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ARTICLE

Partial Bell Polynomials, Falling and Rising Factorials, Stirling Numbers, and Combinatorial Identities

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ABSTRACT

In the paper, the authors collect, discuss, and find out several connections, equivalences, closed-form formulas, and combinatorial identities concerning partial Bell polynomials, falling factorials, rising factorials, extended binomial coefficients, and the Stirling numbers of the first and second kinds. These results are new, interesting, important, useful, and applicable in combinatorial number theory.

KEYWORDS

Connection; equivalence; closed-form formula; combinatorial identity; partial Bell polynomial; falling factorial; rising factorial; binomial coefficient; Stirling number of the first kind; Stirling number of the second kind; problem

1 Preliminaries

In this paper, we use the notation

$$\mathbb{N}=\{1,2,\dots\},\qquad \mathbb{N}_-=\{-1,-2,\dots\},\quad \mathbb{N}_0=\{0,1,2,\dots\},$$

$$\mathbb{Z}=\{0,\pm1,\pm2,\dots\},\qquad \mathbb{R}=(-\infty,\infty),\qquad \mathbb{C}=\left\{x+\mathrm{i}\,y:x,y\in\mathbb{R},\mathrm{i}=\sqrt{-1}\right\}.$$

The partial Bell polynomials, also known as the Bell polynomials of the second kind, in combinatorics can be denoted and defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1,\\ \ell_i\in\{0\}\cup\mathbb{N},\\ \sum_{i=1}^{n-k+1}i\ell_i=n,\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i} \tag{1}$$

for $n \ge k \in \mathbb{N}_0$. See Theorem A on page 134 in [1]. The partial Bell polynomials satisfy the identity

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \,\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2}$$

for $n \ge k \in \mathbb{N}_0$. See page 135 in [1].

The double factorial of negative odd integers $-(2k+1)$ is defined by

$$(-2k-1)!! = \frac{(-1)^k}{(2k-1)!!} = (-1)^k \frac{2^k k!}{(2k)!}, \quad k \in \mathbb{N}_0.$$

The falling factorial $\langle z \rangle_n$ and the rising factorial $(z)_n$ for $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$ can be defined by

$$\langle z \rangle_n = \prod_{k=0}^{n-1}(z-k) = \begin{cases} z(z-1)\cdots(z-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} \tag{3}$$

and

$$(z)_n = \prod_{\ell=0}^{n-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

respectively. It is easy to verify that

$$(-z)_n = (-1)^n \langle z \rangle_n \tag{4}$$

and

$$\langle -z \rangle_n = (-1)^n (z)_n. \tag{5}$$

See page 167 in [2] and related texts in the paper [3].

The Stirling numbers of the first kind $s(n,k)$ for $n \ge k \in \mathbb{N}_0$ can be analytically generated (see page 51 in [1]) by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1$$

and can be explicitly computed (see Corollary 2.3 in [4]) by

$$|s(n+1,k+1)| = n! \sum_{\ell_1=k}^{n} \frac{1}{\ell_1} \sum_{\ell_2=k-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-1}=2}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}} \sum_{\ell_k=1}^{\ell_{k-1}-1} \frac{1}{\ell_k}$$

for $n \ge k \in \mathbb{N}$. The Stirling numbers of the second kind $S(n,k)$ for $n \ge k \in \mathbb{N}_0$ can be analytically generated (see page 51 in [1]) by

$$\frac{(\mathrm{e}^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}$$

and can be explicitly computed (see Theorem A on page 204 in [1]) by

$$S(n,k) = \begin{cases} \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^n, & n > k \in \mathbb{N}_0; \\ 1, & n = k \in \mathbb{N}_0. \end{cases} \tag{6}$$

For more information on the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$, please refer to the papers [5,6] and the monographs [7,8].

