to understand the characteristics of the underlying ST phenomenon.

One way to analyse the ST phenomenon is to detect some meaningful and pertinent patterns in the phenomenon which might explain its overall ST evolution and/or help in better understanding the interactions between all or some of its entities. Once

such patterns are detected, further reasoning can be carried out about the phenomenon. Depending on the requirements of the analysis and the phenomenon being modeled, pertinent patterns might represent the visitation behaviour of animals (Bracis *et al.*, 2018), proximity of trajectories of moving objects (Demšar, Virrantaus, 2010), movement patterns of soccer players (Beernaerts *et al.*, 2020) etc. In these examples, the patterns detected are formalized in terms of quantitative data about the phenomenon. For example, in (Bracis *et al.*, 2018), the authors look at the animal trajectory data to calculate their re-visitation patterns in a specific area. Such patterns are described as the number of times the animal's trajectory passes through an area of interest. The authors then use these patterns and statistical clustering-based methods to determine the frequently visited locations of a turkey vulture. Similarly, (Demšar, Virrantaus, 2010) explores ship trajectory data to find spatio-temporal similarities between the movement patterns of different kinds of ships, and (Beernaerts *et al.*, 2020) uses Qualitative Trajectory Calculus (QTC) (Weghe *et al.*, 2004) to describe the relative movement of soccer players and detect similar movement patterns.

Although useful, such statistical patterns are not able to take into account the underlying structure of the phenomenon and do not naturally incorporate the interactions between different entities of the phenomenon. Graphs, naturally, model the phenomenon in terms of its entities and relations between those entities and, hence, formalize the underlying structure of the phenomenon. Furthermore, graph-based patterns take into account this structural point-of-view and are capable of providing additional information not included within statistical patterns.

In this paper, we take the graph-based approach to model two different kinds of ST phenomena, invasive team sports and urban road traffic. Invasive team sports are sports in which one team has to invade the playing area of another team to score points, such as football, handball, basketball etc. In this paper, we model handball as a ST phenomenon. This paper is an extended version of our previous work (Oberoi, Del Mondo, 2021) in which we applied the graph models to represent the same two phenomena, and discussed various structural patterns which can be detected in both these phenomena using an algorithm we developed based on the idea of subgraph isomorphism for dynamic graphs (Oberoi, 2019).

As compared to our previous work, in this paper, we enhance our graph models with spatial relations (represented as edge labels) between different entities (nodes) for both phenomena. In addition, we incorporate spatial relations in the definition of graph-based patterns themselves, making them spatio-temporal (as opposed to just structural patterns in (Oberoi, Del Mondo, 2021)) in the true sense. Furthermore, we update the mathematical formulation of the problem of subgraph isomorphism in order to take into account the edge labels while searching for suitable solutions. Finally, we report the results obtained by applying the updated representation of spatio-temporal phenomena and spatio-temporal patterns, both modeled as random ST graphs, and the new definition of subgraph isomorphism in our previously proposed algorithm (Oberoi, 2019).

The paper is organised as follows. Section 2 presents the related work about spatial and spatio-temporal graphs as well as explores some existing research related to graph-based pattern detection. Section 3 presents a general dynamic graph model which is applied to both ST phenomena considered in this paper: handball in Section 4 and urban road traffic in Section 5. In addition, Sections 4 and 5 describe various entities and spatial relations between them as well as present some examples of spatio-temporal patterns for both ST phenomena. Section 6 mathematically formalizes the updated definition of subgraph isomorphism for static and dynamic graphs and reports the results obtained by applying this new definition using our algorithm for detecting dynamic patterns in dynamic graphs. Finally, Section 7 concludes the paper while giving future research directions.

2. Related work

Graphs have been used to model different kinds of spatial and spatio-temporal phenomena for last many years. In addition to the structural modeling approach proposed by graph-based models, an interesting discussion is about the kind of analysis that can be carried out once the spatial or spatio-temporal phenomenon is modeled using a graph. In this section, we will discuss some of the existing research related to spatial and spatio-temporal graph models and then we will focus on the issue of graph-based pattern detection and present some of the existing work in this field.

2.1. Spatial graphs

Historically, graphs have been used to model spatial structures for various applications. For example, (B. Jiang *et al*., 2000) proposed a graph representation, called connectivity graph, to model the spatial structure of a building for developing a cognitive point of view of space using the concept of *space syntax*. Another interesting and well studied application of graphs in representing spatial structures is in modeling street networks. Using an approach based on primal graphs (Porta *et al*., 2006), where the nodes of the graph represent the intersections and the edges represent the road segments connecting those intersections, different properties of street networks, such as the topological structure of the network and the *centrality* or the importance of a given street, have been studied (Claramunt, Winter, 2007). In addition to street networks, graphs have also been used to model the spatial layout of a given land in terms of buildings, parcels and road segments, and various topological spatial relations between them (Domingo *et al*., 2013).

2.2. Spatio-temporal graphs

In order to understand the dynamics of a spatio-temporal phenomenon, time has to be added as dimension in the model representing that phenomenon. However, adding the temporal information in the graph-based model is not so straightforward, and many formalisms are proposed in the literature.

A trivial approach is to consider time as an external variable and represent the evolution of the graph using static snapshots at discrete time instants. For example, for modeling communication networks over time, (Bhadra, Ferreira, 2002) consider a set of static snapshots as a model for *evolving* graph (evolving graph is one of the names given to dynamic graphs in the literature). Their model analyzes the communication between different nodes using static graph properties, such as path length and strongly connected components, extended to dynamic graphs.

Considering time as an implicit variable of the model, it can be represented through the events occurring in the graph structure and/or graph attributes. For example, (J. Jiang, Worboys, 2009) propose a spatio-temporal model to describe changes in the topological structures of spatial areal objects. The areal objects are represented as a tree and events represent changes in the tree structure. An event links two consecutive trees, occurring at two consecutive time instants, and simple events can be combined to form a sequence of events representing complex changes.

