

LAPLACE TRANSFORM SOLUTION OF UNSTEADY MHD JEFFRY FLUID FLOW PAST VERTICALLY INCLINED POROUS PLATE

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ABSTRACT

The behavior of unsteady MHD flow of Jeffrey fluid over an inclined porous plate was analyzed in the present article. The governing partial differential equations of the flow phenomena were solved by using powerful mathematical tool Laplace transforms. The variations of velocity, temperature of the flow with respect to dissimilar physical parameters are analyzed through graphs. The parameters of engineering interest are skin friction and Nusselt number. For better understanding of the problem, variations of skin friction and Nusselt number with respect to critical parameters are tabulated.

Keywords: Laplace Transform, MHD, Jeffrey fluid, Unsteady, Inclined, Porous plate

1. INTRODUCTION

Non – Newtonian fluids got good attention by several researchers because of their significant industrial and technical applications. A good number of people have done work on MHD flows. Vijaya N et al.(2018) thermo physical properties of a Casson fluid through an oscillating vertical wall embedded through porous medium. Vedavathi N et al.(2017) investigated the radiation effects, of MHD free convective chemically reacting nano fluid over a semi infinite vertical plate. An analytical investigation has been done by Dharmiah G et al.(2018), for an unsteady, two-dimensional, laminar, boundary layer flow of a viscous incompressible electrically conducting fluid along a semi-infinite vertical permeable moving plate in the presence of Diffusion-thermo and radiation absorption effects. Heat transfer analysis on a stagnation point boundary layer flow of a Jeffrey fluid over a stretching surface was done by Ehtsham A et al.(2019). Dhanalakshmi M et al.(2019) the soret and dufour effect of Kuvshinshiki fluid over MHD free convection flow past a vertical porous plate with heat and mass exchange in companionship of chemical reaction. Kallepalli N.S et al. (2019) studied the behaviour of MHD flow in porous channel under the influence of different critical parameters. Sivaiah G et al. (2019) made a survey of MHD boundary layer flow visco elastic and dissipative fluid past porous plate. Vijaya K et al. (2019) talked about MHD casson fluid flow in presence of radiation soret and chemical reaction effects. Sujatha T et al.(2019) perceived the chemical reaction effect on MHD nanofluid flow over cone and wedge. Ibrahim S.M et al. (2017) investigated heat and mass transfer in a chemically reacting laminar mixed convection flow from a vertical sheet with induced magnetic field. Reddy G.V.R et al. (2018) dicussed the the Soret and Dufour effects on MHD micropolar liquid flow over linearly stretching sheet. Jayaramireddy K et al.(2018) investigated MHD chemically reacting Casson nano fluid flow over non linear permeable stratching sheet in presence of thermal radiation, viscous dissipation. Naga santoshi P et al.(2018) have done analysis on heat and mass transfer of mixed convection MHD Casson Nano fluid flow over a stretching sheet with variable viscosity and non-uniform heat source. Charan kumar G et al.(2018) described the effects of Joule heating and chemical reaction on unsteady MHD mixed convective micro polar fluid over a

stretching sheet in presence of radiation, non-uniform heat source and porous medium. Krishna Y.H et al.(2018) addressed the magneto hydrodynamic flow of a Casson fluid over a permeable stretching sheet in the presence of mass transfer. From above survey in view of their differences with Newtonian fluids, several models of non-Newtonian fluids have been proposed. The familiar and simple model of non-Newtonian fluid is Jeffrey fluid. Sunitha Rani Y et al.(2019) presented the finite element analysis of MHD Jeffrey fluid over inclined plate. As an extension in the present analysis MHD flow of Jeffrey fluid above slanted vertical permeable plate was considered. The governing equations are solved to get the exact analytical solution applying Laplace transform technique. The results are in good agreement with Sunitha Rani Y et al.[2019].

