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EFFECTS OF VISCOUS DISSIPATION AND AXIAL HEAT CONDUCTION ON FORCED CONVECTION DUCT FLOW OF HERSCHEL-BULKLEY FLUID WITH UNIFORM WALL TEMPERATURE OR CONVECTIVE BOUNDARY CONDITIONS

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ABSTRACT

The aim is to study the asymptotic behavior of the temperature field for the laminar forced convection of a Herschel-Bulkley fluid flowing in a circular duct considering both viscous dissipation and axial heat conduction. The asymptotic bulk and mixing Nusselt numbers and the asymptotic bulk and mixing temperature distribution are evaluated analytically in the cases of uniform wall temperature and convection with an external isothermal fluid. In particular, it has been proved that the fully developed value of Nusselt number for convective boundary conditions is independent of the Biot number and is equal to the value of fully developed Nusselt number for uniform wall temperature. The analytical results obtained in the fairly general case of Herschel-Bulkley fluid, which considers both the existence of a yield shear-stress as well as a variable viscosity as a function of shear, are compared with solutions available for the simpler cases of Newtonian fluid and some non-Newtonian fluids with viscous dissipation and/or axial heat conduction taken into account.

Keywords: *Laminar Forced Convection, Viscous Dissipation, Axial Heat Conduction, Asymptotic Behavior, Analytical Methods.*

1. INTRODUCTION

The main goal of this study is to characterize the heat transfer in the case of flow in a cylindrical pipe of fluids whose rheological behavior can be modeled by Herschel-Bulkley law and by considering both the viscous dissipation and the axial heat conduction and this with boundary conditions fixing either a constant temperature at the wall or the value of an exchange coefficient with an external fluid. This type of behavior, which includes the effect of yield stress, corresponds in practice, to an important class of products treated by foods, pharmaceutical, cosmetics, polymers and chemical processing industries (Coussot 2014, Vijaya et *al*. 2020, Revathi et *al*. 2020 and 2021). These products require various heat treatments (heating, cooling) during their processing or their use. The heat treatment often involves heat exchangers with fluid dependent design. The situation considered in this work is that of a laminar flow in horizontal pipe for boundary conditions cases of uniform wall temperature or for a wall heat exchanges by convection with an external fluid for a constant exchange coefficient h_e. This type of geometry corresponds to shell and tube heat exchangers, which makes it very useful.

The problem of heat transfer in cylindrical ducts in case for Newtonian or non-Newtonian fluids has been the subject of abundant literature (see Khatyr and Khalid Naciri 2020). For Newtonian fluids, the cases of constant temperature at the wall or convection with an external fluid with a constant exchange coefficient h_e have been widely studied for various situations with or without viscous dissipation and with or without axial heat conduction, a review is presented by (Goldstein et *al*. 2003). Magyari and Barletta (2007) solved the governing equations analytically by using power series method for laminar forced convection flow of a liquid in the fully developed region of a circular duct with isothermal wall and variable viscosity. Kundu et *al.* (2011) used the Integral Ritz and Variational methods of approximate analytical techniques to obtain a solution for the fully developed laminar Newtonian fluid flow through rectangular channels with constant wall temperature. The Nusselt numbers obtained by different approximate methods are shown as a function of aspect ratio. Astaraki et *al.* (2013) present an analytical solution for fully developed laminar forced convection of Newtonian fluid flowing in a circular duct while neglecting both axial heat conduction and viscous dissipation. The duct walls have a finite width, and the external wall temperature is a sinusoidal function of axial direction. They found that the mean Nusselt number is an increasing monotonic function of Peclet number and a dimensionless frequency.

For non-Newtonian fluids, the effect of viscous dissipation is investigated by Sayed-Ahmed (2000), Sayed-Ahmed and Kishk (2008), Mondal and Mukherjee (2012), Labsi and Benkahla (2016) Chaudhuri and Das (2018), Kiyasatfar and Pourmahmoud (2016) and Kamisli (2020). Sayed-Ahmed (2000) and Sayed-Ahmed and Kishk (2008) presented a numerical solution for laminar heat transfer of Herschel-Bulkley fluids in the thermal entrance region of a square section pipe, assuming the velocity profile established, and by considering the cases of isothermal wall or constant wall heat flux. Mondal and Mukherjee (2012) studied analytically the effect of viscous dissipation on the heat transfer for a shear driven flow for varying degree of asymmetry in the wall heating. Labsi and Benkahla (2016) studied numerically the effect of viscous dissipation on the Herschel-Bulkley fluid with constant physical and rheological properties, flow

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within a pipe of circular cross section, submitted at constant wall temperature. They found that heat transfer is significantly underestimated, when viscous dissipation is neglected, particularly for shear-thinning fluids and for high values of the Hershel-Bulkley number. Chaudhuri and Das (2018) give a semi-analytical solution of the heat transfer including viscous dissipation in the steady flow of a Sisko fluid in cylindrical tubes for constant wall heat flux (heating or cooling) cases. They concluded, in the case of heating, heat transfer coefficient (Nu) decreases with the increase in Brinkman number, but for cooling, Nu increases asymptotically with the increase in the Brinkman number.

Kiyasatfar and Pourmahmoud (2016) studied convective heat transfer for fully developed flows of conducting power-law fluids in square microchannels with presence of transverse magnetic field considering effects of viscous dissipation and joule heating. They found that the effect of Joule heating parameter on Nusselt number is significantly affected by Brinkman number. Kamisli (2020) analyzed the fully developed Herschel-Bulkley fluid flow in planar and circular microducts with constant heat flux considering slip boundary condition and including viscous dissipation. It showed that, at larger value of the Brinkman number and the flow behavior index (n), the influences of viscous dissipation and slip coefficient on irreversibility distribution ratio, entropy generation and Nusselt number are significant. Consequently, they should be considered for the designing of microfluidic heat exchangers when non-Newtonian fluids are used.