(Casteigts *et al.*, 2012) proposed a general unifying model of time-varying graphs (another name for dynamic graphs) considering time as an integral part of the model. The proposed model enables the representation of a time-varying graph as a sequence of static snapshots and also, more importantly, as the sequence of "shorter" time-varying graphs, each representing the temporal subgraph of the whole graph. This model helps to define the temporal metrics of the time-varying graph (Santoro *et al.*, 2011).

Furthermore, some interesting work exists related to applications where time is aggregated over an interval and an aggregated graph model is proposed to represent the spatio-temporal phenomenon. For example, (Costes *et al.*, 2015) propose an aggregated graph to model the changes in the street network over time. The idea of using an aggregated graph is to avoid the replication of nodes and edges for each time instant and model the attributes of nodes and edges as time-series over the interval (George, Shekhar, 2008).

2.3. Graph-based pattern detection

Since our focus in this paper is on graph-based pattern detection, here, we present some of the existing work in this field.

From a general point of view, graph-based pattern detection is divided into exact and inexact pattern detection (Conte *et al.*, 2004). The idea of exact pattern detection is to find the exact mapping between the nodes of two graphs, generally called pattern graph and target graph. If the objective is to find an injective node mapping between pattern and target graphs, the problem is referred to as subgraph isomorphism, whereas if the objective is to find bijective node mapping, the problem is called graph isomorphism. For clarification, injective node mapping means that all the nodes of the pattern graph are mapped to at most one node of the target graph. However, it is possible to have some nodes of the target graph which are not mapped to any node of

the pattern graph. Bijective node mapping requires, in addition, that all the nodes of the target graph are also mapped to exactly one node of the pattern graph.

On the other hand, inexact pattern detection is used when exact detection is not possible or is not fruitful. In this case, the objective is to find the solution which is as close as possible to the given pattern graph. The "closeness" of the found solution is defined in terms of the *graph edit distance*, i.e., the number of edit operations needed to transform one graph into the other.

In this paper, we will focus only on the problem of exact pattern detection, especially the case of subgraph isomorphism. This problem has well been studied for static graphs (Cordella *et al.*, 2004; Bonnici *et al.*, 2013), and recently, the case of dynamic/time-varying graphs has also seen a growing interest. (Kovanen *et al.*, 2011) and (Paranjape *et al.*, 2017) propose pattern detection algorithms for dynamic graphs, while representing them using an aggregated graph model, whose edges are time-stamped with the times when they are present. The temporal order of the edges is taken into account by the proposed pattern detection algorithms. Recently, an algorithm for subgraph isomorphism called RI (Bonnici *et al.*, 2013), originally developed for static graphs, has been extended to the case of dynamic graphs (Micale *et al.*, 2021). This algorithm also considers the aggregated graph model, with time-stamped edges, of dynamic graphs.

Taking a similar approach, in our previous work (Oberoi, 2019), we proposed an algorithm to solve the problem of subgraph isomorphism in case of dynamic pattern and dynamic target graphs by modeling them as a sequence of static snapshots. The core of our algorithm was an existing algorithm, called VF3 (Carletti *et al.*, 2018), developed for the case static pattern and target graphs. VF3 uses a search space modeled as a tree with the possibility of backtracking while searching for suitable node matchings. We extended the algorithm VF3 to detect dynamic patterns which persist over time. However, our algorithm did not include the edge labels between different nodes while searching for graph mappings.

The objective of this paper is to update our definition of subgraph isomorphism for dynamic graphs to also include edge labels, representing the spatial relations between the entities of the modeled phenomenon, and modify the implementation of our algorithm accordingly. Although we do not go into details of the algorithm, we report the results obtained using the new implementation in later sections.

3. General dynamic graph model

Before diving into the specifics of the two applications considered in this paper, we start our discussion with the formalization of a general dynamic graph model which would be applied throughout. The field of dynamic graphs is highly inter-disciplinary due to which, as discussed above, many formalisms have been proposed in the literature which deal with and model graph dynamics (Holme, Saramäki, 2012; Casteigts *et al.*, 2012; Wehmuth *et al.*, 2015).

The choice of the dynamic graph model to be used depends on the characteristics of the phenomenon to be represented. Both handball and urban road traffic are highly dynamic phenomena in the sense that the structure of the graph representing these phenomena cannot be known *a priori*, i.e., the nodes and edges can be added or removed at any time according to the evolution of these phenomena. In addition, we could have nodes and edges which persist over time. The dynamic graph model applied in this paper is inspired from the existing model proposed in (Casteigts *et al.*, 2012) since it is able to handle these modeling requirements.

Formally, we define a general dynamic graph as

$$\mathcal{G} = (X, \ E, \ \mathcal{T}, \ \rho_X, \ \rho_E, \ \psi_X, \ \psi_E) \tag{1}$$

where, X and E are sets of nodes and edges, respectively, $\mathcal{T}$ is the lifetime, $\rho_X : X \times \mathcal{T} \to \{0, \ 1\}$ is node presence function, $\rho_E : E \times \mathcal{T} \to \{0, \ 1\}$ is edge presence function, ψ_X is node labeling function and ψ_E is edge labeling function. The presence function indicates whether a given node (or an edge) is present at a given time and the labeling function assigns labels to nodes (and edges). The sets X and E contain the nodes and edges, respectively, which have been present in the graph $\mathcal{G}$ at least once during its lifetime $\mathcal{T}$.

As in case of (Casteigts *et al.*, 2012), our model (theoretically) considers continuous time domain, which implies that graph structure can vary at any time during $\mathcal{T}$. In addition, it can handle both instantaneous and persisting nodes (and edges) since the definition of the presence function provides an ambiguity on the time when a node (or an edge) can be present during $\mathcal{T}$. Hence, this model is suitable for representing highly dynamic phenomena. However, as proposed by the original authors (Casteigts *et al.*, 2012), it is possible to represent the dynamic graph as a sequence of static snapshots. For our purposes, and in the rest of the paper, we use this formalisation of dynamic graphs.