The extended binomial coefficient $\binom{z}{w}$ for $z,w\in\mathbb{C}$ is defined in [9] by

$$\binom{z}{w}=\begin{cases}\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z\notin\mathbb{N}_-,\quad w,z-w\notin\mathbb{N}_-\\ 0, & z\notin\mathbb{N}_-,\quad w\in\mathbb{N}_- \text{ or } z-w\in\mathbb{N}_-\\ \frac{\langle z\rangle_w}{w!}, & z\in\mathbb{N}_-,\quad w\in\mathbb{N}_0\\ \frac{\langle z\rangle_{z-w}}{(z-w)!}, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_0\\ 0, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_-\\ \infty, & z\in\mathbb{N}_-,\quad w\notin\mathbb{Z}\end{cases}$$

in terms of the falling factorial $\langle z\rangle_w$, which is defined by (3), and the classical Euler's gamma function $\Gamma(z)$, which can be defined (see Chapter 3 in [10]) by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!\,n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

On page 206 in [1] and on page 165 in [7], there are two relations

$$\langle z\rangle_n=\sum_{\ell=0}^{n}s(n,\ell)z^\ell,\quad z\in\mathbb{C},\quad n\in\mathbb{N}_0 \tag{7}$$

and

$$n!\binom{z}{n}=\sum_{\ell=0}^{n}s(n,\ell)z^\ell,\quad n\in\mathbb{N}_0,\quad z\in\mathbb{C}. \tag{8}$$

The falling factorial $\langle z\rangle_n$ and the rising factorial $(z)_n$ can be represented by

$$\langle z\rangle_n=n!\binom{z}{n}=\frac{\Gamma(z+1)}{\Gamma(z-n+1)}\quad\text{and}\quad (z)_n=(-1)^n n!\binom{-z}{n}=\frac{\Gamma(z+n)}{\Gamma(z)}$$

for $z\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

In this paper, we will collect, discuss, and find out several connections, equivalences, closed-form formulas, and combinatorial identities concerning partial Bell polynomials $\mathrm{B}_{n,k}$, falling factorials $\langle z\rangle_n$, rising factorials $(z)_n$, extended binomial coefficients $\binom{z}{w}$, and the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$.

2 Equivalences

Among the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$, the falling factorial $\langle\alpha\ell\rangle_n$, and extended binomial coefficient $\binom{\alpha\ell}{n}$, there are the following beautiful equivalences.

Theorem 2.1. For $n\ge k\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}$, we have

$$\sum_{\ell=k}^{n}s(n,\ell)\alpha^\ell S(\ell,k)=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n \tag{9}$$

and

$$\sum_{\ell=k}^{n} s(n,\ell)\alpha^{\ell} S(\ell,k) = (-1)^k \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{\alpha\ell}{n}. \tag{10}$$

Proof. In Remark 3.1 of [11], the formula

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \tag{11}$$

for $n \ge k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ was concluded. An equivalence of the formula (11) is

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \langle\alpha\ell\rangle_n \tag{12}$$

for $n \ge k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, which was proved in Theorems 2.1 and 4.1 of [2].

The formulas (11) and (12) can be rewritten respectively as

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \begin{cases} (-1)^k \frac{\lambda^n n!}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{\ell/\lambda}{n}, & \lambda \ne 0 \\ S(n,k), & \lambda = 0 \end{cases}$$

and

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = (-1)^k \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{\alpha\ell}{n} \tag{13}$$

for $n \ge k \in \mathbb{N}_0$ and $\alpha, \lambda \in \mathbb{C}$, as done in Remark 7 of [12].

Considering the formulas (6), (7), and (8) in the formulas (12) or (13), we can derive

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \sum_{j=k}^{n} s(n,j)\alpha^j S(j,k) \tag{14}$$

for $n \ge k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$.

Combining (12), (13), and (14) results in

$$\begin{aligned} \mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) &= \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \langle\alpha\ell\rangle_n \\ &= (-1)^k \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{\alpha\ell}{n} \qquad (15) \\ &= \sum_{\ell=k}^{n} s(n,\ell)\alpha^{\ell} S(\ell,k) \end{aligned}$$

for $n \geq k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. The equivalences in (9) and (10) are thus proved. The proof of Theorem 2.1 is complete.