Alves et *al*. (2015) using a semi-analytical method, (Cruz et *al*. 2012, Baptista et *al*. 2014), calculated the values of heat transfer coefficients for laminar flow of non-Newtonian fluids (Herschel-Bulkley, Bingham, Casson, Carreau-Yasuda) in pipes with constant wall temperature while neglecting axial heat conduction. Comparing with accurate numerical results, the estimated errors are below 7,4 % for Herschel-Bulkly fluids ($n \le 1{,}5$), and 3,5 % for the cases analyzed. Mendes et *al*. (2018) used the same method in the case of fully developed flow between parallel plates subject to constant wall heat flux and constant temperature. The error was found to be small, below 3,4%, except for the fluids with yield stress for which the maximum error increased to 8,4 % for the cases analyzed.

More Recently, Coelho and Poole (2021) analyzed analytically the effect of viscous dissipation on the heat transfer in laminar fully developed flow of a Herschel-Bulkley fluid between parallel plates in the cases constant and asymmetric wall heat fluxes. They show that the effect of variables power-law index and stress ratio on the Nusselt number is greater as the Brinkman number decreases.

The boundary condition of convective heat transfer with an external fluid is analyzed analytically by Lin et *al.* (1983) for Newtonian fluid and experimentally and numerically by Javaherdeh and Devienne (1999) for Herschel-Bulkley fluids. Lin et *al.* (1983) used the eigenfunction series expansion technique to determine the effect of viscous dissipation on thermal entrance heat transfer in laminar pipe flows. They showed that the effects of viscous dissipation for convective boundary condition are similar to those for uniform wall boundary condition for large outside Nusselt number ($Nu_0 = UR/k$, with U is the outside heat transfer coefficient, R the pipe radius and k is the thermal conductivity of the fluid in pipe). Javaherdeh and Devienne (1999) studied the heat transfer for Herschel-Bulkley fluids with thermally depending consistency flowing in a horizontal cylindrical duct submitted to a wall cooling by an external counter current flow with constant exchange coefficient h_e. They concluded, for the tested fluids, that the thermodependence of the apparent viscosity only slightly modifies the temperature field. However, one can observe a measurable deformation of the dynamic field linked to an increase in consistency near the cold wall.

The case of exchange by convection with an external isothermal fluid has not yet been analyzed analytically. However, note that this case is a special case of variable flux at the wall. The interest of the analytical solution is given by Letelier et *al.* (2017) “Given the difficulties still plaguing numerical methods to determine the flow and thermal fields of non-Newtonian fluids analytical solutions provide valuable guidance to the behavior of complex fluids.”

In this context, the solution presented in this paper is, to the best of our knowledge, an original analytical solution of the fully developed forced convection of Herschel-Bulkley fluid flowing in circular duct with boundary conditions fixing either a constant temperature at the wall or the value of an exchange coefficient with an external fluid and by taking into account viscous dissipation and axial heat conduction. It will be shown that the values of Nu_∞^* and Nu_∞ obtained in previous work (Khatyr and Khalid Naciri 2020), for particular values of Br_∞, coincide with those obtained when the wall temperature is constant or with those obtained when the wall exchanges heat by convection with an external fluid.

The organization of this article is as follows: section two is devoted to the mathematical analysis. The third section presents the results in the cases uniform wall temperature and convective boundary conditions.

2. ANALYSIS

Let us consider a Herschel-Bulkley fluid of constant physical properties flowing in a circular duct of radius r_0, submitted to a uniform temperature or to heat exchanges by convection with an external isothermal fluid. The flow is supposed to be steady, laminar, fully developed and axisymmetric.

The fully developed velocity profile for a laminar pipe flow of a Herschel-Bulkley fluid is given as follows (Nouar et *al.* 1994)

$$u(r) = \begin{cases} \frac{u_m}{\omega}\left(1 - \left(\frac{\frac{r}{r_0} - a}{1-a}\right)^{m+1}\right) & if \quad r_c \le r \le r_0 \\ \frac{u_m}{\omega} & if \quad 0 \le r \le r_c \end{cases} \tag{1}$$

where $\omega = 1 - 2\left(\frac{a(1-a)}{m+2} + \frac{(1-a)^2}{m+3}\right)$, $m = 1/n$ is the inverse of exponent index n (with $n > 0$; for $n = 1$: Bingham fluid), $a = \frac{\tau_c}{\tau_w} = \frac{r_c}{r_0}$ is the dimensionless radius of the plug flow region (with $0 \le a \le 1$), τ_c the yield shear stress, τ_w the wall shear stress, r the radial coordinate, r_c the yield radius and u_m the mean velocity value.

The energy equation is given by (Bejan 1984, Nouar et *al.* 1995)

$$\rho c_p u \frac{\partial T}{\partial x} = \lambda \frac{\partial^2 T}{\partial x^2} + \frac{\lambda}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + K\left|\frac{du}{dr}\right|^n \frac{du}{dr} + \tau_c \frac{du}{dr} \tag{2}$$

where ρ, λ, K and c_p are the density of fluid, thermal conductivity, the consistency index, and the specific heat at constant pressure, respectively and x is the axial coordinate.

It should be noted that in Eq. (2), viscous dissipation only occurs in the part $r_c \le r \le r_0$ for which the radial velocity gradient is non-zero. However, this equation remains valid throughout the section as far as for $0 \le r < r_c$ the velocity gradient is zero and therefore the dissipation term vanishes.

The boundary conditions associated to Eq. (2) of course relates to the values of temperature imposed on the walls or for the heat exchange coefficient with the external fluid, but they must also specify the thermally established flow situation to be considered, which is not obvious for the considered cases.

Note that the condition that leads to an asymptotic thermally developed region in the case of the forced convection problem considered above is defined (Bejan 1984, Barletta and Zanchini 1995) as follow

$$\lim_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)} = \lim_{x\to+\infty} \Theta\left(\frac{r}{r_0}, \frac{x}{2r_0 Pe}\right) = \Theta_\infty(r/r_0) \tag{3}$$

$$\lim_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_m(x)} = \lim_{x\to+\infty} \Theta^*\left(\frac{r}{r_0}, \frac{x}{2r_0 Pe}\right) = \Theta_\infty^*(r/r_0) \tag{4}$$

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where $T_w(x)$, $T_b(x)$ and $T_m(x)$ are the wall temperature, the bulk temperature and the mixing temperature, respectively, $Pe = 2r_0 u_m \rho c_p/\lambda$ is the Peclet number, $\Theta_\infty(r/r_0)$ and $\Theta^*_\infty(r/r_0)$ are the asymptotic dimensionless temperature which are continuous and differentiable functions of r.