Given this general dynamic graph model, in the following sections we will focus on defining the entities and relations between those entities for both handball and urban road traffic. The set of entities and corresponding relations between them will define the sets of nodes and edges, X and E respectively, of graph $\mathcal{G}$.

4. Use case: invasive team sports (handball)

Although in this paper we focus on the game of Handball, the proposed graph formalization can be applied to other invasive team sports as well such as football, kabaddi, basketball etc.

4.1. Entities constituting handball

An entity of a phenomenon is a "thing" (physical/abstract) which plays a significant role in the evolution of that phenomenon (Chen, 1976). Entities may have multiple attributes and some of those attributes could be shared between multiple entities.

However, it is important to be able to identify each entity using a unique attribute, its identity (Homsby, Egenhofer, 1997).

In case of team sports, the most significant entity is the *Player* entity for which different types of attributes like height, weight, age, games played etc. can be defined. Each player can have these attributes with different values, and the jersey number of the player uniquely identifies the players of both teams.

Another significant entity which affects the movement of the players, and in turn the evolution of the game, is the *Ball*. Considering ball as a separate entity allows us to formalize its movement as well as understand the behavior of the players around the one who possesses the ball. If, on the other hand, the ball is considered as only an attribute of the player who possesses it, its significance as the trigger which motivates the coordination of the players to act as functional sub-unit, as noted in (Passos *et al.*, 2011), gets reduced. Hence, we consider the ball as a separate entity.
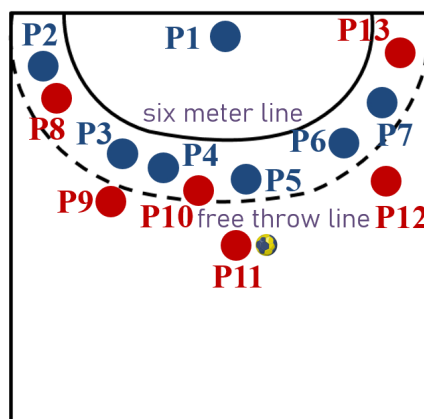

Figure 1. Entities constituting the game of handball. Players of attacking (red) and defending (blue) team as well as the ball possessed by player P11 are shown

Furthermore, different areas on the playing field, categorized as *Zone* entity, could be considered as well. In case of handball, we consider two zones - Free Throw (FT) zone which is the area between the six meter line and free throw line and Nine Meter (9m) zone which lies beyond the free throw line and extends towards the middle of the field. The objective behind considering different areas of the playing field as entities is to be able to locate different players on the field and to understand their movement during a match. Figure 1 shows the game of handball with the players of both teams and the ball. The zone demarcation can be deducted using the position of six meter and free throw lines.

4.2. Relations between entities

Having discussed the three types of entities we consider, now let us describe the relations between them. These relations highlight the interactions between different

entities and are necessary to understand the evolution of the game. It is noteworthy that we only consider binary relations, i.e., those which can be defined between two distinct entities. Table 1 lists all the relations we consider between different types of entities.

Table 1. Different relations between the entities of handball

Entity	Relation with	Relations
Player	Player	Distance (D), Relative Trajectory (RT)
Player	Zone	Topological (T)
Player	Ball	Possession (Po), Pass (Pa)

Depending on the type of entities, we define spatial and/or non-spatial relations between them. Starting with the *Player* entity, the most significant spatial relation which affects the behavior of players during the match is their interpersonal distance (Passos *et al*., 2011). The *distance (D)* between two players can be considered in precise quantitative terms or can be abstracted into qualitative terms like *very close*, *close*, *far*, etc. depending on the considered level of detail (Clementini *et al*., 1997). Furthermore, since players are moving entities, their movement is described using their trajectories during a match. To compare the trajectories of two players, Qualitative Trajectory Calculus (QTC) (Weghe *et al*., 2004) can be applied using which the *relative trajectory (RT)* between two players can be compared. This gives further information about the movement of the players by considering if they move in the same or opposite directions. Considering the moving objects as points, QTC describes the relative movement of objects by using symbols like "-" (one object moves towards the other), "+" (one object moves away from the other) and "0" (no relative movement). Recently, QTC has been applied to describe the relative movement of football players (Beernaerts *et al*., 2020).

Next, we consider the *Player* and *Zone* entities. The position of the player on the field can be described in qualitative terms by defining the spatial *topological (T)* relations (Randell *et al*., 1992) between the player and the zone. The qualitative topological relations take an abstract point of view to describe the position of the player. For example, considering the RCC8 formalism proposed in (Randell *et al*., 1992) for describing topological relations between two regions, a player can be said to be Externally Connected (EC), Non-Tangential Proper Part (NTPP) etc. of a zone (FT or 9m) on the field. In addition to RCC8, other formalisms of topological relations, such as the one proposed in (Clementini *et al*., 1993), can also be applied.

Finally, considering the entities *Player* and *Ball*, we define two non-spatial relations between them. The first is the *possession (Po)* relation which describes if a player possesses the ball. Since it has only two possible values (True or False), it can be implemented using a Boolean variable. Secondly, we define *Pass (Pa)* relation which describes a ball pass between two players. This relation can be implemented using a tuple $< From, To >$ which contains the identity of both players involved in the pass. For example, a ball passed between players $P1$ and $P2$ can be represented

as $< P1, \; P2 >$ and added in the graph as an edge label between nodes $Ball$ and $P1$ as well as nodes $Ball$ and $P2$.

4.3. Sets of nodes and edges

Given the semantics behind the entities and their corresponding relations in case of handball, the set of nodes X and set of edges E in equation 1 (Section 3) can be formalized. Considering the set P of players, the ball B and the set Z of zones, the set of nodes is written as

$$X = P \cup Z \cup B \tag{2}$$

In handball, there are seven players (six field players and one goal keeper) in each team. In addition, we consider two zones (FT and 9m) and one Ball, which makes the total number of nodes in the graph to be $|X| = 17$.

