



BOUNDARY LAYER STAGNATION-POINT FLOW OF CASSON FLUID AND HEAT TRANSFER TOWARDS A SHRINKING/STRETCHING SHEET

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ABSTRACT

The steady boundary layer stagnation-point flow of Casson fluid and heat transfer towards a shrinking/stretching sheet is studied. Appropriate similarity transformations are employed to transform the governing partial differential equations into the self-similar ordinary differential equations and those are then solved numerically using very efficient shooting method. The numerical computations are carried out for several values of parameters involved (especially, velocity ratio parameter and Casson parameter) to know the possibility of similarity solution for the boundary layer stagnation-point flow. It is found that the range of velocity ratio parameter for which similarity solution exists is unaltered for any change in Casson parameter, though the skin friction changes with Casson parameter. Thus, the possibility of similarity solution for Casson fluid flow is same as that of Newtonian fluid flow.

Keywords: *Boundary layer stagnation-point flow, Casson fluid, heat transfer, shrinking/stretching sheet.*

1. INTRODUCTION

Derivation of boundary layer equations for the flow and their solutions using similarity transformations is among the most successful idealization in the history of fluid mechanics [Schlichting and Gersten (2000)]. With the help of this boundary layer theory, the flows of various types of fluids (Newtonian and different non-Newtonian fluids) have been successfully mathematically modeled and the derived equations are solved. The obtained results are in excellent agreement with experimental observations. However, many fluids of industrial importance are of non-Newtonian type. It is now generally recognized that, in real industrial applications, non-Newtonian fluids are more appropriate than Newtonian fluids, due to their applications in petroleum drilling, polymer engineering, certain separation processes, manufacturing of foods and paper and some other industrial processes [Mustafa *et al.* (2011), Cortell (2008)]. Therefore, the analysis of flow dynamics of non-Newtonian fluids is extremely important.

For non-Newtonian fluids, various types of nonlinear relationship between stress and the rate of strain are observed and it is difficult to express all those properties of several non-Newtonian fluids in a single constitutive equation. Consequently, several non-Newtonian fluid models [Fox *et al.* (1969), Wilkinson (1970), Djukic (1974), Rajagopal (1980), Rajagopal and Gupta (1981), Dorier and Tichy (1992), Zhou and Gao (2007), Cui *et al.* (2010) and Bhattacharyya and Layek (2011a)] have been proposed depending on various physical characters. Casson fluid is one of such non-Newtonian fluids, which behaves like an elastic solid and for this fluid, a yield shear stress exists in the constitutive equation. Fredrickson (1964) investigated the steady flow of a Casson fluid in a tube. Mustafa *et al.* (2011) studied the unsteady boundary layer flow and heat transfer of a Casson fluid over a moving flat plate with a parallel free stream using homotopy analysis method (HAM).

On the other hand, boundary layer flows of non-Newtonian fluids caused by a stretching sheet have vast applications in several manufacturing processes such as extrusion of molten polymers through a slit die for the production of plastic sheets, hot rolling, wire and fiber coating, processing of food stuffs, metal spinning, glass-fiber production and paper production [Hayat *et al.* (2008a)]. During the processes, the rate of cooling has an important bearing on the properties of the final product. Hence, the quality of the final product depends on the rate of heat transfer from the stretching surface. The viscous fluid flow due to a stretching flat sheet was first investigated by Crane (1970). The pioneering work of Crane was extended by Rajagopal *et al.* (1984) by taking viscoelastic fluid and also Siddappa and Abel (1985) discussed some other important aspects of flow of non-Newtonian fluid over stretching sheet. Sankara and Watson (1985) studied micropolar fluid flow over a stretching sheet. Troy *et al.* (1987) established the uniqueness of solution of the flow of second order fluid over a stretching sheet. Andersson and Dandapat (1991) reported the flow behaviour of a non-Newtonian power-law fluid over a stretching sheet.

Hiemenz (1911) first reported the stagnation point flow towards a flat plate. It is worthwhile to note that the stagnation flow appears whenever the flow impinges to any solid object and the local fluid velocity at a point (called the stagnation-point) is zero. Chiam (1994) extended the works of Hiemenz (1911) replaced the solid body a stretching sheet with equal stretching and straining velocities and he was unable to obtain any boundary layer near the sheet. Whereas, Mahapatra and Gupta (2001) re-investigated the stagnation-point flow towards a stretching sheet considering different stretching and straining velocities and they found two different kinds of boundary layers near the sheet depending on the ratio of the stretching and straining constants. Some other important aspects of stagnation-point flow of Newtonian fluid are discussed by Nazar *et al.* (2004), Layek *et al.* (2007), Nadeem *et al.* (2010), Bhattacharyya *et al.* (2011a,2012a,b) and

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Salem and Fathy (2012). Mahapatra *et al.* (2009) studied the boundary layer magnetohydrodynamic (MHD) stagnation-point flow of an electrically conducting non-Newtonian power-law fluid towards a stretching surface. Van Gorder and Vajravelu (2010) explained the hydromagnetic stagnation point flow of a second grade fluid over a stretching sheet. Recently, Hayat *et al.* (2012) analyzed the mixed convection stagnation-point flow of a non-Newtonian Casson fluid over a stretching sheet considering convective boundary conditions.