3 Simpler Closed-Form Formulas

When taking $\alpha = \pm 1, \pm 2, \frac{1}{2}$ in (9) and (10) respectively, we can derive several simpler closed-form combinatorial identities.

Theorem 3.1. For $n \geq k \in \mathbb{N}_0$, we have

$$\sum_{j=k}^{n} s(n,j) 1^j S(j,k) = \binom{0}{n-k}, \tag{16}$$

$$\sum_{j=k}^{n} s(n,j) 2^j S(j,k) = \frac{n!}{k!} \binom{k}{n-k} 2^{2k-n}, \tag{17}$$

$$\sum_{j=k}^{n} s(n,j) \left(\frac{1}{2}\right)^j S(j,k) = (-1)^{n+k} \frac{[2(n-k)-1]!!}{2^n} \binom{2n-k-1}{2(n-k)}, \tag{18}$$

$$\sum_{j=k}^{n} s(n,j)(-1)^j S(j,k) = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}, \tag{19}$$

and

$$\sum_{j=k}^{n} s(n,j)(-2)^j S(j,k) = (-1)^{n+k} \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{n+2\ell-1}{n}. \tag{20}$$

Proof. By the definition (1), we can easily deduce that, for $n \geq k \in \mathbb{N}_0$,

$$\mathrm{B}_{n,k}(1,0,\dots,0) = \mathrm{B}_{n,k}(\langle 1\rangle_1, \langle 1\rangle_2, \dots, \langle 1\rangle_{n-k+1}) = \sum_{j=k}^{n} s(n,j) 1^j S(j,k) = \binom{0}{n-k}, \tag{21}$$

where we used the relation (15). See also pages 167–168 in [2]. The identity (16), which recovers the first one in (28) below, is thus proved.

In Theorem 5.1 of [13] and in Section 3 of [14], the formula

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n} \tag{22}$$

was established for $n \geq k \in \mathbb{N}_0$, where we assumed $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \in \mathbb{N}_0$. Making use of the identity (2), we can derive from the formula (22) that

$$\mathrm{B}_{n,k}(2,2,0,\dots,0) = \mathrm{B}_{n,k}(\langle 2\rangle_1, \langle 2\rangle_2, \dots, \langle 2\rangle_{n-k+1}) = \sum_{j=k}^{n} s(n,j) 2^j S(j,k) = \frac{n!}{k!} \binom{k}{n-k} 2^{2k-n} \tag{23}$$

for $n \geq k \in \mathbb{N}_0$. The identity (17) is thus verified.

In the proof of Theorem 3.2 in [2], it was obtained that

$$\begin{aligned} \mathrm{B}_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right) &= \sum_{j=k}^{n} s(n,j)\left(\frac{1}{2}\right)^j S(j,k) \\ &= (-1)^{n+k}\frac{[2(n-k)-1]!!}{2^n}\binom{2n-k-1}{2(n-k)} \end{aligned} \tag{24}$$

for $n \ge k \in \mathbb{N}_0$. The identity (18) is thus proved.

Replacing α by $-\alpha$ in (14) and utilizing (5) give

$$\mathrm{B}_{n,k}((\alpha)_1,(\alpha)_2,\dots,(\alpha)_{n-k+1}) = (-1)^n\sum_{j=k}^{n} s(n,j)(-\alpha)^j S(j,k) \tag{25}$$

for $n \ge k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Taking $\alpha = -\frac{1}{2}$ in the formula (25) and making use of the identity

$$\mathrm{B}_{n,k}((-1)!!,1!!,3!!,\dots,[2(n-k)-1]!!) = [2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}$$