Furthermore, the set of all possible relations considered between the three types of entities (Table 1) is

$$\Delta = \{D, \; RT, \; T, \; Po, \; Pa\} \tag{3}$$

with each element Γ of Δ being a set of its possible values (like $D = \{close, far\}$).

With this, the set E of edges can be written as

$$E = \{(x, \; y) \mid x \, \gamma \, y\}, \; \gamma \in \Gamma, \; \Gamma \in \Delta, \forall \, x, \; x \in P \; \wedge \; \forall \, y, \; y \in X \tag{4}$$

which means that an edge is allowed to exist between a player and any other entity type and not between the ball and the zone. Here, γ represents the value of the element Γ present in set Δ of all possible relations. For example, γ could represent the value *close* of Distance (D) relation. It is assumed that $\rho_X(x) = 1$ and $\rho_X(y) = 1$, i. e., the corresponding nodes are present in the graph at the time when the edge is added.

4.4. Spatio-temporal patterns in handball

Having clarified the entities in handball as well as different types of relations between those entities, in this section, we will present different kinds of spatio-temporal patterns which can be defined for handball. These patterns represent the tactics which the players of the attacking team develop during their training. Depending on the temporal granularity considered, the patterns described can be further detailed with intermediate game situations. However, we restrict ourselves to three timestamps.

4.4.1. Cross pass between players

A common tactic of the attacking team for passing the ball between players is cross passing. A representation of cross pass is shown in Figure 2. The idea of cross pass is to confuse the defending team and pass the ball in the opposite direction of the player movement. In Figure 2, at time t_1, player $P4$ possesses the ball and is moving towards player $P5$ who is also moving towards player $P4$. Their relative trajectories
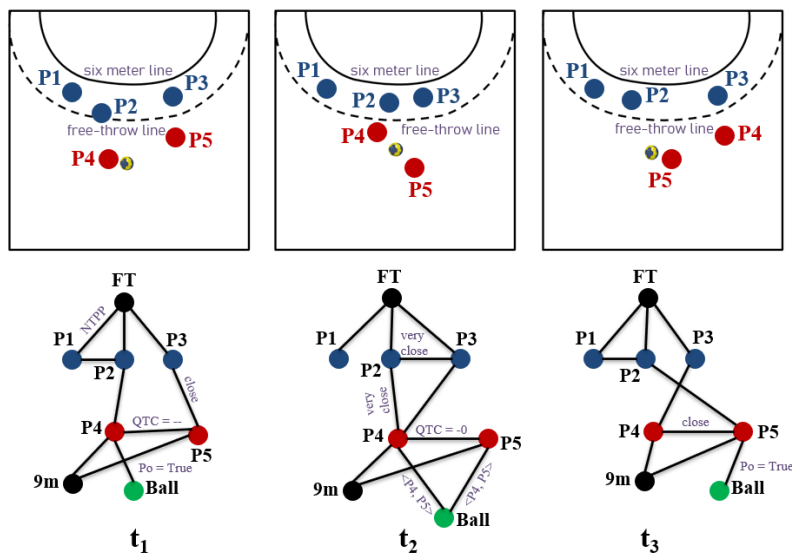
Figure 2. A representation of cross pass between players of the attacking team. All nodes and all relations are not shown due to space constraints. The Ball *node is shown in green in the graph*

are shown using QTC relation "- -" in the corresponding graph. At time t_2, player $P5$ is stationary and the player $P4$ moves towards $P5$, as represented by QTC relation "- 0", the ball is passed to player $P5$. The *Pass* relation is shown using the tuple $< P4, \ P5 >$ in the graph at time t_2. Player $P4$ continues his/her movement and at time t_3, player $P5$ is left with the ball. Then, depending on the situation, player $P5$ can either pass the ball to another player or try to take a shot.

4.4.2. *Position switch between players*

Since the handball is a highly dynamic game, one tactic to confuse the defending team is to pass the ball to one player and then switch the position on the field with another player who will take the shot. Since this happens very rapidly, it is difficult to cover the player who will take the shot. A representation of this tactic is shown in Figure 3. In the figure, player $P5$ has the ball at time t_1 and players $P1$ and $P2$ of the defending team try to cover this player. The notion of "covering" the player $P5$ by the players $P1$ and $P2$ is represented with the presence of the edge $(P1, P2)$ at time t_1. It can be further detailed by considering the Distance relation *close* between $P1$ and $P2$. At time t_2, player $P5$ starts moving towards player $P6$, passes the ball to player $P6$ and switches his/her position with player $P4$ at time t_3. The position switch becomes evident by the presence of edges $(P1, P5)$ and $(P2, P4)$ at time t_3 which were not present at time t_2. Then player $P4$ can try to move past the defending players and try to take a shot. Different relations between entities of the game are shown in the figure as well.
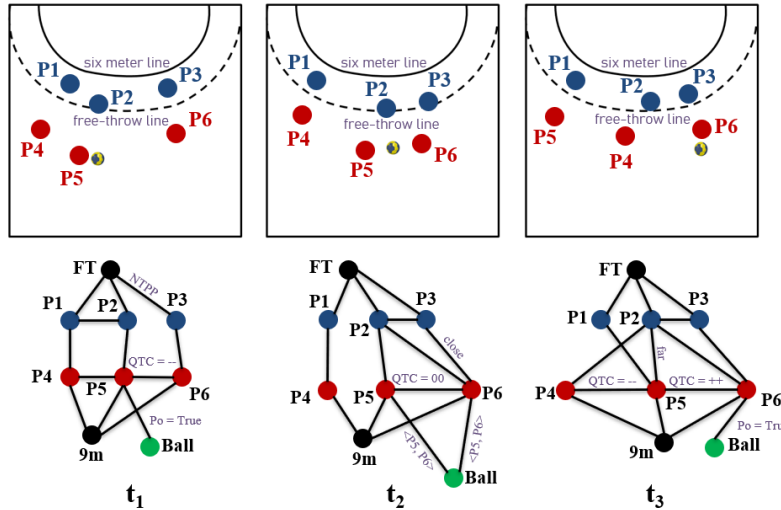
Figure 3. A representation of position switch between players of the attacking team. All nodes and all relations are not shown due to space constraints. The Ball *node is shown in green in the graph*

5. Use case: urban road traffic

Another ST phenomenon we consider in this paper is the urban road traffic. In our previous work (Oberoi, 2019), we proposed a dynamic graph model to represent urban road traffic and applied it to define structural traffic patterns, i.e., graph patterns without considering relations between different entities of road traffic. In this section, we will enhance the definition of those patterns by considering different types of relations between various entities of road traffic. As in the previous section, we will start with the definition of the entities which make up road traffic and list various relations between those entities. Then we will discuss some examples of spatio-temporal patterns in urban road traffic defined in terms of those entities and corresponding relations.