In contrast, the flow due to a shrinking sheet [Wang (1990)] exhibits quite different behaviour from the forward stretching sheet flow. This shrinking sheet flow is essentially one of backward flows described by Goldstein (1965). The generated vorticity due to shrinking makes the nature of the flow interesting. In their study, Miklavčič and Wang (2006) established the requirement of adequate amount of fluid mass suction through the porous sheet to maintain the steady boundary layer flow of Newtonian fluid due to shrinking of porous flat sheet. Actually, fluid mass suction suppresses the vorticity generated due to shrinking of the sheet, inside the boundary layer. Later, numerous important properties of shrinking sheet flows of Newtonian fluid are discussed by Fang and Zhang (2009), Fang *et al.* (2009,2010), Bhattacharyya and Pop (2011) and Bhattacharyya (2011a,b,c). Hayat *et al.* (2008b,2010) reported the MHD flow and mass transfer of a upper-convected Maxwell fluid over a porous shrinking sheet in presence of chemical reaction and they also obtain an analytic solution of flow of non-Newtonian second grade fluid due to shrinking sheet in a rotating frame. Bhattacharyya *et al.* (2012c) showed the effects of thermal radiation on micropolar fluid flow and heat transfer on a porous shrinking sheet. The non-Newtonian power-law fluid flow past a permeable shrinking sheet with fluid mass transfer was studied by Fang *et al.* (2012). Most importantly, Bhattacharyya *et al.* (2013a,b) recently investigated the boundary layer flow of Casson fluid over a permeable stretching/shrinking sheet without and with magnetic field effect.

The boundary layer stagnation-point flow on a shrinking sheet is interesting for its unusual nature and Wang (2008) illustrated those characters by his study of two-dimensional stagnation point flow of Newtonian fluid towards a shrinking sheet. Later, Bhattacharyya and Layek (2011b) explained the effects of suction/blowing and thermal radiation on boundary layer stagnation-point flow and heat transfer past a shrinking sheet and Bhattacharyya *et al.* (2011b) reported the slip effects on steady stagnation-point flow and heat transfer over a shrinking sheet. The influence of external magnetic field on stagnation-point flow over a shrinking sheet was described by Mahapatra *et al.* (2011) and Lok *et al.* (2011). Fan *et al.* (2010) obtained analytic homotopy solutions of unsteady stagnation-point flow and heat transfer over a shrinking sheet, where as, Bhattacharyya (2011d,2013) found the numerical solutions of that flow problem. In addition, Bhattacharyya (2011e), Bachok *et al.* (2011), Rosali *et al.* (2011), Bhattacharyya and Vajravelu (2012), Bhattacharyya *et al.* (2012d), Van Gorder *et al.* (2012) and Mahapatra *et al.* (2012) and Mahapatra and Nandi (2013) explored various important aspects of stagnation-point flow due to shrinking sheet for Newtonian fluid. From literature, it can be found that not much attention is given to the stagnation-point flow of non-Newtonian fluid on shrinking sheet. Ishak *et al.* (2010) and Yacob *et al.* (2011) discussed the steady boundary layer stagnation-point flow of micropolar fluid past a stretching/shrinking sheet. Nazar *et al.* (2011) presented the stagnation-point flow and heat transfer towards a shrinking sheet in a nanofluid. Khan *et al.* (2012) proposed a mathematical model for the unsteady stagnation point flow of a linear viscoelastic fluid bounded by a stretching/shrinking sheet.

The increasing use of several non-Newtonian fluids in processing industries has given a strong motivation to understand their behavior in several transport processes. Therefore, in this investigation, the steady boundary layer stagnation-point flow of an incompressible Casson fluid and heat transfer towards a shrinking/stretching sheet are studied. The governing partial differential equations are converted into the nonlinear ordinary differential equations using the suitable similarity transformations. The transformed self-similar ODEs are solved by

shooting method, an efficient numerical method [Mahapatra and Nandi (2013), Ishak *et al.* (2010), Yacob *et al.* (2011)] for solving boundary value problem. Then a graphical analysis is presented to show the existence and uniqueness of solution and to elaborately discuss the characters of the flow and heat transfer for the variation of physical parameters.

2. FLOW ANALYSIS

Consider the steady two-dimensional stagnation-point flow of incompressible Casson fluid induced by a shrinking/stretching sheet located at $y=0$, the flow being confined in $y>0$ (Fig. 1). It is assumed also that the rheological equation of state for an isotropic and incompressible flow of a Casson fluid can be written as [Nakamura and Sawada (1988), Mustafa *et al.* (2011), Bhattacharyya *et al.* (2013a)]:

$$\tau_{ij} = \begin{cases} (\mu_B + p_y/\sqrt{2\pi})2e_{ij}, & \pi > \pi_c \\ (\mu_B + p_y/\sqrt{2\pi_c})2e_{ij}, & \pi < \pi_c, \end{cases} \quad (1)$$

where μ_B is plastic dynamic viscosity of the non-Newtonian fluid, p_y is the yield stress of fluid, π is the product of the component of deformation rate with itself, namely, $\pi=e_{ij}e_{ij}$, e_{ij} is the (i,j) -th component of the deformation rate and π_c is critical value of π based on non-Newtonian model.