The bulk value of temperature field is defined as

$$T_b(x) = \frac{2}{u_m r_0^2}\int_0^{r_0} T(r,x)u(r)rdr \quad (5)$$

The mixing value of temperature field is defined as

$$T_m(x) = \frac{2}{u_m r_0^2}\int_0^{r_0}\left(T(r,x)u(r) - \alpha\frac{\partial T}{\partial x}\right)rdr = T_b(x) - \frac{4}{r_0 Pe}\int_0^{r_0}\frac{\partial T}{\partial x}rdr \quad (6)$$

where α is the thermal diffusivity of the fluid.

Thus, the boundary conditions can be fixed by giving the values of temperature or coefficient of exchanges at the wall to which one adds the relations (3) and (4).

In addition, if condition (3) and (4) holds, the asymptotic values of the Nusselt number Nu_∞ and Nu^*_∞ exists (Bejan 1984, Barletta and Zanchini 1995) and are given by

$$\lim_{x\to+\infty}\mathrm{Nu} = 2r_0\lim_{x\to+\infty}\frac{\left.\frac{\partial T}{\partial r}\right|_{r=r_0}}{T_w(x)-T_b(x)} = -2r_0\left.\frac{d\Theta_\infty}{dr}\right|_{r=r_0} = \mathrm{Nu}_\infty \quad (7)$$

$$\lim_{x\to+\infty}\mathrm{Nu}^* = 2r_0\lim_{x\to+\infty}\frac{\left.\frac{\partial T}{\partial r}\right|_{r=r_0}}{T_w(x)-T_m(x)} = -2r_0\left.\frac{d\Theta^*_\infty}{dr}\right|_{r=r_0} = \mathrm{Nu}^*_\infty \quad (8)$$

Introducing the dimensionless quantities (Barletta 1997)

$$X = \frac{x}{2r_0 Pe}, R = \frac{r}{r_0}, U(R) = \frac{u(r)}{u_m}, \theta = \lambda r_0^{n-1}\frac{T-T_{0b}}{Ku_m^{n+1}} \quad (9)$$

Eq. (2) can be rewritten in the dimensionless form

$$\frac{\partial}{\partial R}\left[R\frac{\partial\theta}{\partial R}\right] = \frac{RU}{4}\frac{\partial\theta}{\partial X} - \frac{R}{4Pe^2}\frac{\partial^2\theta}{\partial X^2} + \frac{a}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n R\frac{dU}{dR} - R\left|\frac{dU}{dR}\right|^{n-1}\left(\frac{dU}{dR}\right)^2 \quad (10)$$

Integrating Eq. (10) over the interval $0 \le R \le 1$ and by using the condition at $R = 0$ $\left(\left.\frac{\partial\theta}{\partial R}\right|_{R=0} = 0\right)$ yields

$$\frac{d\theta_m}{dX} = \frac{2^{3-n}}{Br(X)} + \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (11)$$

where $\theta_m(X)$ is the mixing value of the dimensionless temperature $\theta(R,X)$ and $Br(X)$ is a local Brinkman number defined as

$$Br(X) = \frac{Ku_m^{n+1}}{(2r_0)^n q_w(X)} \quad (12)$$

with $q_w(X)$ is the wall heat flux given by, when taking into account Eq. (9)

$$q_w(X) = \lambda\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = \frac{Ku_m^{n+1}}{r_0^n}\left.\frac{\partial\theta}{\partial R}\right|_{R=1} \quad (13)$$

Finally, Eqs. (10), (11) and (13) are used in the following to determine analytically the asymptotic temperature field while Eqs. (7) and (8) enables to evaluate values of the asymptotic Nusselt numbers Nu_∞ and Nu^*_∞ in both cases of uniform wall temperature and convection with an external isothermal fluid.

3. THERMAL ASYMPTOTIC BEHAVIOR

In this section, the behavior of the established laminar forced convection is determined, for both cases of uniform wall temperature $T_w = cte$ and of convection, through the pipe wall, with external fluid at a reference temperature T_f and a constant convection coefficient h_e.

3.1 Case of Constant Wall Temperature

To evaluate the asymptotic bulk and mixing temperature distribution (Θ_∞, Θ^*_∞) and the asymptotic bulk and mixing Nusselt numbers (Nu_∞, Nu^*_∞) in the case of uniform wall temperature $T_w = cte$ we express, by taking into account Eq. (8), the wall heat flux q_w in the fully developed region as follow

$$q_w = \lambda\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = \frac{\lambda\mathrm{Nu}^*}{2r_0}(T_w - T_m) \quad (14a)$$

By inserting Eq. (9) into Eq. (14a) we obtain

$$q_w = \frac{\lambda\mathrm{Nu}}{2r_0}(T_w - T_m) = \frac{Ku_m^{n+1}}{2r_0^n}\mathrm{Nu}^*(\theta_w - \theta_m) \quad (14b)$$

By substituting the Eq. (12) in the Eq. (11) we have

$$\frac{d\theta_m}{dX} = \frac{2^3 r_0^n}{Ku_m^{n+1}}q_w(\mathrm{X}) + \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (15)$$

Using Eq. (14b), the Eq. (15) becomes

$$\frac{d\theta_m}{dX} = 4\mathrm{Nu}^*(\theta_w - \theta_m) + \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (16)$$

In the case of constant wall temperature $\theta_w = cte$, we can express the Eq. (16) in the following form

$$\frac{d(\theta_w-\theta_m)}{dX} = -4\mathrm{Nu}^*(\theta_w - \theta_m) - \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (17)$$

For a fully developed temperature field $\mathrm{Nu}^* = \mathrm{Nu}^*_\infty$, and by integration of the Eq. (17) we obtain

$$\theta_w - \theta_m(\mathrm{X}) = \mathrm{C}_1\,exp(-4\mathrm{Nu}^*_\infty X) - \frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (18)$$

Where C_1 is an integration constant, determined by setting $X = L_{fd}$ in Eq. (18) such as

$$\mathrm{C}_1 = \theta_w - \theta_m(X = L_{fd}) + \frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (19)$$

where $X = L_{fd}$ is any given position for which the value of the dimensionless mixing temperature is known and where the thermally established flow hypothesis can be assumed to be valid. Generally, L_{fd} could be defined as a section where a matching is made between the solution at the end of the thermal entry region and that at the beginning of the thermally established flow zone (Hsu 1971).