5.1. Entities constituting urban road traffic

Figure 4 shows different entities of urban road traffic at an intersection. As in case of handball, we consider various types of entities for urban road traffic. In our model, most significant entities which make up the road traffic are dynamic entities such as vehicles, pedestrians, and bicycles since our objective is to study their behaviour and their movement and understand the resulting ST evolution of traffic. In addition, there are other static entities which directly affect the flow of traffic. We consider various static entities like buildings, vertical structures (such as sign boards, traffic signals, lane dividers, guard rails etc.), road markings (such as stop line, zebra crossing, center line etc.), road sides (like footpath, cycle track, roadside parking etc.), road segments, and intersections in our model.
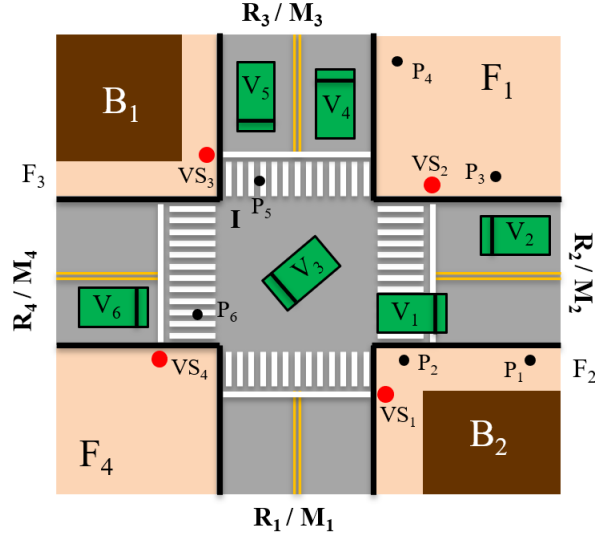
Figure 4. Entities constituting urban road traffic. The entities shown are Vehicles (V), Pedestrians (P), Buildings (B), Vertical structures (VS), Road Markings (M), Road Segments (R), Road Sides (F), and Intersection (I)

5.2. Relations between entities

Having described which real-world objects form the set of entities in our model, let us now discuss various spatial relations we consider between these entities. Table 2 lists spatial relations between different entity types considered in our model.

We distinguish between vehicle-centric relations and non-vehicle-centric relations. Vehicle-centric relations are defined between a vehicle and any other entity. Between Vehicle-Vehicle, Vehicle-Pedestrian and Vehicle-Bicycle entities, we consider Topological (T), Orientation (O), Relative Trajectory (RT), Relative Speed (RS), and Distance (D) relations. Topological relation describes if two vehicles (or pedestrians/bicycles) touch each other (for example, in case of any accidents). Orientation relation describes relative orientation such as *front*, *left*, *left-front* etc. (Hernández, 1993) between these entities. Relative Speed and Relative Trajectory, defined using QTC (Weghe *et al*., 2004), describe if a vehicle/pedestrian/bicycle is moving faster or slower than another vehicle/pedestrian/bicycle and their relative direction of movement, respectively. Distance relation describes the qualitative distance between two entities. Similarly for Vehicle-Building and Vehicle-Vertical Structure, we consider Topological, Orientation and Distance relations. In case of Vehicle-Road Marking and Vehicle-Road Segment, only Topological relation is considered to highlight if a vehicle covers or touches a road marking.

Furthermore, non-vehicle-centric relations between entities such as Pedestrian, Roadside, Bicycle, Road Segments and Road Markings are considered to add com-

Table 2. Spatial relations between different entities of urban road traffic

Entity	Relation with	Relations
Vehicle	Vehicle	$\{T,\ O,\ RT,\ RS,\ D\}$
Vehicle	Building	$\{T,\ O,\ D\}$
Vehicle	Vertical Structure	$\{T,\ O,\ D\}$
Vehicle	Road Marking	$\{T\}$
Vehicle	Roadside	$\{T,\ D\}$
Vehicle	Road Segment	$\{T\}$
Vehicle	Pedestrian	$\{T,\ O,\ RT,\ RS,\ D\}$
Vehicle	Bicycle	$\{T,\ O,\ RT,\ RS,\ D\}$
Vehicle	Intersection	$\{T,\ D\}$
Pedestrian	Roadside	$\{T\}$
Bicycle	Roadside	$\{T\}$
Pedestrian	Road Segment	$\{T\}$
Bicycle	Road Segment	$\{T\}$
Pedestrian	Road Marking	$\{T\}$
Bicycle	Road Marking	$\{T\}$
Road Segment	Road Marking	$\{T\}$

plementary information about the spatial interactions between them. For these entities, we consider only Topological relation to describe if a pedestrian/bicycle is on the road segment or on the road side and which road marking is present on which road segment.