2. ANALYSIS OF THE PROBLEM

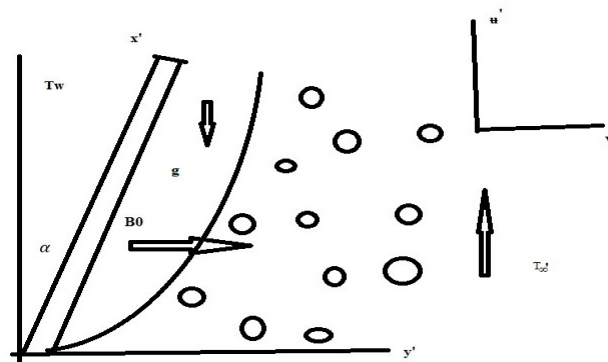


Fig. 1 Geometry of the problem

The plate is taken along X' - direction and y' is taken normal to the plate. The plate is considered of infinite length along X' direction and so all the physical parameters will not depend on X' . The velocity components along X' and y' direction are taken as u' and v'

respectively. T'_W and T'_∞ are the temperatures at wall and free stream temperatures. A uniform magnetic field of strength B_0 is applied normal to the plate. The transverse magnetic field and magnetic Reynolds number are assumed to be very small so that the produced magnetic field is negligible. An unsteady MHD free convective, incompressible, electrically conducting fluid flow over an inclined permeable plate was considered. The equations narrating the flow are given as follows.

$$\frac{\partial v'}{\partial y'} = 0 \quad (v' \text{ is constant}) \quad (1)$$

$$\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} = g\beta(T' - T'_\infty) \cos \alpha + \left(\frac{v}{1+\lambda} \right) \frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_0^2 u'}{\rho} - \frac{vu'}{k'} \quad (2)$$

$$\frac{\partial T'}{\partial t'} + v' \frac{\partial T'}{\partial y'} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T'}{\partial y'^2} \quad (3)$$

The associated boundary conditions of the flow are

$$u' = 0, v' = -v_0, T' = T'_W + \varepsilon(T'_W - T'_\infty) e^{i\omega t'} \text{ at } y' = 0 \quad (4)$$

$$u' \rightarrow 0, T' \rightarrow T'_\infty \text{ as } y' \rightarrow \infty \quad (5)$$

The non dimensional variables and parameters are

$$y = \frac{y'v_0}{v}, t = \frac{t'v_0^2}{4v}, \omega = \frac{4v\omega'}{v_0^2}, u = \frac{u'}{v_0}, M = \frac{\sigma B_0^2 v}{\rho v_0^2} \quad (6)$$

$$\theta = \frac{T' - T'_\infty}{T'_W - T'_\infty}, Pr = \frac{v\rho c_p}{\kappa}, Gr = \frac{vg\beta(T'_W - T'_\infty)}{v_0^3}, k = \frac{k'v_0^2}{v^2}$$

Using the (6) in equations (2) and (3)

$$\frac{1}{4} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = Gr \cos \alpha \theta + \left(\frac{1}{1+\lambda} \right) \frac{\partial^2 u}{\partial y^2} - \left(M + \frac{1}{k} \right) u \quad (7)$$

$$\frac{1}{4} \frac{\partial \theta}{\partial t} - \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} \quad (8)$$

The transformed boundary conditions are

$$u = 0, \theta = 1 + \varepsilon e^{i\omega t} \text{ at } y = 0 \quad (9)$$

$$u \rightarrow 0, \theta \rightarrow 0 \text{ as } y \rightarrow \infty \quad (10)$$

3. Laplace Transform Solution

Laplace transform technique was employed to get exact solution of the problem. Applying Laplace transform for the equations (7) and (8) becomes ordinary differential equations of second order.

$$\frac{d^2 u}{dy^2} - D \frac{du}{dy} - (e - sf)u = G \cos(\alpha) \theta(y, s) \quad (11)$$

$$\frac{d^2 \theta}{dy^2} + Pr \frac{d\theta}{dy} - \frac{Prs}{4} \theta = -\frac{Pr}{4} \theta(y, 0) \quad (12)$$

and the boundary conditions (9) and (10) becomes

$$u(y, s) = 0, \theta(y, s) = \frac{1}{s} + \varepsilon \frac{1}{s - i\omega} \text{ at } y = 0 \quad (13)$$

$$u(y, s) \rightarrow 0, \theta(y, s) \rightarrow 0 \text{ as } y \rightarrow \infty \quad (14)$$