Hence the Eq. (18) is written

$$\theta_w - \theta_m = \left[\theta_w - \theta_m(L_{fd}) + \frac{2}{Nu^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right]exp(-4\mathrm{Nu}^*_\infty X) - \frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \quad (20)$$

Substituting Eq. (20) into Eq. (14b), q_w is expressed in the fully developed region by

$$q_w = \frac{Ku_m^{n+1}\mathrm{Nu}^*_\infty}{2r_0^n}\left\{\left[\theta_w - \theta_m(L_{fd}) + \frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right]exp(-4\mathrm{Nu}^*_\infty X) - \frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right\} \quad (21)$$

Taking into account Eq. (19), Eq. (12) is written

$$Br(X) = \frac{Ku_m^{n+1}}{(2r_0)^n q_w(X)} = \frac{1}{2^{n-1}\mathrm{Nu}^*_\infty\left\{\left[\theta_w-\theta_m(L_{fd})+\frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right]exp(-4\mathrm{Nu}^*_\infty X)-\frac{2}{\mathrm{Nu}^*_\infty}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right\}} \quad (22)$$

It is easily checked that the condition $\lim_{X\to+\infty} Br(X) = Br_\infty$ (See Khatyr and Khalid Naciri in 2020) is satisfied with the non-vanishing values of Br_∞ such as

$$Br_\infty = -\frac{(1-a)^{n+1}}{2^n}\left(\frac{\omega}{m+1}\right)^n \tag{23}$$

However, $\Theta_\infty^*(R)$, $\Theta_\infty(R)$, Nu_∞^* and Nu_∞ are given by Khatyr and Khalid Naciri 2020, with Br_∞ expressed by the Eq. (23)

$$\Theta_\infty^*(R) = \frac{f(1)-f(R)}{f(1)} \tag{24}$$

$$\Theta_\infty(R) = \frac{f(1)-f(R)}{f(1)-\frac{1}{Pe^2}\left[2^{3-n}+\frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n Br_\infty\right]} = \frac{f(1)-f(R)}{f(1)} \tag{25}$$

$$\mathrm{Nu}_\infty^* = \mathrm{Nu}_\infty = -2\left.\frac{d\Theta_\infty^*}{dR}\right|_{R=1} = \frac{2^{1-n}}{f(1)} \tag{26}$$

where $f(1)$ is the expression of $f(R)$ at the wall $R = 1$ obtained analytically and given in Appendix.

3.2 Case of Convective Boundary Conditions

In this case, the boundary conditions relate to heat exchange through the duct wall with an external fluid which is at a constant temperature T_f, and with a uniform convection coefficient h_e. This situation is in fact a special case of the general category of flows with spatially variable wall heat flux, but with the constraint that the heat exchange coefficient remains constant and that the temperature of the external fluid is kept constant.

Introducing the Biot number, $Bi = \frac{h_e r_0}{\lambda}$, the wall heat flux is expressed as

$$q_w = \frac{\lambda Bi}{r_0}\left(T_f - T_w\right) \tag{27}$$

By using Eq. (9), the Eq. (27) yields

$$q_w = \frac{Ku_m^{n+1}}{r_0^n} Bi\left(\theta_f - \theta_w\right) \tag{28}$$

Similarly, the Eq. (14-b) gives

$$\theta_w = \frac{2r_0^n}{Ku_m^{n+1}Nu^*} q_w(X) + \theta_m \tag{29}$$

The substitution of the Eq. (29) in the Eq. (28) gives

$$q_w = \frac{\mathrm{Nu}^*}{\mathrm{Nu}^*+2Bi}\frac{Ku_m^{n+1}}{r_0^n} Bi\left(\theta_f - \theta_m\right) \tag{30}$$

Taking into account Eqs. (12) and (30), the Brinkman number is given by

$$Br(X) = \frac{\mathrm{Nu}^*+2Bi}{2^n \mathrm{Nu}^* Bi(\theta_f-\theta_m)} \tag{31}$$

By inserting the Eq. (31) into the Eq. (11) we have

$$\frac{d\theta_m}{dX} = \frac{8\mathrm{Nu}^* Bi(\theta_f-\theta_m)}{\mathrm{Nu}^*+2Bi} + \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \tag{32}$$

But we have a uniform reference temperature, θ_f, so we can write

$$\frac{d(\theta_f-\theta_m)}{dX} = -\frac{8\mathrm{Nu}^* Bi(\theta_f-\theta_m)}{\mathrm{Nu}^*+2Bi} - \frac{8}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \tag{33}$$

For a fully developed temperature field $\mathrm{Nu}^* = \mathrm{Nu}_\infty^*$, and by integration of the Eq. (33) we obtain

$$\theta_f - \theta_m(X) = \mathrm{C}_2 exp\left(-\frac{8\mathrm{Nu}_\infty^* Bi}{\mathrm{Nu}_\infty^*+2Bi}X\right) - \frac{1}{(1-a)^{n+1}}\frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\left(\frac{m+1}{\omega}\right)^n \tag{34}$$

Where C_2 is an integration constant, determined by replacing $X = L_{fd}$ in Eq. (34) such as

$$\mathrm{C}_2 = \theta_f - \theta_m\left(L_{fd}\right) + \frac{1}{(1-a)^{n+1}}\frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\left(\frac{m+1}{\omega}\right)^n \tag{35}$$

with L_{fd} is the duct length from the point of duct heating $(X = 0)$ at which the temperature profile first becomes fully developed (see Shah and London 1978).