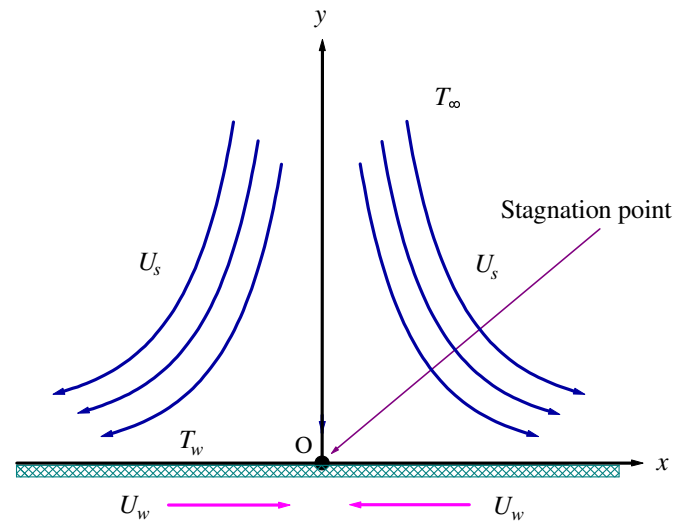


Fig. 1 Physical sketch of the problem.

Under above conditions the boundary layer equations for steady stagnation-point flow towards a shrinking/stretching sheet can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_s \frac{dU_s}{dx} + \nu \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2}, \quad (3)$$

where u and v are the velocity components in x and y directions respectively, x is distance along the sheet, y is distance perpendicular to the sheet, $U_s=ax$ is the straining velocity of the stagnation-point flow with $a(>0)$ being the straining constant, ν is the kinematic fluid viscosity and $\beta = \mu_B\sqrt{2\pi_c}/p_y$ is the non-Newtonian or Casson parameter.

The boundary conditions for the velocity components are

$$u = U_w, v = 0 \text{ at } y = 0; u \rightarrow U_s \text{ as } y \rightarrow \infty, \quad (4)$$

where $U_w=cx$ is shrinking/stretching velocity of the sheet with c being the shrinking/stretching constant and $c<0$ corresponds to shrinking and $c>0$ corresponds to stretching.

Table 1 Values of $f''(0)$ for several values of c/a for $\beta=\infty$ (Newtonian fluid case) for stretching sheet.

c/a	Present study	Wang (2008)	Ishak <i>et al.</i> (2010)
0	1.2325878	1.232588	1.232588
0.1	1.1465608	1.14656	1.146561
0.2	1.0511299	1.05113	1.051130
0.5	0.7132951	0.71330	0.713295
1	0	0	0

Table 2 Values of $f''(0)$ for several values of c/a for $\beta=\infty$ (Newtonian fluid case) for shrinking sheet.

c/a	Present study		Wang (2008)		Ishak <i>et al.</i> (2010)	
	First solution	Second solution	First solution	Second solution	First solution	Second solution
-0.25	1.4022405		1.40224		1.402241	
-0.5	1.4956697		1.49567		1.495670	
-0.75	1.4892981		1.48930		1.489298	
-1	1.3288169	0	1.32882	0	1.328817	0
-1.1	1.1866806	0.0492286	1.186681	0.049229	1.186681	0.049229
-1.15	1.0822316	0.1167023	1.08223	0.116702	1.082231	0.116702
-1.2	0.9324728	0.2336491	0.932474	0.233650	0.932474	0.233650
-1.24	0.7066020	0.4356712				
-	0.5842915	0.5542856	0.55430		0.584295	0.554283
1.2465	0.5745268	0.5639987				
1.24657						

The following relations for u and v are introduced:

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}, \tag{5}$$

where ψ is the stream function.

For relations in (5), the equation (2) is satisfied automatically and the equation (3) takes the following form:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U_s \frac{dU_s}{dx} + v(1+1/\beta) \frac{\partial^3 \psi}{\partial y^3}. \tag{6}$$

Also, the boundary conditions in (4) reduce to

$$\frac{\partial \psi}{\partial y} = U_w, \quad \frac{\partial \psi}{\partial x} = 0 \text{ at } y = 0; \quad \frac{\partial \psi}{\partial y} \rightarrow U_s \text{ as } y \rightarrow \infty. \tag{7}$$

The dimensionless variable for the stream function is introduced as:

$$\psi = \sqrt{av} \, x f(\eta), \tag{8}$$

where the similarity variable η is given by $\eta = y\sqrt{a/v}$.

Using the relation (8) and the similarity variable, the equation (6) finally takes following self-similar form:

$$(1+1/\beta)f''' + ff'' - f'^2 + 1 = 0, \tag{9}$$

where primes denote differentiation with respect to η .