Solving the equations (11) and (12) along with the boundary conditions we obtain the expressions for the solutions of (11) and (12) as

$$u(y, s) = e^{-\frac{Dy}{2}} \left\{ Pr G \cos(\alpha) \left[\frac{M_1}{\sqrt{1+Prs-K}} + \frac{M_2}{\sqrt{1+Prs-L}} \right] + \varepsilon \left[\frac{M_5}{\sqrt{1+Prs-K}} + \frac{M_6}{\sqrt{1+Prs-L}} \right] + \left[\frac{M_7}{\sqrt{1+Prs-N_4}} - \frac{M_7}{\sqrt{1+Prs+N_4}} \right] \right\} \quad (15)$$

$$\theta(y, s) = e^{-\frac{yPr}{2}} \left(\frac{1-k}{s} + \varepsilon \frac{1}{s-i\omega} \right) e^{-\frac{yPr\sqrt{1+Prs}}{2}} + \frac{k}{s} e^{-yPr\sqrt{1+Prs}} \quad (16)$$

Applying inverse Laplace transform for equations (15) and (16) we get the expressions for velocity and temperature distributions of the flow as follows

$$u(y, t) = e^{-\frac{Dy}{2}} Pr G \cos \alpha \left[2KM_1 e^{-y\sqrt{N_1} + K^2 t} + 2LM_2 e^{-y\sqrt{N_1} + L^2 t} \right] - \varepsilon \left[2KM_5 e^{-y\sqrt{N_1} + L^2 t} + 2LM_6 e^{-y\sqrt{N_2} + L^2 t} + 2N_4 M_7 e^{-y\sqrt{N_3} + N_4^2 t} \right] + Pr G \cos \alpha e^{-\frac{yPr}{2}} \left[M_1 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + Kc (K^2 t - Ky) \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - K\sqrt{t} \right) \right] \right. \quad (17)$$

$$+ M_2 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + Le (L^2 t - Ly) \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - L\sqrt{t} \right) \right]$$

$$+ M_3 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + e^{(t-y)} \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - \sqrt{t} \right) \right]$$

$$- M_3 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} - e^{(t-y)} \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - \sqrt{t} \right) \right]$$

$$+ \varepsilon e^{-\frac{yPr}{2}} \left[M_5 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + Kc (K^2 t - Ky) \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - K\sqrt{t} \right) \right] \right]$$

$$+ M_6 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + Le (L^2 t - Ly) \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - L\sqrt{t} \right) \right]$$

$$\begin{aligned}
 &+M7 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} + N_4 e^{(N_4^2 t - N_4 y)} \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} - N_4 \sqrt{t} \right) \right] \\
 &-M7 \left[\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2 Pr}{16t}} - N_4 e^{(N_4^2 t - N_4 y)} \operatorname{erfc} \left(\frac{y\sqrt{Pr}}{4\sqrt{t}} + N_4 \sqrt{t} \right) \right] \\
 \theta(y,t) = &e^{-b} \left[\frac{Pr}{2} \left\{ e^{-b+t} \operatorname{erfc} \left(\frac{b}{2\sqrt{t}} - \sqrt{t} \right) + e^{b+t} \operatorname{erfc} \left(\frac{b}{2\sqrt{t}} + \sqrt{t} \right) \right\} \right] \\
 &+ \epsilon e^{-b} \left[\frac{Pr}{2} \left\{ e^{-bN_4 + N_4^2 t} \operatorname{erfc} \left(\frac{b}{2\sqrt{t}} - N_4 \sqrt{t} \right) + e^{b+t} \operatorname{erfc} \left(\frac{b}{2\sqrt{t}} + N_4 \sqrt{t} \right) \right\} \right]
 \end{aligned}$$

4. SKIN FRICTION AND RATE OF HEAT TRANSFER

The physical parameters of engineering interest are skin friction and Nusselt number. These parameters are useful to narrate the fluid flow and exchange of temperature near to the plate. The non dimensional skin friction and Nusselt number are

$$sf = - \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad (19)$$

$$Nu = - \left(\frac{\partial \theta}{\partial y} \right)_{y=0} \quad (20)$$

5. RESULTS AND DISCUSSIONS

The equations narrating the flow are resolved by employing Laplace transform method, to get exact analytical solutions for velocity and temperature. The influence of various physical parameters on fluid flow were analyzed graphically. The variations in skin friction, Nusselt number with respect to different physical parameters were illustrated through tables. The range of the variable y was taken in between 0 and 6. It is observed that the profiles of velocity and temperature fields reflecting the boundary conditions of the problem.