Inserting the Eq. (35) into the Eq. (34) makes it possible to write

$$\theta_f - \theta_m(X) = \left[\theta_f - \theta_m\left(L_{fd}\right) + \frac{1}{(1-a)^{n+1}}\frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\left(\frac{m+1}{\omega}\right)^n\right] exp\left(-\frac{8\mathrm{Nu}_\infty^* Bi}{\mathrm{Nu}_\infty^*+2Bi}X\right) - \frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n \tag{36}$$

Substituting Eq. (36) into Eq. (30), q_w is expressed in the fully developed region by

$$q_w = \frac{Ku_m^{n+1}}{r_0^n}\left\{\frac{\mathrm{Nu}_\infty^*}{\mathrm{Nu}_\infty^*+2Bi}Bi\left[\theta_f - \theta_m\left(L_{fd}\right) + \frac{1}{(1-a)^{n+1}}\frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\left(\frac{m+1}{\omega}\right)^n\right] exp\left(-\frac{8\mathrm{Nu}_\infty^* Bi}{\mathrm{Nu}_\infty^*+2Bi}X\right) - \frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right\} \tag{37}$$

By employing Eqs. (37) and (12), one obtains

$$Br(X) = \frac{Ku_m^{n+1}}{(2r_0)^n q_w(X)} = \frac{1}{2^n\left\{\frac{\mathrm{Nu}_\infty^*}{\mathrm{Nu}_\infty^*+2Bi}Bi\left[\theta_f-\theta_m\left(L_{fd}\right)+\frac{1}{(1-a)^{n+1}}\frac{\mathrm{Nu}_\infty^*+2Bi}{\mathrm{Nu}_\infty^* Bi}\left(\frac{m+1}{\omega}\right)^n\right] exp\left(-\frac{8\mathrm{Nu}_\infty^* Bi}{\mathrm{Nu}_\infty^*+2Bi}X\right)-\frac{1}{(1-a)^{n+1}}\left(\frac{m+1}{\omega}\right)^n\right\}} \tag{38}$$

For large values of X, it is easy to check that $Br(X)$ given by Eq. (38) is reduced to that given by Eq. (23) as in the case $T_w = constant$. Consequently, $\Theta_\infty^*(R)$, $\Theta_\infty(R)$, Nu_∞^* and Nu_∞ will be expressed respectively by Eqs. (24), (25) and (26) and do not depend on the value of Bi. This can be interpreted as convective boundary condition with Bi tends to infinity. By taking into account of Eq. (12), the Eq. (38) ensures that, when $Bi \to \infty$, the wall heat flux is a finite value. Therefore, on account of Eq. (27), T_w must tend to T_f.

3.3 Results and Discussion

In the Newtonian fluid case ($a = 0$ and $n = 1$), with uniform wall temperature or convection with an external isothermal fluid and if viscous dissipation is taken into account, one finds $\mathrm{Nu}_\infty = 48/5 = 9.6$ for every nonzero value of the Brinkman number. This value coincides with the result reported in the literature (Lin et *al.* 1983). Let us recall that in the usual case of the Newtonian flow in a pipe with a circular cross-section, the Nusselt number $\mathrm{Nu}_\infty = 3.665$ when both the axial conduction and the viscous dissipation are neglected and that this number varies slightly when the axial conduction is considered, while the dissipation viscous is neglected.

For the case of power-law fluid ($a = 0$) and while axial conduction is neglected, Table 1 represents the values of Nu_∞ for different values of $n = 1/3, 1/2$ (pseudoplastic fluids), $n = 1$ (Newtonian fluids) and $n = 3/2, 3$ (dilatant fluids). This table shows that the obtained results are in agreement with those of Barletta (1997) and Jambal et *al.* (2005). Note that the axial heat conduction is neglected in the results of Barletta (1997) but taken into account for those of Jambal et *al.* (2005). Our results consider both the viscous dissipation and the axial heat conduction.

These results show that the asymptotic Nusselt number is nearly independent of the Peclet number Pe. The inference is that in the case of a fluid with variable viscosity, and for the considered boundary conditions of uniform wall temperature or convection with an external isothermal fluid, the infinite Nusselt number Nu_∞ is nearly independent of the axial conduction effect, which remains low in the thermally established zone, and is a decreasing function of the index of the fluid. The viscous dissipation acts in a similar way to that noted in the Newtonian case, i.e. the value of the infinite Nusselt number Nu_∞ is more than doubled when the brinkman number is non-zero and the

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effect of axial conduction on Nu_∞ remains weak, and moreover this value of Nu_∞ decreases as a function of the index of the fluid.

Table 1 Values of Nu_∞ for different values of n compared with those of Barletta (1997) and Jambal et *al.* (2005) for power-law fluids ($a = 0$)

n	Barletta (1997)	Jambal et *al.* (2005)	Present study
1/3	13.714228	-----	13.71428
1/2	-----	11.67	11.6667
1	9.6	9.6	9.6
3/2	------	8.905	8.9032
3	8.205129	------	8.205129

For Herschel-Bulkley fluid, Eq. (23) indicates that $Br_\infty = -\frac{(1-a)^{n+1}}{2^n}\left(\frac{\omega}{m+1}\right)^n$ is a function of the structure index n and of the ratio a of yield shear stress to wall shear stress.

The Eq. (26) shows that the asymptotic bulk Nusselt number Nu_∞ and the asymptotic mixing Nusselt number Nu_∞^* become equal when $X \to +\infty$, thus highlighting the vanishing impact of the axial heat conduction at infinity on average temperature and bulk Nusselt number (see Barletta and Zanchini 1995) as underlined in the case of the power law fluid.