The boundary conditions reduce to

$$f(\eta) = 0, f'(\eta) = c/a \text{ at } \eta = 0; f'(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty, \tag{10}$$

where c/a is the velocity ratio parameter.

3. HEAT TRANSFER

For the temperature distribution in the flow field, the governing energy equation can be written as:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2}, \tag{11}$$

where T is the temperature, κ is the thermal conductivity, ρ is the fluid density and c_p is the specific heat.

The appropriate boundary conditions are

$$T = T_w \text{ at } y = 0; T \rightarrow T_\infty \text{ as } y \rightarrow \infty, \tag{12}$$

where T_w is the constant temperature at the sheet and T_∞ is the free stream temperature assumed to be constant.

Next, the dimensionless temperature θ is introduced as:

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{13}$$

Using (8), (13) and the similarity variable, the equation (11) reduces to $\theta'' + Pr f \theta' = 0$,

where primes denote differentiation with respect to η and $Pr = c_p \mu / \kappa$ is the Prandtl number.

The boundary conditions for θ are obtained from (12) as:

$$\theta(\eta) = 1 \text{ at } \eta = 0; \theta(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty. \tag{15}$$

4. NUMERICAL METHOD FOR SOLUTION

The self-similar equations (9) and (14) along with boundary conditions (10) and (15) are solved using shooting method [Bhattacharyya *et al.* (2011c,d) and Bhattacharyya (2012)] by converting them to an initial value problem (IVP). In this method, it is necessary to choose a suitable finite value of $\eta \rightarrow \infty$, say η_∞ . The following system is set:

$$\left. \begin{aligned} f' &= p, \\ p' &= q, \\ q' &= (p^2 - fq - 1)/(1+1/\beta), \end{aligned} \right\} \tag{16}$$

$$\left. \begin{aligned} \theta' &= z, \\ z' &= -Prfz \end{aligned} \right\} \tag{17}$$

with the boundary conditions

$$f(0) = 0, p(0) = c/a, \theta(0) = 1. \tag{18}$$

In order to integrate the IVP (16) and (17) with (18), the values for $q(0)$ i.e. $f''(0)$ and $z(0)$ i.e. $\theta'(0)$ are required, but no such values are given at the boundary. The suitable guess values for $f''(0)$ and $\theta'(0)$ are chosen and then integration is carried out. Then the calculated values for f' and θ at $\eta_\infty=15$ (say) are compared with the given boundary conditions $f'(15)=1$ and $\theta(15)=0$ and the estimated values, $f''(0)$ and $\theta'(0)$ are adjusted to give a better approximation for the solution. A series of values for $f''(0)$ and $\theta'(0)$ are taken and the fourth order classical Runge-Kutta method with step-size $\Delta\eta=0.01$ is applied. The above procedure is repeated until the asymptotically converged results within a tolerance level of 10^{-5} are obtained.

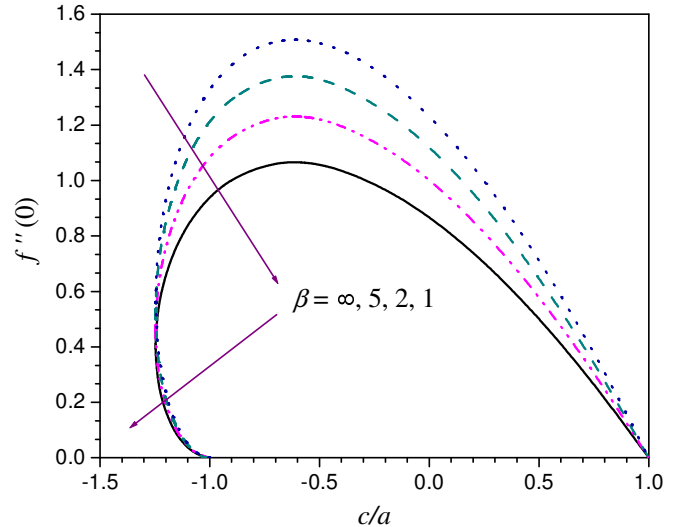


Fig. 2 The values of $f''(0)$ vs. c/a for different values of β .

5. RESULTS AND DISCUSSIONS

The numerical computations have been carried out using above-described shooting method for several values of the physical parameters arised in the study: such as, the velocity ratio parameter (c/a), the Casson parameter (β) and the Prandtl number (Pr). Then acquired results are presented in graphs (Fig. 2-Fig. 14) to explain the existence

and uniqueness of solution for the flow, as well as, the variations in velocity and temperature fields. Also, to validate the numerical scheme, a comparison of results is made. Table 1 and Table 2 show that the values of $f''(0)$ for $\beta=\infty$ (i.e., Newtonian fluid case) are in a favorable agreement with previously published data in the literature by Wang (2008) and Ishak *et al.* (2010).