Figure – 2 displays the velocity profiles for different values of inclined angle. It reflects the rise in angle of inclination slow down the velocity of the fluid, because of buoyancy force due to gravity. Figure -3 shows that increase of Jeffery fluid parameter increases fluid velocity. Figure – 4 depicts increase of thermal grashof number enhances fluid velocity. This is due to the fact increment of grashof number increases the buoyancy forces that lead to rise in liquid velocity. Figure – 5 shows the impact of permeability parameter on fluid flow. This is because porous medium opposes the fluid flow. Lorentz forces resist the fluid flow which was accompanied with magnetic parameter. Therefore increase of magnetic parameter declines the fluid velocity. This was depicted in figure – 6. Figure – 7 shows the impact of Prandtl number on velocity profiles. Increase of Prandtl number results in decrease in thermal conductivity of the fluid, and hence temperature of the fluid decreases.

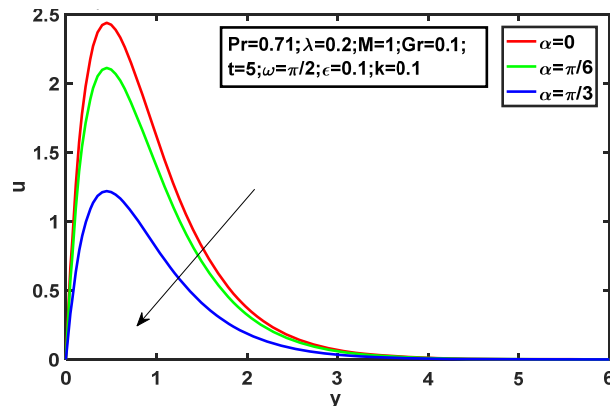