Note that even if, as shown by equations (25) and (26), there is no explicit dependence of the asymptotic Nusselt number Nu_∞ and the asymptotic temperature field $\Theta_\infty(R)$ on the brinkman number Br_∞, there is an implicit dependence through the variations of the core radius a and the power-law exponent n. The effect of viscous dissipation remains important at infinity and varies if n and a change.

The results of the analytical values of Nu_∞ are represented in Tables 2 for $0 < n < 1$ and in Table 3 for $n \geq 1$. We also notice that for Herschel-Bulkley fluid when power-law exponent n decreases the asymptotic Nusselt number Nu_∞ increases (see Tables 2 and 3). This improvement is obviously linked to an increase in wall shear. Furthermore, Fig. 1 shows that the asymptotic Nusselt number Nu_∞ is an increasing function of a while it decreases as a function of n.

Table 2 Values of Nu_∞ for different values of n ($0 < n < 1$)

n	$a = 0$	$a = 0.2$	$a = 0.4$	$a = 0.6$	$a = 0.8$
0.2	17.7778	21.0546	26.6709	38.1301	68.7401
0.25	15.750	18.4896	23.2376	32.9991	62.1518
0.3	14.3267	16.6819	20.8145	29.3821	54.7527
0.4	12.6923	14.7225	18.0178	25.2142	46.8930
0.5	11.6667	13.2733	16.2323	22.5597	42.2609
0.6	10.9804	12.3829	15.0305	20.7769	38.3866
0.7	10.4887	11.7404	14.1616	19.4907	35.9094
0.8	10.1191	11.2542	13.5027	18.5174	34.0372
0.9	9.8309	10.8730	12.9852	17.7544	32.5716

Table 3 Values of Nu_∞ for different values of n ($n \geq 1$)

n	$a = 0$	$a = 0.2$	$a = 0.4$	$a = 0.6$	$a = 0.8$
1	9.6	10.56573	12.56744	17.14062	31.7893
1.25	9.1828	10.0061	11.8049	16.0206	29.2128
1.3	9.1177	9.9181	11.6794	15.8367	28.9097
1.35	9.0575	9.8365	11.5734	15.6877	28.5982
1.4	9.0041	9.7641	11.4745	15.5371	28.3215
1.45	8.9519	9.6931	11.3752	15.3954	28.0512
1.5	8.9032	9.6298	11.2910	15.2629	27.8098
3	8.205129	8.659926	9.960690	13.33702	24.10196

The asymptotic behavior of the temperature profile $\Theta_\infty(R)$ is described in Fig. 2(a)-(c) and Fig. 3(a)-(c), respectively for $n = 1/3, 1, 3$ and for $a = 0, 0.4, 0.8$. These figures show that the gradient of $\Theta_\infty(R)$ increases in the vicinity of the wall ($R = 1$), for n fixed, when the core radius a increases (see Fig. 2(a)-(c)) and for a fixed, when the power-law exponent n decreases.

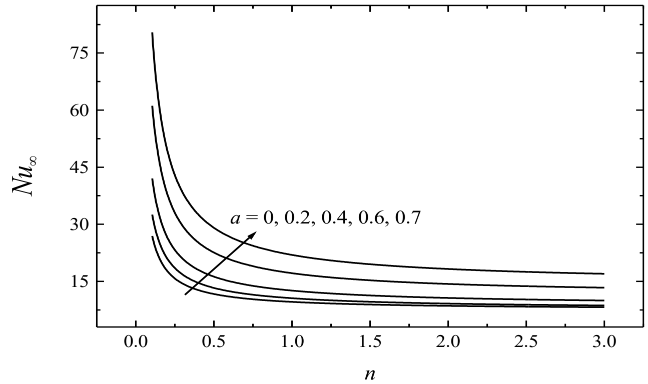
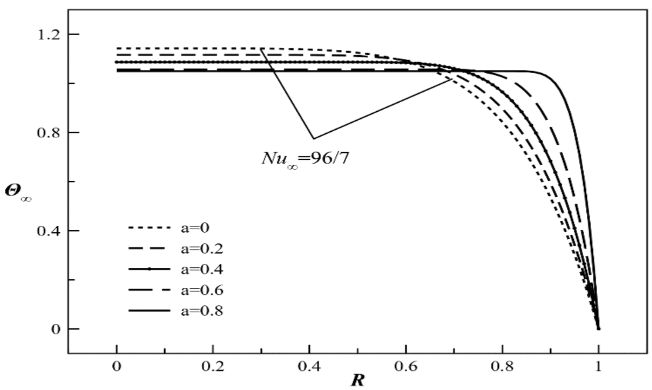
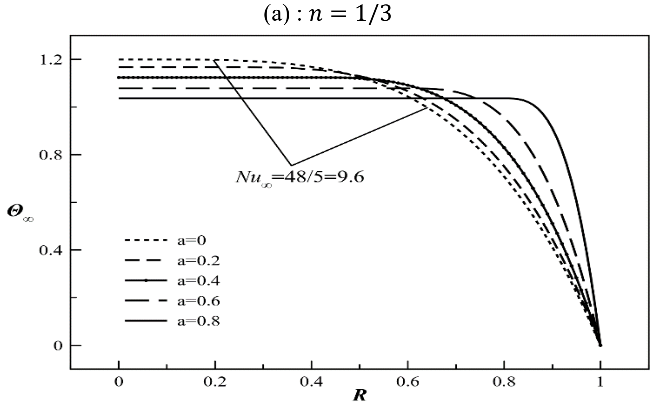


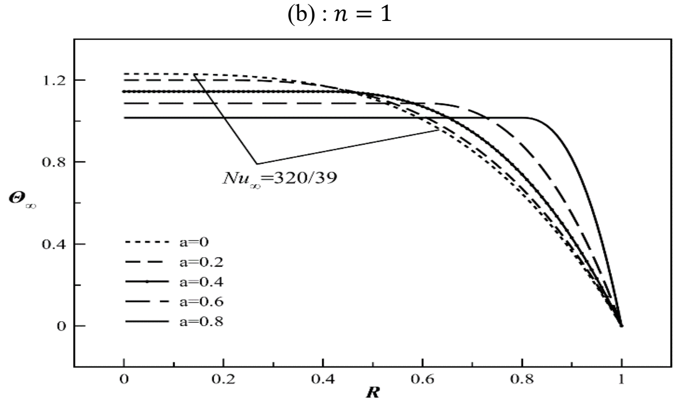
Fig. 1 Evolution of the Nu_∞ versus n for various values of a.