in Section 1.5 of [15] and in Theorem 1.2 of [16], we derive

$$\begin{aligned} \mathrm{B}_{n,k}\left(\left(-\frac{1}{2}\right)_1,\left(-\frac{1}{2}\right)_2,\dots,\left(-\frac{1}{2}\right)_{n-k+1}\right) &= \frac{(-1)^k}{2^n}\mathrm{B}_{n,k}((-1)!!,1!!,3!!,\dots,[2(n-k)-1]!!) \\ &= (-1)^n\sum_{j=k}^{n} s(n,j)\left(\frac{1}{2}\right)^j S(j,k) \\ &= \frac{(-1)^k}{2^n}[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)} \end{aligned}$$

for $n \ge k \in \mathbb{N}_0$. The identity (18) is proved again.

Employing the relation (25) and using the identities

$$\mathrm{B}_{n,k}(1!,2!,3!,\dots,(n-k+1)!) = \frac{n!}{k!}\binom{n-1}{k-1}$$

and

$$\mathrm{B}_{n,k}(2!,3!,\dots,(n-k+2)!) = \frac{n!}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\binom{n+2\ell-1}{n}$$

in Sections 1.3 and 1.9 of [15] and in Lemma 6 of [17], we acquire

$$\begin{aligned}
\mathrm{B}_{n,k}((1)_1,(1)_2,\dots,(1)_{n-k+1}) &= \mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!) \\
&= (-1)^n \sum_{j=k}^{n} s(n,j)(-1)^j S(j,k) \\
&= \frac{n!}{k!}\binom{n-1}{k-1}
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\mathrm{B}_{n,k}((2)_1,(2)_2,\dots,(2)_{n-k+1}) &= \mathrm{B}_{n,k}(2!,3!,\dots,(n-k)!) \\
&= (-1)^n \sum_{j=k}^{n} s(n,j)(-2)^j S(j,k) \\
&= \frac{n!}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\binom{n+2\ell-1}{n}.
\end{aligned} \tag{27}$$

The identities (19) and (20) are thus derived. The proof of Theorem 3.1 is complete.

Remark 3.1. We can regard those identities from (17) to (20) in Theorem 3.1 as generalizations of the orthogonality relations

$$\sum_{j=0}^{n} s(n,j)S(j,k) = \sum_{j=0}^{n} S(n,j)s(j,k) = \binom{0}{n-k}, \quad n \ge k \in \mathbb{N}_0 \tag{28}$$

listed on page 171 in [7].

Theorem 3.2. For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k!\,\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1}) &= \sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\alpha\ell\rangle_n \\
&= n!\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\alpha\ell}{n} \\
&= \sum_{k=0}^{n}(-1)^k k!\sum_{j=k}^{n} s(n,j)\alpha^j S(j,k) \\
&= \langle-\alpha\rangle_n
\end{aligned} \tag{29}$$

and

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k!\,\mathrm{B}_{n,k}((\alpha)_1,(\alpha)_2,\dots,(\alpha)_{n-k+1})&=\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}(\alpha\ell)_n\\&=(-1)^n n!\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{-\alpha\ell}{n}\\&=(-1)^n\sum_{k=0}^{n}(-1)^k k!\sum_{j=k}^{n}s(n,j)(-\alpha)^j S(j,k)\\&=(-\alpha)_n.\end{aligned}\tag{30}$$

Proof. Combining (12) and (14) yields

$$(-1)^k k!\sum_{j=k}^{n}s(n,j)\alpha^j S(j,k)=\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n,\quad n\ge k\in\mathbb{N}_0.$$