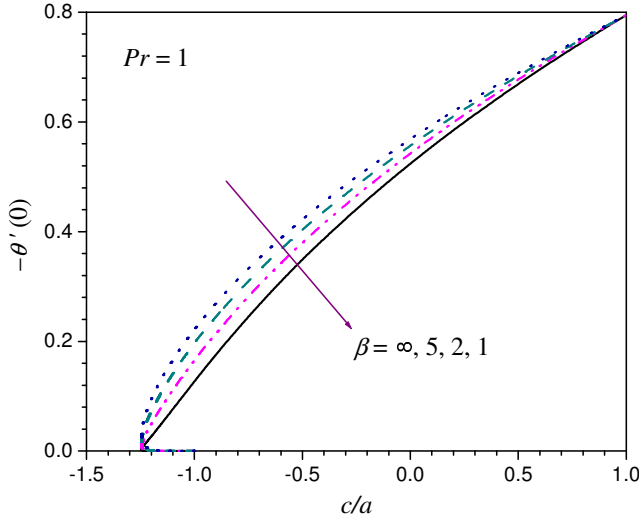


Fig. 3 The values of $-\theta'(0)$ vs. c/a for different values of β .

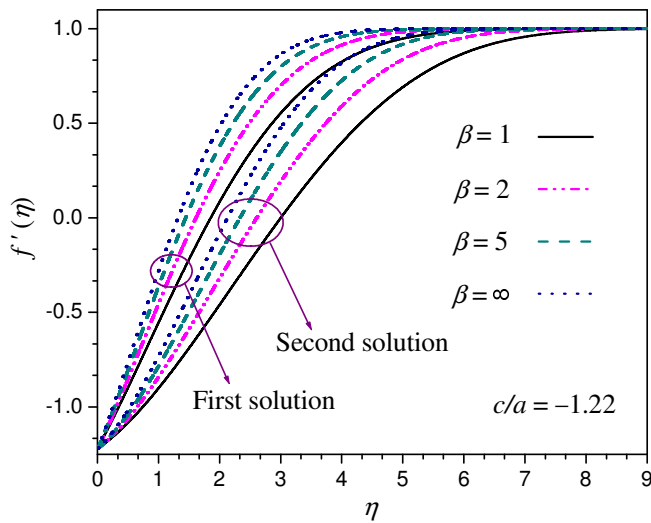


Fig. 4 The effects of β on dual velocity profiles $f'(\eta)$.

The stagnation-point flow over a shrinking sheet was investigated by Wang (2008) for Newtonian fluid and he found that the self-similar solution of boundary layer flow is possible only if the velocity ratio parameter c/a satisfies the inequality $c/a \geq -1.2465$. In this study, it is obtained that for $\beta=\infty$ i.e., for Newtonian fluid the boundary layer exists if $c/a \geq -1.24657$, which is similar to that of Wang (2008). It has also noted that for $\beta=\infty$ the solution is of dual nature for $-1.24657 \leq c/a \leq 0$, the solution is unique for $c/a > 0$ and for $c/a < -1.24657$ no solution is found. Due to decrease in β i.e., for Casson fluid, the existence range does not alter and the similarity solution of boundary layer is obtained when $c/a \geq -1.24657$. Therefore, it is worth noting that there is no change occurred in the solution range of c/a due to variation in the Casson parameter β . For all values of β , dual solutions exist for $-1.24657 \leq c/a \leq 0$, the solution is unique for $c/a > 0$ and no similarity solution is found for $c/a < -1.24657$. Thus, there exists dual self-similar solutions in some situations of shrinking sheet case and for stretching sheet case the solution is always unique. These all phenomena can be observed in Fig. 2 and Fig. 3 of $f''(0)$ and $-\theta'(0)$ vs. c/a , those are related to wall skin friction coefficient and the heat transfer coefficient respectively. Though there is no change in the solution range of c/a , but

the value of $f''(0)$ decreases with decrease of β for first and second solutions in dual solutions case and unique solution case. Similar effects is observed for the values of $-\theta'(0)$. Hence, for Casson fluid flow over a shrinking sheet near a stagnation-point dual solutions and unique solution are found in the same ranges as that of Newtonian fluid case, but Casson parameter affects the wall skin-friction coefficient as well as the heat transfer coefficient. To know the detailed effects of Casson parameter on the flow, the dimensionless curves related to velocity, temperature and their gradients are plotted.

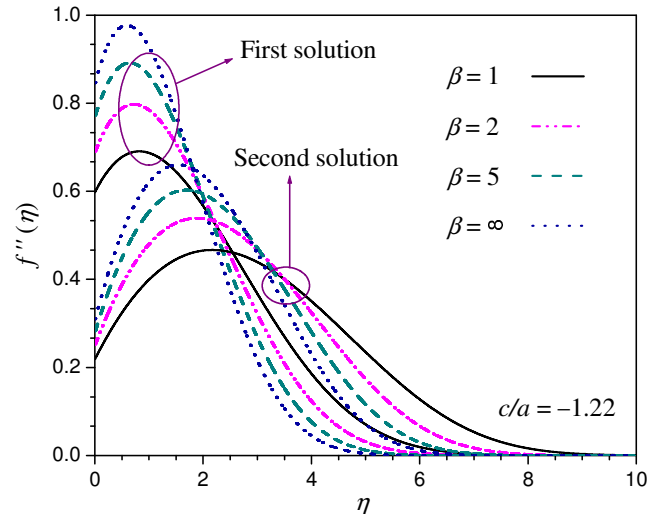


Fig. 5 The effects of β on dual velocity gradient profiles $f''(\eta)$.