(a) : $n = 1/3$



(b) : $n = 1$



(c) : $n = 3$

Fig. 2 Evolution the Θ_∞ versus R for various values of a :
(a) : $n = 1/3$, (b) : $n = 1$, , (c) : $n = 3$

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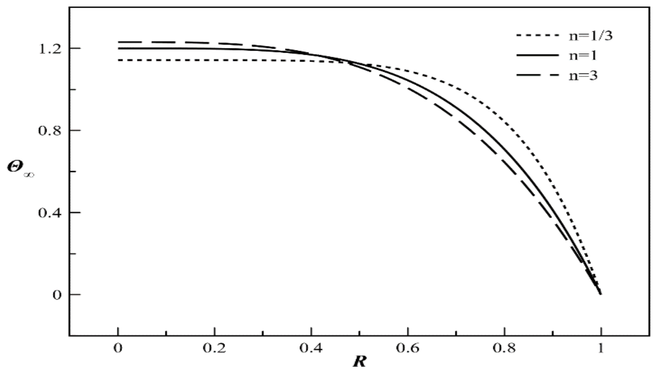
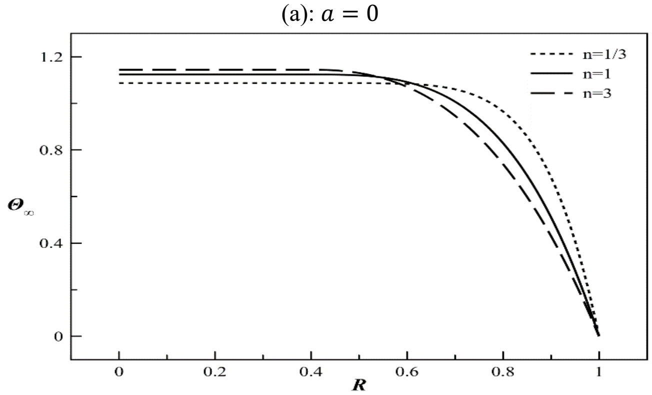
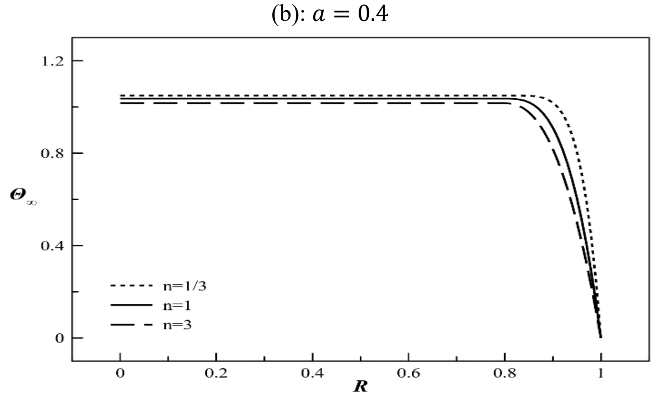


(a): $a = 0$



(b): $a = 0.4$



(c): $a = 0.8$

Fig. 3 Evolution of the Θ_∞ versus R for various values of n : (a) : $a = 0$, (b) : $a = 0.4$, (c) : $a = 0.8$

4. CONCLUSION

Forced laminar convection for a Herschel-Bulkley fluid in a circular tube with a fully developed velocity profile is studied analytically by including both the viscous dissipation and the axial heat conduction with two types of boundary conditions, namely : constant wall temperature and convection with an external isothermal fluid.

The results show that the asymptotic Nusselt number Nu_∞ is almost independent of the Peclet number Pe and is an increasing function of core radius a while it decreases as a function of the index n of the fluid. It can be deduced that the effect of axial conduction is negligible in the thermally fully developed region. The effect of viscous dissipation remains important at infinity and varies if n and a change. In particular, it was shown that the asymptotic value of the Nusselt number in the case of convective boundary conditions is independent of the Biot number. The comparison between our theoretical results and those of the literature in the Newtonian fluid case and non-Newtonian fluid case power law fluid shows good agreement.

NOMENCLATURE

a	Ratio of yield shear stress to wall shear stress
Bi	Biot number, $h_e r_0/\lambda$
$Br(X)$	Local Brinkman number, $\frac{K u_m^{n+1}}{(2r_0)^2 q_w(X)}$
c_p	Specific heat at constant pressure
f	Function of R employed in Appendix
h_e	Convection coefficient with a fluid external to the tube wall
K	Consistency index
m	Inverse of power-law exponent, $1/n$
n	Power-law exponent
Nu	Bulk Nusselt number, $2r_0 q_w/[\lambda(T_w - T_b)]$
Nu*	Mixing Nusselt number, $2r_0 q_w/[\lambda(T_w - T_m)]$
Pe	Peclet number, $2r_0 u_m \rho c_p/\lambda$
q_w	Wall heat flux
r	Radial coordinate
r_c	Yield radius
r_0	Radius of the tube
R	Dimensionless radial coordinate, r/r_0
T	Temperature
T_0	Inlet temperature distribution
T_f	Reference temperature of a fluid external to the tube wall
u	Velocity component in the axial direction
u_m	Mean axial velocity
U	Dimensionless axial velocity, u/u_m
x	Axial coordinate
X	Dimensionless axial coordinate, $x/(2r_0 Pe)$