5.3. Set of nodes and edges

With the information about various entities of road traffic and corresponding spatial relations between, we can formalize the sets of nodes and edges in equation 1 (Section 3) for this phenomenon. Given set of Vehicles (V), set of Buildings (B), set of Vertical Structures (VS), set of Road Marking (M), set of Roadsides (F), set of Road Segments (R), set of Pedestrians (P), set of Bicycles (H) and set of Intersections (I), we formalize the set of nodes

$$X = V \ \cup \ B \ \cup \ VS \ \cup \ M \ \cup \ F \ \cup \ R \ \cup \ P \ \cup \ H \ \cup \ I \qquad (5)$$

Furthermore, set of all possible relations between the entities of urban road traffic can be written as

$$\Delta = \{T,\ O,\ RT,\ RS,\ D\} \qquad (6)$$

As in case of handball, the set of edges in the graph representing road traffic is formalized as

$$E = \{(x,\ y) \mid x\ \gamma\ y\},\ \gamma \in \Gamma,\ \Gamma \in \Delta, \forall\ x,\ y \mid (x \in V\ \wedge\ y \in X)$$
$$\vee\ (x \in P \cup H\ \wedge\ y \in F \cup R \cup M)\ \vee\ (x \in R\ \wedge\ y \in M) \quad (7)$$

which means that an edge can exist between a Vehicle and any other entity, between a pedestrian/bicycle and roadside/road segment/road marking, or between a road segment and a road marking. It is assumed that the corresponding nodes between which an edge is added are present in the graph at a given time.

5.4. Spatio-temporal patterns in urban road traffic

Similar to handball, we use the graph model of urban road traffic to define spatio-temporal traffic patterns in this section. These patterns represent the movement of dynamic entities with respect to other dynamic and static entities over time. The patterns described in this section can be refined depending on the level of detail required.
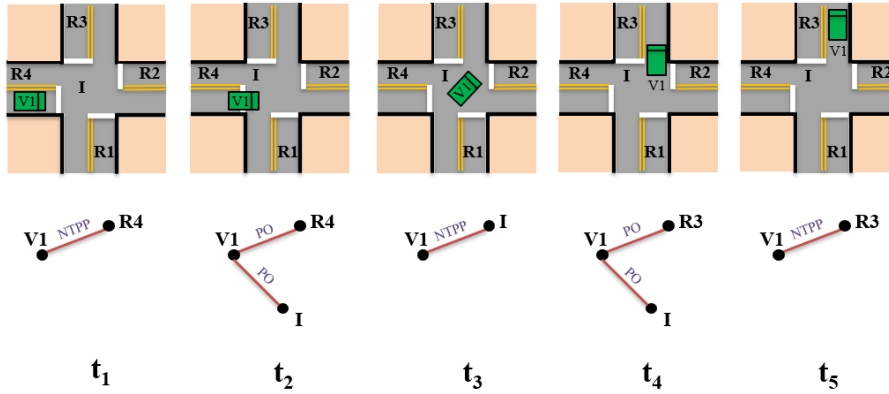

Figure 5. A representation of the movement of a vehicle $V1$ over an intersection I and turning left

5.4.1. Vehicle crossing an intersection

One of the basic movement behaviour of vehicles is crossing an intersection and turning left or right or going straight. This behaviour is shown using the proposed graph model in Figure 5. At time t_1, the vehicle $V1$ is moving on road segment $R4$ towards the intersection I. The corresponding topological relation (Non-tangential proper part (NTPP) in RCC8 formalism (Randell *et al.*, 1992)) between the vehicle and the road segment is shown. At time t_2, the vehicle enters the intersection and partially overlaps (PO, defined in RCC8) both the intersection and the road segment. The vehicle continues moving, turns left towards road segment $R3$ at times t_3, t_4, and t_5, and its spatial relations with different static entities keep evolving with its movement.

As mentioned before, we are not limited to RCC8 to describe the topological relations and other approaches such as (Clementini *et al*., 1993) can also be used.

5.4.2. *Pedestrian crossing a road segment as the vehicle approaches*

Considering the movement of another important dynamic entity (Pedestrian), a traffic pattern is shown in Figure 6. In this case, the pedestrian $P1$ is crossing the road segment $R1$ while the vehicle is moving towards him/her. At time t_1, the pedestrian is on the road side $F1$ and on the *left* (orientation relation) of vehicle $V1$. At time t_2, the spatial relation between the vehicle and pedestrian has changed since the pedestrian has moved and is about to start crossing the road. However, the graph structure remains the same. At time t_3, the pedestrian has come in front of the vehicle, on the road segment and on the road marking (zebra crossing) and the corresponding spatial relations are updated. This type of pattern can also be used to detect dangerous situations, like if the pedestrian crosses the road without using the zebra crossing, the corresponding edge between nodes $P1$ and $M1$ will not be there in the graph at time t_3. The absence of this edge indicates a dangerous traffic situation.
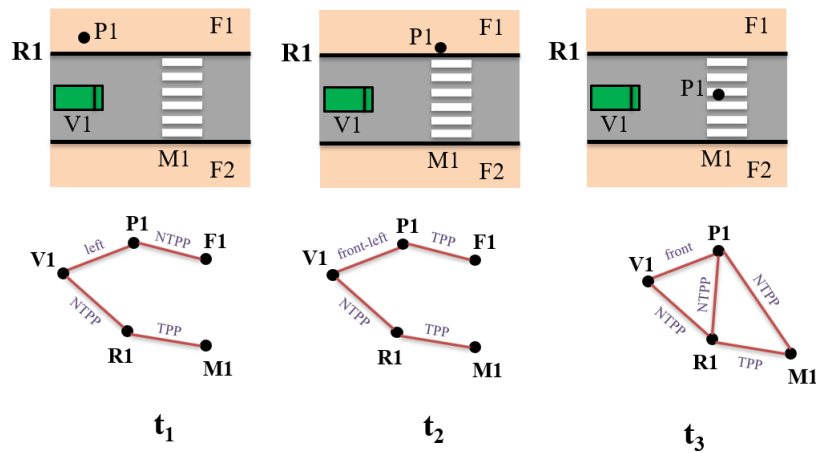

Figure 6. A representation of the movement of a pedestrian $P1$ crossing a road segment $R1$ while vehicle $V1$ is approaching

6. Spatio-temporal pattern detection in spatio-temporal graphs

In previous sections, we described the graph models to represent highly dynamic spatio-temporal phenomena - handball and urban road traffic. Then we used the same formalization to define certain graph-based spatio-temporal patterns which can be detected in both these phenomena. In this section, we will formalize the problem of detecting such spatio-temporal patterns, in the graph representing the phenomenon (called target graph), as the problem of subgraph isomorphism (SI) for dynamic graphs. We extend our previous definition of the problem (Oberoi, Del Mondo,

2021) to include the edge labels in both pattern and target graphs. After mathematically defining the problem of SI for static and dynamic graphs, we will report the results obtained using our algorithm, proposed in (Oberoi, 2019), on random dynamic graphs having varying edge labels (relations). Since we do not yet have the real data for any of the phenomena discussed above, testing our approach using random graphs is our only option since we can vary the graph paramaters, such as number of nodes, number of snapshots and the distribution of edge labels, for both pattern and target graphs, in order to test the performance of our algorithm.