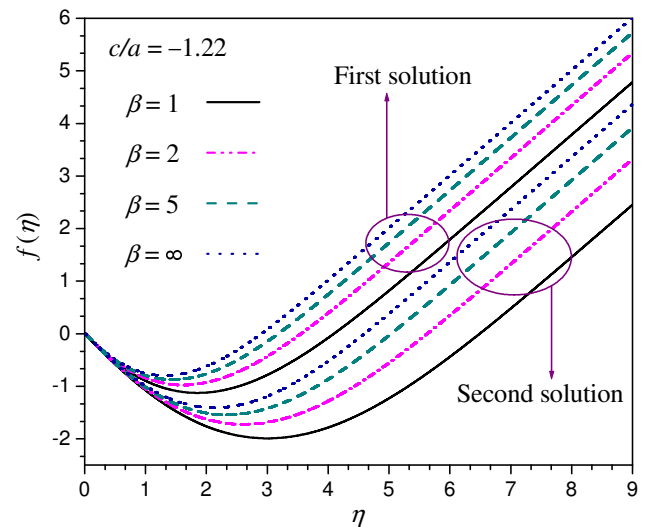


Fig. 6 The effects of β on $f(\eta)$.

The dimensionless velocity, velocity gradient, stream function, temperature and temperature gradient are depicted in Fig. 4-Fig. 10 for different values of Casson parameter β . The velocity at a point decreases with decreasing values of β for both solutions in dual solutions (Fig. 4) and also for unique solution (Fig. 9 and Fig. 11). The velocity gradient decreases near the sheet with decreasing β , but faraway from the sheet it increases (Fig. 5). Also, it is important to note that the decrease in Casson parameter makes the velocity boundary layer thickness larger. Thus, the velocity boundary layer thickness for Casson fluid is larger than that of Newtonian fluid. It happens because of plasticity of Casson fluid. When Casson parameter decreases the plasticity of the fluid increases, which causes the increment in velocity boundary layer thickness. The dimensionless stream function profiles (Fig. 6) show the back flow character of stagnation point flow over a shrinking sheet. The dimensionless temperature increases with the decrease in β for all cases, both solutions in dual solutions (Fig. 7) and unique solution (Fig. 10). Similar to the velocity boundary layer, due to

increase in plasticity of fluid the thermal boundary layer thickness increases with decreasing β , which can be confirmed from the temperature gradient profiles in Fig. 8. Here also, in dual solutions the boundary layer thickness for second solution is thicker.

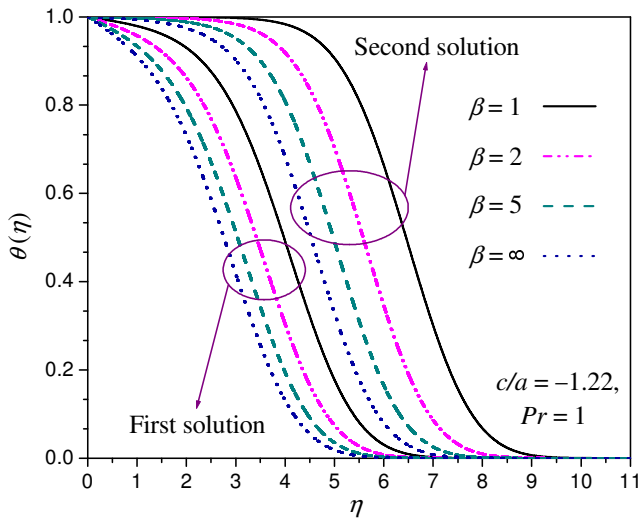


Fig. 7 The effects of β on dual temperature profiles $\theta(\eta)$.

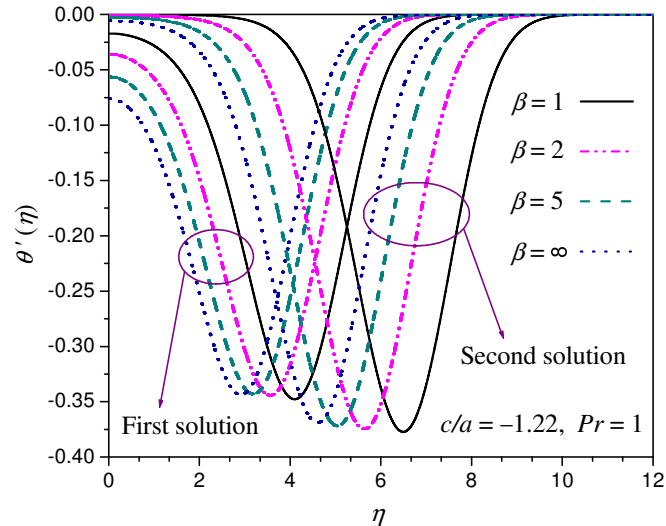


Fig. 8 The effects of β on dual temperature gradient profiles $\theta'(\eta)$.