Fig. 2 Variation of velocity for dissimilar values of α

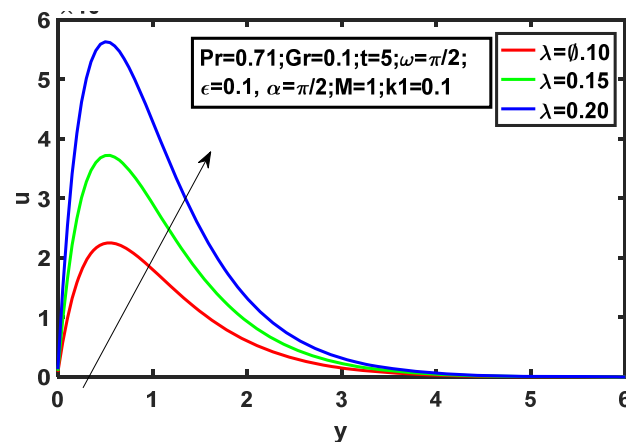


Fig. 3 Variation of velocity for dissimilar values of λ

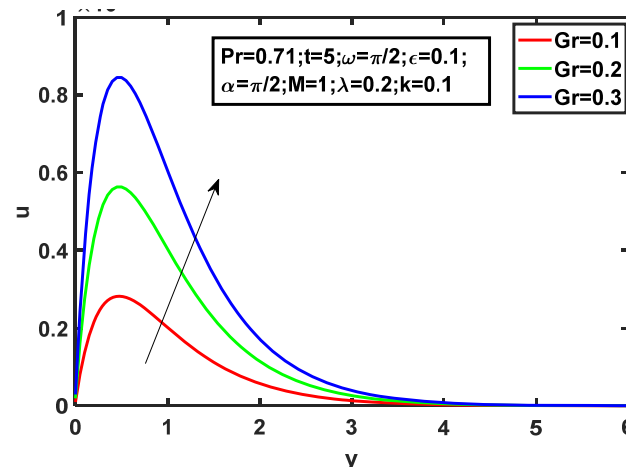


Fig. 4 Variation of velocity for dissimilar values of Gr

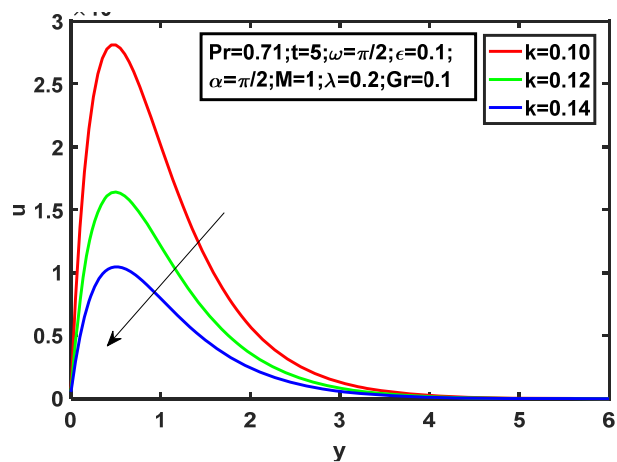


Fig. 5 Variation of velocity for dissimilar values of k

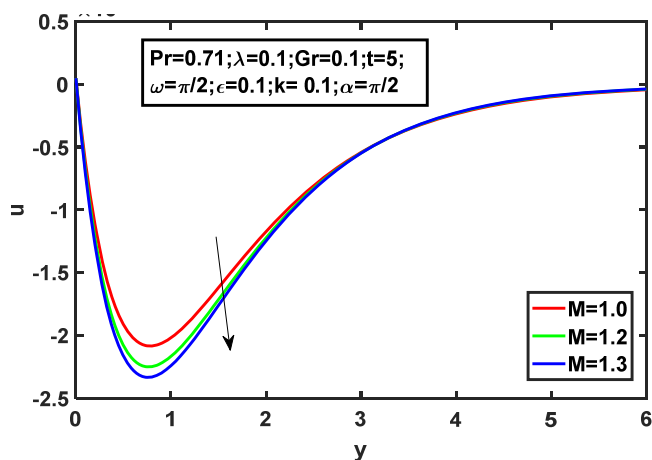


Fig. 6 Variation of velocity for dissimilar values of M

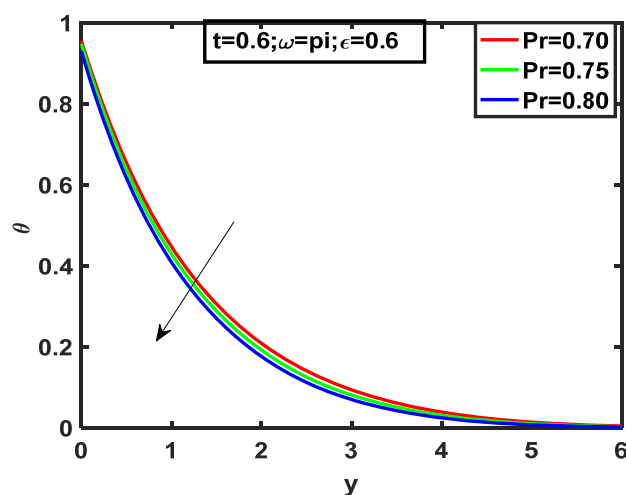


Fig. 7 Variation of temperature for dissimilar values of Pr

Table 1 Variation of skin friction for different physical parameters

Gr	k	M	Pr	α	λ	sf
0.1	0.1	1	0.71	0	0.1	1.6396
0.2	0.1	1	0.71	0	0.1	3.2793
0.3	0.1	1	0.71	0	0.1	4.9190
0.1	0.1	1	0.71	0	0.1	1.6396
0.1	0.2	1	0.71	0	0.1	-0.7401
0.1	0.3	1	0.71	0	0.1	-0.7821
0.1	0.1	1	0.71	0	0.1	1.6396
0.1	0.1	2	0.71	0	0.1	2.8005
0.1	0.1	3	0.71	0	0.1	4.2873
0.1	0.1	1	0.71	0	0.1	1.6396
0.1	0.1	1	4	0	0.1	5.1381
0.1	0.1	1	7	0	0.1	6.5435
0.1	0.1	1	0.71	0	0.1	1.6396
0.1	0.1	1	0.71	0	0.2	-0.7401
0.1	0.1	1	0.71	0	0.3	-0.7821