6.1. Problem formulation

6.1.1. SI for static graphs

We start by defining the problem of SI in case of static graphs. Consider a static graph $G = (X, E, \psi_X, \psi_E)$ with a set of nodes X, set of edges $E \subset X \times X$ and node and edge labeling functions ψ_X and ψ_E, respectively. Given a pair of static pattern graph $H = (X_H, E_H, \psi_{X(H)}, \psi_{E(H)})$ and static target graph $G = (X_G, E_G, \psi_{X(G)}, \psi_{E(G)})$, the problem of SI is to find an injective mapping μ between the nodes of the two graphs. Mathematically, $\mu \subset X_H \times X_G$ represents a set of pairs of mapped nodes of pattern and target graphs. Following equations give the formal definition of the problem.

$$\forall u \in X_H, \exists\, v \in X_G \mid (u, v) \in \mu \tag{8}$$

$$\forall u, u' \in X_H, u \neq u' \mid \{(u, v), (u', v')\} \in \mu \Leftrightarrow v \neq v' \tag{9}$$

$$\forall (u, u') \in E_H, \exists\, (v, v') \in E_G \mid \{(u, v), (u', v')\} \in \mu \tag{10}$$

$$\forall u, u' \in X_H, \forall (v, v') \in E_G \mid \{(u, v), (u', v')\} \in \mu \Leftrightarrow (u, u') \in E_H \tag{11}$$

Equations 8 and 9 state that all distinct nodes of the H must be mapped to some (but distinct) nodes of G. Equation 10 states that all edges of H are mapped to some edges of G given that the corresponding nodes of the two graphs are mapped as well. Lastly, Equation 11 verifies the "induced" version of SI by checking that a mapped edge of G has a corresponding edge in H. In addition, using the labeling functions ψ_X and ψ_E, we can add additional constraints to check that the mapped nodes and edges have same labels, as follows:

$$\forall u \in X_H, \ (u, v) \in \mu, \ v \in X_G \Rightarrow \psi_{X(H)}(u) = \psi_{X(G)}(v) \tag{12}$$

$$\forall u, u' \in X_H, \ \{(u, v), (u', v')\} \in \mu, \ v, v' \in X_G \Rightarrow \psi_{E(H)}(u, u') = \psi_{E(G)}(v, v') \tag{13}$$

If all conditions mentioned above are satisfied, we represent SI between graphs H and G as $\mu : H \hookrightarrow G$. Otherwise, we use the notation $H \not\hookrightarrow G$.

6.1.2. Extending SI to dynamic graphs

In Section 3, we defined a dynamic graph as $\mathcal{G} = (X, \; E, \; \mathcal{T}, \; \rho_X, \; \rho_E, \; \psi_X, \; \psi_E)$. Simplifying the model by not explicitly considering the presence functions, we can define a dynamic pattern graph as $\mathcal{H} = (X_{\mathcal{H}}, E_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}}, \psi_{X(\mathcal{H})}, \; \psi_{E(\mathcal{H})})$ and a dynamic target graph as $\mathcal{G} = (X_{\mathcal{G}}, E_{\mathcal{G}}, \mathcal{T}_{\mathcal{G}}, \psi_{X(\mathcal{G})}, \; \psi_{E(\mathcal{G})})$. We consider a snapshot-based model for both dynamic pattern and target graphs where $\mathcal{H} = \{H_1, \; H_2, \; ..., \; H_m\}$ has m snapshots and $\mathcal{G} = \{G_1, \; G_2, \; ..., \; G_n\}$ has n snapshots, with $m \leq n$. Then the problem of SI can be extended to dynamic graphs, i.e.,

– Each snapshot of $\mathcal{H}$ must be mapped to some snapshot of $\mathcal{G}$ while respecting the equations of induced SI mentioned above. Formally,

$$\forall H_i \in \mathcal{H}, \; 1 \leq i \leq m \; \exists \, \mu_{ij} : H_i \hookrightarrow G_j \mid G_j \in \mathcal{G}, \; 1 \leq j \leq n \qquad (14)$$

where, μ_{ij} represents the mapping between snapshots $H_i \in \mathcal{H}$ and $G_j \in \mathcal{G}$.

– The temporal order of $\mathcal{H}$ and $\mathcal{G}$ must be respected, i.e., if a snapshot $H_i, 1 \leq i \leq m$ of $\mathcal{H}$ is mapped to a snapshot $G_j, 1 \leq j \leq n$ of $\mathcal{G}$, then the next snapshot H_{i+1} of $\mathcal{H}$ must be mapped to some snapshot G_f of $\mathcal{G}$ where $f > j$. Formally,

$$\forall H_i \in \mathcal{H}, \; 1 \leq i \leq m, \; \mu_{ij} : H_i \hookrightarrow G_j, \; G_j \in \mathcal{G}, \; 1 \leq j \leq n \Leftrightarrow$$
$$\exists \, \mu_{(i+1)f} : H_{i+1} \hookrightarrow G_f \mid G_f \in \mathcal{G}, \; j < f \leq n \quad (15)$$