Accordingly, similar to arguments in Lemma 2.2 of [18], we acquire

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k!\sum_{j=k}^{n}s(n,j)\alpha^j S(j,k)&=\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n\\&=\sum_{\ell=1}^{n}(-1)^\ell\left[\sum_{k=\ell}^{n}\binom{k}{\ell}\right]\langle\alpha\ell\rangle_n\\&=\sum_{\ell=1}^{n}(-1)^\ell\binom{n+1}{\ell+1}\sum_{m=0}^{n}s(n,m)(\alpha\ell)^m\\&=\sum_{m=0}^{n}s(n,m)\alpha^m\sum_{\ell=1}^{n}(-1)^\ell\binom{n+1}{\ell+1}\ell^m\\&=\sum_{m=0}^{n}s(n,m)(-\alpha)^m\\&=\langle-\alpha\rangle_n\end{aligned}$$

for $n\in\mathbb{N}$, where we used the relation

$$\sum_{k=\ell}^{n}\binom{k}{\ell}=\binom{n+1}{\ell+1},\quad \ell,n\in\mathbb{N},\tag{31}$$

which is a special case $x=0$ and $r=\ell$ of the identity

$$\sum_{k=0}^{n}\binom{k+x}{r}=\binom{n+x+1}{r+1}-\binom{x}{r+1}$$

in the formula (1.48) on pages 27–28 of [8], we used the relation (7) twice, and we used the equality

$$\sum_{\ell=1}^{n}(-1)^{\ell}\binom{n+1}{\ell+1}\ell^{m}=\sum_{\ell=1}^{m}(-1)^{\ell}\binom{n+1}{\ell+1}\ell^{m}=(-1)^{m}, \quad m,n\in\mathbb{N}, \tag{32}$$

which is a special case $r=1$ and $p=m$ of the identity

$$\sum_{k=0}^{rp-r+1}(-1)^{k}\binom{rp+1}{k}\binom{rp-k}{r}^{p}=1$$

in the formula (X.5) on page 132 of [8]. Further applying relations in (15), we conclude those relations in (29).

Replacing α by $-\alpha$ in (29), using the identities (4), (5), and (2) in sequence, and simplifying lead to (30). The proof of Theorem 3.2 is thus complete.

Remark 3.2. The last equality in (29) can be rewritten as

$$\sum_{k=0}^{n}(-1)^{k}k!\sum_{j=k}^{n}s(n,j)\alpha^{j}S(j,k)=\sum_{j=0}^{n}s(n,j)\alpha^{j}\sum_{k=0}^{j}S(j,k)[(-1)^{k}k!]=\langle-\alpha\rangle_{n}. \tag{33}$$

Theorem 12.1 on page 171 of [7] reads that, if b_α and a_k are a collection of constants independent of n, then

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n=\sum_{k=0}^{n}s(n,k)a_k.$$

Applying Theorem 12.1 on page 171 in [7] to the second equality in (33), we find

$$\sum_{k=0}^{n}S(n,k)\langle-\alpha\rangle_{k}=\alpha^{n}\sum_{j=0}^{n}S(n,j)[(-1)^{j}j!].$$

Considering the explicit formula (6) and utilizing (31) and (32), we arrive at

$$\sum_{k=0}^{n}S(n,k)[(-1)^{k}k!]=\sum_{k=1}^{n}\sum_{\ell=1}^{k}(-1)^{\ell}\binom{k}{\ell}\ell^{n}=\sum_{\ell=1}^{n}(-1)^{\ell}\ell^{n}\sum_{k=\ell}^{n}\binom{k}{\ell}=\sum_{\ell=1}^{n}(-1)^{\ell}\ell^{n}\binom{n+1}{\ell+1}=(-1)^{n}$$

for $n\in\mathbb{N}$. Therefore, we obtain

$$\sum_{k=0}^{n}S(n,k)\langle-\alpha\rangle_{k}=(-\alpha)^{n}, \quad n\in\mathbb{N}_0, \quad \alpha\in\mathbb{C},$$

which is a recovery of the well-known relation

$$\sum_{k=0}^{n} S(n,k)\langle\alpha\rangle_k = \alpha^n, \quad n\in\mathbb{N}_0, \quad \alpha\in\mathbb{C}$$

in the equation (1.27) on page 19 of [10].