Table 2 Variation of Nusselt number with Prandtl number

Pr	t	nu
0.70	0.6	0.8804
0.75	0.6	1.0109
0.80	0.6	1.15055

6. Conclusions

Analysis of unsteady MHD Jeffery fluid flow was done in the present article. Laplace transform technique was employed for the governing equations of the flow. The governing equations are first transformed into a set of normalized equations and then solved analytically applying Laplace transform technique to obtain the general solution. The expressions for velocity and temperature were obtained exponential and error functions. For better understanding of the problem the outcome of the phenomenon was illustrated graphically to explain the behavior of the fluid flow under the influence of different physical parameters. The velocity profiles are starting at origin and converging to zero far away from the plate. Whereas temperature profiles starting at 1 at origin and far away from the plate coinciding with the axis. This shows that the obtained solution is satisfying the boundary conditions of the present problem. Change in skin friction and Nusselt numbers with respect to different critical parameters was shown with tables. During the analysis the following conclusions were drawn. The present work can be extended by incorporating Newtonian heating for the temperature equation.

1. Angle of inclination α declines the speed the fluid
2. Fluid velocity escalates with rise in Jeffery parameter.
3. Thermal grashof number enhances velocity of the fluid
4. Permeability declines the fluid velocity
5. Magnetic parameter lowers fluid velocity
6. Temperature distribution of the fluid drops down for growing values of Prandtl number.
7. From Table 1, we can observe that increase of Prandtl number and Magnetic parameter raises skin friction.
8. Rise in Jeffery parameter lowers skin friction
9. Increase in permeability parameter lowers skin friction
10. Nusselt number increases for growing values of Prandtl number

Tables 1 and 2 analyze the change in skin friction and Nusselt number under the influence of different physical parameters.

Nomenclature

u'	Velocity component in x' – direction
v'	Velocity component in y' – direction
u	Dimensionless velocity
c_p	Specific heat at constant pressure
g	Acceleration due to gravity
T	Dimensional temperature
T_w	Wall dimensional temperature
T_∞	Free stream dimensional temperature
θ	Dimensionless temperature
P_r	Prandtl number
B_0	Uniform magnetic field
k	Permeability parameter
k'	Permeability of the porous media
M	Magnetic parameter
t'	Dimensional time
t	Dimensionless time
Gr	Thermal grashof number
Greek symbols	
ν	Kinematic viscosity
α	Angle of inclination
β	Volumetric coefficient of thermal expansion
ω	The constant
ϵ	A real positive constant <1
λ	Jeffrey parameter
σ	Electrical conductivity
ρ	Fluid density

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Appendix

$$a = Pr, b = \frac{yPr}{2}, c = yPr, d = \frac{1}{1+\lambda}, e = \frac{M + \frac{1}{k}}{1+\lambda}$$

$$f = \frac{1}{4(1+\lambda)}, G = \frac{Gr}{1+\lambda}, H = d^2 + 4e,$$

$$A = \frac{Pr^2}{4} + \frac{f}{a}, B = \frac{Pr^2}{2} - \frac{dPr}{2}, C = \frac{Pr^2}{4} - \frac{dPr}{2} - \left(\frac{ac+f}{a}\right)$$

$$K = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, L = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$M_1 = \frac{1}{(K-L)(K^2-1)}, M_2 = \frac{1}{(L-K)(L^2-1)}, M_4 = -M_3$$

$$M_5 = \frac{1}{(K-L)(K-N_4)(K+N_4)}, M_6 = \frac{1}{(L-K)(L-N_4)(L+N_4)}$$

$$M_7 = \frac{1}{2(N_4-K)(N_4-L)N_1}, M_8 = -M_7$$

$$N_1 = H - \frac{(K^2-1)f}{a}, N_2 = H - \frac{(L^2-1)f}{a}, N_3 = H - i\omega f,$$

$$N_4 = \sqrt{1 + ai\omega}$$