– It is possible to have some snapshots of $\mathcal{G}$ to which no snapshots of $\mathcal{H}$ are matched. Such snapshots represent *noise*. We include such snapshots of target graph to be more realistic in terms of the sampling rate. For real-world applications, depending on this sampling rate, it is possible to have some snapshots in which no matching is found. Formally,

$$\exists \, G_j \in \mathcal{G}, \; 1 \leq j \leq n \mid H_i \not\hookrightarrow G_j, H_i \in \mathcal{H}, \; 1 \leq i \leq m \qquad (16)$$

6.2. Experiments and results

Having mathematically formalized the problem of SI for dynamic graphs, in this section we report the results of the experiments we conducted using our algorithm, first proposed in (Oberoi, 2019), to detect dynamic patterns in dynamic graphs. The details of the algorithm are out of scope of this paper. In our previous work (Oberoi, Del Mondo, 2021), we reported the results of the average algorithm runtime considering only structural graph properties such as number of nodes, number of graph snapshots, graph density and number of node labels. In this paper, since we extended the definition of subgraph isomorphism to include relations as edge labels in both pattern and target graphs, we conducted experiments to see the effect of the distribution of such edge labels on the average algorithm runtime.

All the experiments were conducted on a PC with Intel Core-i7 3.6GHz CPU and 32GB RAM running Ubuntu 18.04 LTS. Our algorithm is implemented in C++ and

was compiled with GNU C++ compiler (version 7.5.0) with level-3 optimisation. For each experiment, we considered the timeout of 10 mins. If the execution of the algorithm terminated before timeout, we report the value of average algorithm runtime (in seconds), otherwise we report the result as NULL. It is noteworthy that for all the experiments we do not consider any node labels (so all nodes are semantically equal) and only consider the edge labels.

We conducted the experiments considering different scenarios using random dynamic graphs. Firstly, we consider that all the relations on all the edges of both dynamic pattern and dynamic target graphs are the same. This is not a real scenario but it helps to understand the limits of our algorithm. The results for this experiment are reported in Table 3. One can see that none of the experiments in this scenario return any result before timeout. This shows that having same value of edge labels for each edge in both graphs drastically affects the performance of the algorithm since there are multiple solutions/matchings possible but it takes too long to find them. Hence, the types of edge labels considered should be carefully examined when applying this approach to real graphs.

Table 3. Average algorithm runtime considering same edge labels (relations) for all edges in both dynamic pattern and dynamic target graphs. NULL values mean that the algorithm did not terminate before timeout of 10 mins

Target Graph		Pattern Graph		Avg. Time (in secs)
Nodes	No. of snapshots	Nodes	No. of snapshots	
50	50	10	10	NULL
		10	30	NULL
		50	10	NULL
		50	30	NULL
100	50	10	10	NULL
		10	30	NULL
		50	10	NULL
		50	30	NULL

For the next set of experiments, we consider that the all the edges of the pattern graph have the same edge labels, whereas the edges of the target graph have uniformly distributed edge labels. The results for this experiment are reported in Table 4. Finally, for the last set of experiments, we considered that the edge labels for both dynamic pattern and dynamic target graphs are uniformly distributed (Table 5). For both these sets of experiments, we see similar results since for both these experiments we find zero matchings between the pattern and target graph pairs. This is due to the fact that the edge labels are taken into account while calculating the matchings and since they are not same between pattern and target graphs, we do not get any matching.

The results described in this paper, combined with the results in our previous work (Oberoi, Del Mondo, 2021), give comprehensive details about the algorithm performance depending on both structural and semantic properties of dynamic pattern and target graphs. These results form a baseline for implementing our algorithm with real spatio-temporal graphs in the future.

Table 4. Average algorithm runtime considering same edge labels (relations) for the dynamic pattern graph and uniformly distributed edge labels for the dynamic target graph

Target Graph		Pattern Graph		Avg. Time (in secs)
Nodes	No. of snapshots	Nodes	No. of snapshots	
		10	10	0.0128
50	50	10	30	0.0065
		50	10	0.0057
		50	30	0.0038
		10	10	0.0639
100	50	10	30	0.0341
		50	10	0.0579
		50	30	0.0289

Table 5. Average algorithm runtime considering uniformly distributed edge labels (relations) for both dynamic pattern and dynamic target graphs

Target Graph		Pattern Graph		Avg. Time (in secs)
Nodes	No. of snapshots	Nodes	No. of snapshots	
		10	10	0.0122
50	50	10	30	0.0063
		50	10	0.0058
		50	30	0.0038
		10	10	0.0651
100	50	10	30	0.0340
		50	10	0.0606
		50	30	0.0282

7. Conclusion and future work

In this paper, we discussed the idea of using spatio-temporal graphs to model spatio-temporal phenomena. Considering two distinct, highly dynamic phenomena - invasive team sports (focusing on handball) and urban road traffic, we highlighted the entities and spatial relations between those entities for both phenomena. A generic dynamic graph model is proposed which is applicable to model both invasive team sports and urban road traffic.

The reason behind using a graph-based model to represent a spatio-temporal phenomenon is the existing plethora of graph algorithms and analysis tools which can be directly applied to analyze the phenomenon. In this paper, we focus on pattern detection as an approach for analyzing the evolving behaviour of various entities and their corresponding spatial interactions. In case of handball, the spatio-temporal patterns represent the tactics which the players of the attacking team develop during training, whereas, in case of urban road traffic, such spatio-temporal patterns represent the movement of different dynamic entities such as vehicles and pedestrians. Once such graph-based patterns are defined, they can be detected in the phenomenon. We formalize the problem of pattern detection as the problem of subgraph isomorphism

for dynamic graphs, and we discuss the results of our algorithm for detecting spatio-temporal patterns in spatio-temporal graph representing the phenomenon. The ideas explored in this paper are applicable to other types of spatio-temporal phenomena as well.

In the future, our objective is to apply our algorithm for pattern detection on real data about invasive team sports and urban road traffic. For handball, we have generated a spatio-temporal graph using a single video sequence of a game between France and Czech-Republic and are currently in the process of detecting spatio-temporal patterns. For urban road traffic, we plan to use a traffic simulator like CARLA (Dosovitskiy *et al.*, 2017) to generate the corresponding spatio-temporal graph.

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