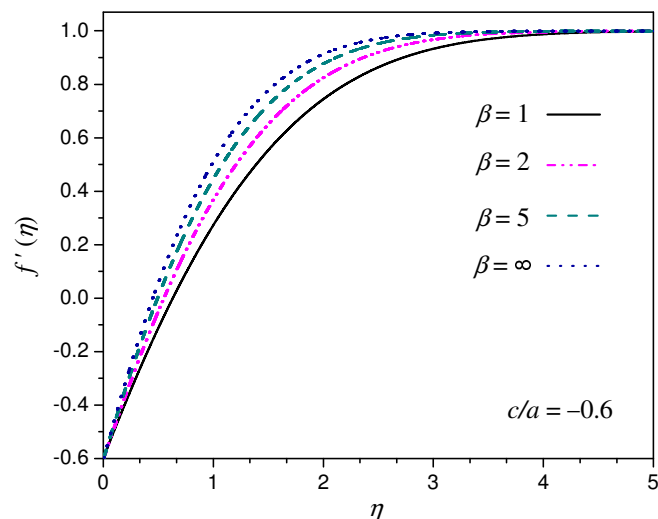


Fig. 9 The effects of β on the unique velocity profiles $f'(\eta)$ for shrinking sheet.

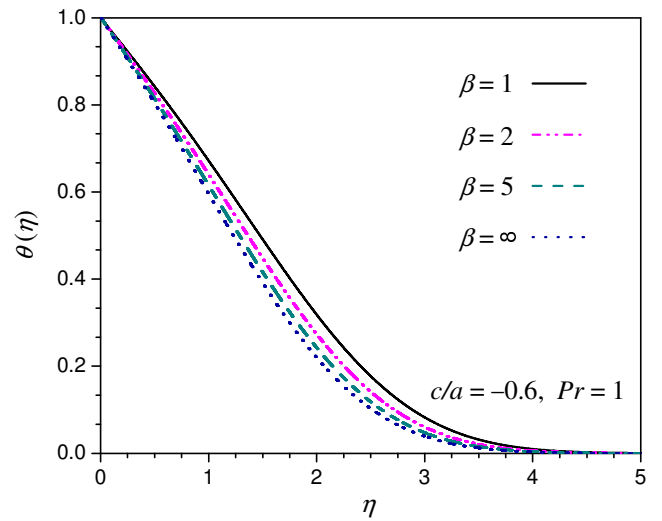


Fig. 10 The effects of β on the unique temperature profiles $\theta(\eta)$ for shrinking sheet.

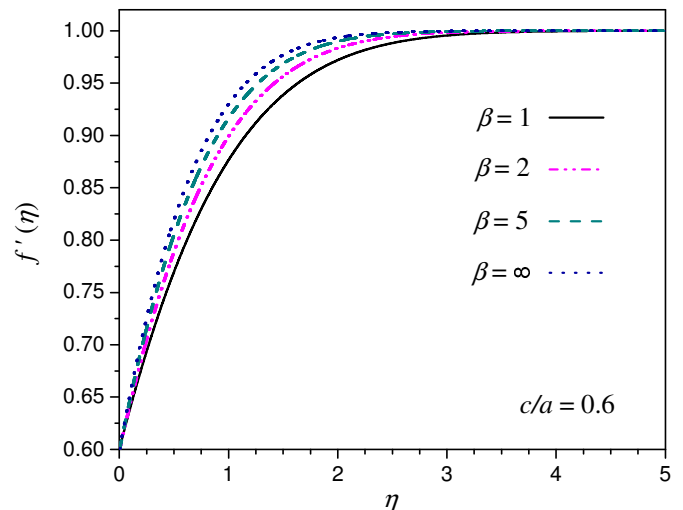


Fig. 11 The effects of β on the unique velocity profiles $f'(\eta)$ for stretching sheet.

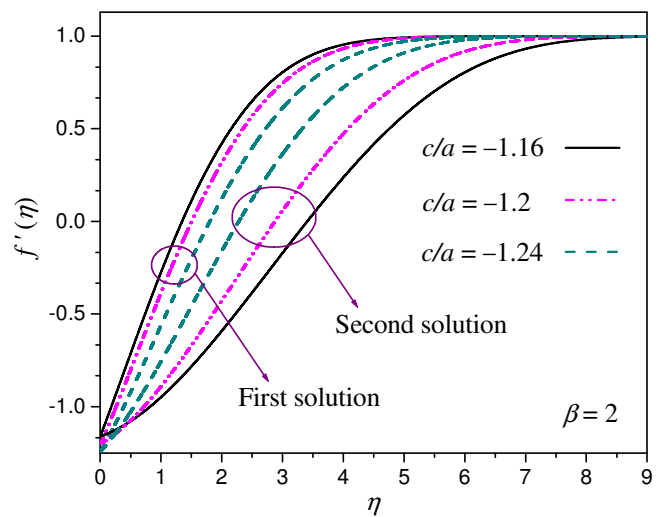


Fig. 12 The effects of c/a on dual velocity profiles $f'(\eta)$.