Greeks Symbols

λ	Thermal conductivity of fluid
ω	Dimensionless parameter, $\omega = 1 - 2\left[\frac{a(1-a)}{m+2} + \frac{(1-a)^2}{m+3}\right]$
ρ	Fluid density
τ_c	Yield shear stress
τ_w	Wall shear stress
θ	Dimensionless temperature, $\lambda r_0^{n-1}(T - T_{0b})/K u_m^{n+1}$
θ_f	Dimensionless temperature, $\lambda r_0^{n-1}(T_f - T_{0b})/K u_m^{n+1}$
Θ	Dimensionless bulk temperature, $(T_w - T)/(T_w - T_b)$
Θ^*	Dimensionless mixing temperature, $(T_w - T)/(T_w - T_m)$

Subscripts

b	Bulk quantity
m	Mixing quantity
w	Wall condition
∞	Quantity evaluated for $X \to +\infty$
fd	Fully developped

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APPENDIX

Where

If $\boldsymbol{m} \in \mathbb{N}^*$:

$$f(R) = \frac{2^{3-n}}{4\omega} \frac{a^{m+3}}{(1-a)^{m+1}(m+2)(m+3)} \left[\sum_{k=0}^{m+2} \frac{(m+3)!(-1)^k a^k}{k!(m+3-k)!(m+3-k)} + (-1)^{m+3}\ln(a)\right] + C_3 \quad \text{if} \quad 0 \le R \le a$$

and

$$f(R) = \frac{1}{2^n\omega}\left[\frac{a(1-a)}{m+2}\left(\frac{R-a}{1-a}\right)^{m+2} + \frac{(1-a)^2}{m+3}\left(\frac{m}{m+2}\left(\frac{R-a}{1-a}\right)^{m+3} + \frac{2}{(m+2)(1-a)^{m+3}}\left[\sum_{k=0}^{m+2}\left(\frac{(m+3)!(-1)^k a^k}{k!(m+3-k)!}\frac{R^{m+3-k}}{m+3-k}\right) + (-1)^{m+3}a^{m+3}\ln(R)\right]\right)\right] + C_3 \quad \text{if} \quad a \le R \le 1$$

with

$$C_3 = \frac{1}{2^n\omega^2(m+2)(m+3)}\left\{2(m+1)(1-a)^2\left[-\frac{a^2}{2(m+2)} + \frac{(1-a)(am+2a-m)}{(m+3)(m+5)} + \frac{a(5a+2am-2m-3)}{(m+4)(m+5)}\right] - \frac{4}{(1-a)^{m+1}}\left[(-1)^{m+3}a^{m+3}\left(\frac{a^2-1}{4} + \frac{(1-a)^2}{(m+2)(m+3)}\left[1 + \frac{(-1)^{m+3}\ln(a)}{(1-a)^{m+3}}\right]\right) + \sum_{k=0}^{m+2}\frac{(m+3)!(-1)^k a^k}{k!(m+3-k)!(m+3-k)}\left(\frac{2+(m+3-k)a^{m+5-k}}{2(m+5-k)} - \frac{1}{(1-a)^{m+1}}\left[\frac{(-1)^{m+3}a^{m+3}}{(m+2)(m+3)} + \sum_{p=0}^{m+1}\frac{(m+1)!(-1)^p a^p}{p!(m+1-p)!}\frac{1-a^{2m+6-k-p}}{2m+6-k-p}\right]\right)\right]\right\}$$

$$f(1) = -\frac{1}{2^{n+1}}\left[1 - \frac{1}{\omega} + \frac{(1-a)^2}{\omega(m+2)(m+3)}\left(1 - \frac{1}{(1-a)^{m+3}}\sum_{k=0}^{m+2}\frac{(m+3)!(-1)^k a^k}{k!(m+3-k)!(m+3-k)}\right)\right] + C_3$$

but

If $\boldsymbol{m} \in \mathbb{Q}^*$:

$$f(R) = \frac{2^{3-n}}{4\omega}\frac{a^{m+3}}{(1-a)^{m+1}(m+2)(m+3)}\sum_{k=0}^{\infty}\frac{P(m+3-k)(-1)^k}{k!(m+3-k)^2} + C_3 \quad \text{if} \quad 0 \le R \le a$$

and

$$f(R) = \frac{1}{2^n\omega}\left[\frac{a(1-a)}{m+2}\left(\frac{R-a}{1-a}\right)^{m+2} + \frac{(1-a)^2}{(m+2)(m+3)}\left(m\left(\frac{R-a}{1-a}\right)^{m+3} + \frac{2}{(1-a)^{m+3}}\sum_{k=0}^{\infty}\frac{P(m+3-k)(-1)^k a^k}{k!(m+3-k)^2}R^{m+3-k}\right)\right] + C_4 \quad \text{if} \quad a \le R \le 1$$

with

$$C_3 = \frac{1}{2^n\omega^2(m+2)(m+3)}\left\{2(m+1)(1-a)^2\left[-\frac{a^2}{2(m+2)} + \frac{(1-a)(am+2a-m)}{(m+3)(m+5)} + \frac{a(5a+2am-2m-3)}{(m+4)(m+5)}\right] - \frac{4}{(1-a)^{m+1}}\sum_{k=0}^{\infty}\left[\frac{P(m+3-k)(-1)^k a^k}{k!(m+3-k)^2}\left(\frac{2+(m+3-k)a^{m+5-k}}{2(m+5-k)} - \frac{1}{(1-a)^{m+1}}\sum_{p=0}^{\infty}\frac{P(m+1-p)(-1)^p a^p}{p!(m+1-p)!}\frac{1-a^{2m+6-k-p}}{2m+6-k-p}\right)\right]\right\}$$

$$f(1) = -\frac{1}{2^{n+1}}\left[1 - \frac{1}{\omega} + \frac{(1-a)^2}{\omega(m+2)(m+3)}\left(1 - \frac{1}{(1-a)^{m+3}}\sum_{k=0}^{\infty}\frac{P(m+3-k)(-1)^k a^k}{k!(m+3-k)^2}\right)\right] + C_3$$

$$\forall m, \quad \begin{array}{llll} if & k = 0 & then & P(m) = m \\ if & k \ge 1 & then & P(m-k) = (m-k)P\big(m-(k-1)\big) \end{array}$$