Next, the focus is concentrated to the effects of the velocity ratio parameter c/a and the Prandtl number Pr on the velocity and temperature distributions in Casson fluid flow. In Fig. 12 and Fig. 13 the influence of c/a on dual velocity and temperature profiles are presented respectively. Similar to Newtonian fluid case, here for Casson fluid two opposite effects are observed in two solutions. For first solution, boundary layer thicknesses (velocity and thermal) increase with increasing magnitude of c/a and in second solution those decrease. The dual temperature profiles for several values of Prandtl number are demonstrated in Fig. 14. The thermal boundary layer thickness decreases with increasing Prandtl number with a temperature crossing over.

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NOMENCLATURE

a	straining constant
c/a	velocity ratio parameter
c	shrinking/stretching constant
c_p	specific heat
f	dimensionless stream function
f'	dimensionless velocity
Pr	Prandtl number
p	a variable
p_y	yield stress of fluid
q	a variable
T	temperature
T_w	constant temperature at the sheet
T_∞	constant free stream temperature
U_s	straining velocity of the stagnation-point flow
U_w	shrinking/stretching velocity of the sheet
u	velocity component in x direction
v	velocity component in y direction
x	distance along the sheet
y	distance perpendicular to the sheet
z	a variable

Greek symbols

β	non-Newtonian/Casson parameter
η	similarity variable
η_∞	finite value of η
κ	thermal conductivity
μ_B	plastic dynamic viscosity of the non-Newtonian fluid
π	product of the component of deformation rate with itself
π_c	critical value of π
ν	kinematic fluid viscosity
ρ	fluid density
ψ	stream function
θ	dimensionless temperature

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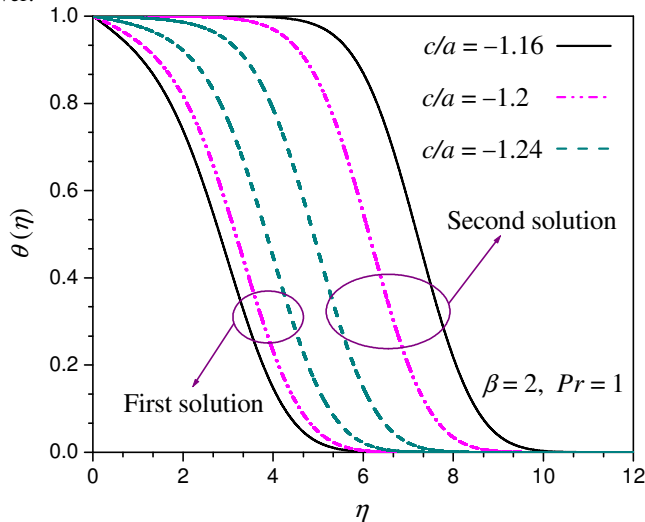


Fig. 13 The effects of a/c on dual temperature profiles $\theta(\eta)$.

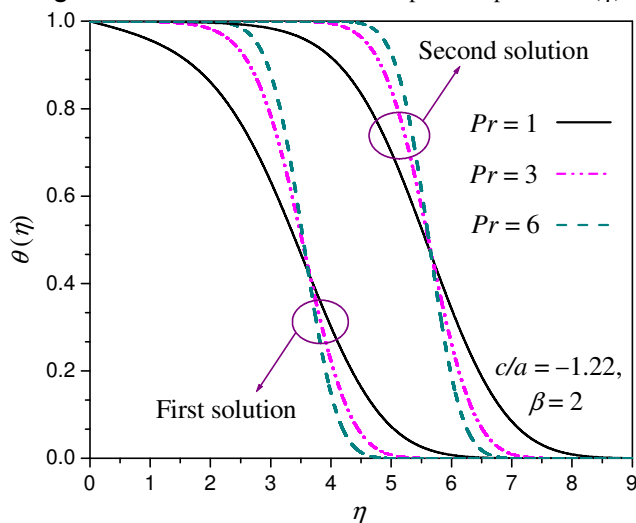


Fig. 14 The effects of Pr on dual temperature profiles $\theta(\eta)$.

6. CONCLUSIONS

The objective of this investigation is to emphasize on similarity solution properties of the boundary layer stagnation-point flow of Casson fluid and heat transfer towards a shrinking/stretching sheet. An analysis of existence and uniqueness of boundary layer self-similar solution of transformed equations is made based on numerical computations using shooting method. The study explores that similar to Newtonian case, the self-similar solution is of dual nature in some situations of shrinking sheet case, for stretching sheet case the solution is always unique. Also, it is found that the velocity and thermal boundary layer thicknesses are larger for Casson fluid than that of Newtonian fluid.

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