4 Several Combinatorial Identities

In items (3.163) and (3.164) on pages 91–92 of [8], we find two identities

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{\ell} = \frac{n}{\ell}\binom{n-\ell-1}{\ell-1}2^{n-2\ell}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k/2}{\ell} = (-1)^\ell 2^{n-2\ell}\left[\binom{2\ell-n-1}{\ell-1}-\binom{2\ell-n-1}{\ell}\right].$$

Lemma 2.2 in [18] reads that

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell/2}{n} = \begin{cases}0, & k>n\in\mathbb{N}_0;\\ (-1)^n k!\,\frac{[2(n-k)-1]!!}{(2n)!!}\binom{2n-k-1}{2(n-k)}, & n\ge k\in\mathbb{N}_0.\end{cases} \tag{34}$$

We can also find some discussions and alternative proofs for these three identities at the sites https://math.stackexchange.com/q/1098257 and https://math.stackexchange.com/q/4235171.

Theorem 4.1. For $n,k\in\mathbb{N}_0$, the identities

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell}{n} = \begin{cases}0, & k>n\in\mathbb{N}_0;\\ (-1)^k\frac{k!}{n!}\binom{0}{n-k}, & n\ge k\in\mathbb{N}_0,\end{cases} \tag{35}$$

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{2\ell}{n} = \begin{cases}0, & k>n\in\mathbb{N}_0;\\ (-1)^k\binom{k}{n-k}2^{2k-n}, & n\ge k\in\mathbb{N}_0,\end{cases} \tag{36}$$

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{-\ell}{n} = \begin{cases}0, & k>n\in\mathbb{N}_0;\\ (-1)^{n+k}\binom{n-1}{k-1}, & n\ge k\in\mathbb{N}_0,\end{cases} \tag{37}$$

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{-2\ell}{n} = \begin{cases}0, & k>n\in\mathbb{N}_0;\\ (-1)^n\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{n+2\ell-1}{n}, & n\ge k\in\mathbb{N}_0,\end{cases} \tag{38}$$

and the identity (34) are valid.

Proof. For the case $n\ge k\in\mathbb{N}_0$, these identities follow from Theorem 3.1, the equivalence (10), and simplifying.

For the case $k > n \in \mathbb{N}_0$, making use of the relation 8 and utilizing the explicit formula (6), we acquire

$$\begin{aligned}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\alpha\ell}{n} &= \frac{1}{n!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\left[n!\binom{\alpha\ell}{n}\right] \\ &= \frac{1}{n!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{q=0}^{n}s(n,q)(\alpha\ell)^{q} \\ &= \frac{1}{n!}\sum_{q=0}^{n}s(n,q)\alpha^{q}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\ell^{q} \\ &= (-1)^{k}\frac{k!}{n!}\sum_{q=0}^{n}s(n,q)\alpha^{q}S(q,k)\end{aligned}$$

for all $k,n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, where 0^0 was regarded as 1. Therefore, it is clear that

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\alpha\ell}{n} = 0, \quad k > n \in \mathbb{N}_0, \quad \alpha \in \mathbb{C}.$$

The proof of Theorem 4.1 is complete.

Remark 4.1. The identity (35) can be simplified as

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\ell}{n} = \begin{cases} 0, & k \neq n \\ (-1)^{k}, & n = k \end{cases}$$

for $n,k \in \mathbb{N}_0$.

The identities (36), (37), and (39) in Theorem 4.1 are probably new.

Theorem 4.2. For $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{n}(-1)^{k}k!\binom{0}{n-k} = (-1)^{n}n!, \tag{39}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{k}{n-k}2^{2k} = (-1)^{n}2^{n}(n+1), \tag{40}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{n-1}{k-1} = \begin{cases} 1, & n = 0; \\ -1, & n = 1; \\ 0, & n \geq 2, \end{cases} \tag{41}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+2k-1}{n}\binom{n+1}{k+1}=\begin{cases}1, & n=0;\\ -2, & n=1;\\ 1, & n=2;\\ 0, & n\ge 3,\end{cases} \tag{42}$$

$$\sum_{\ell=0}^{n}(-1)^\ell\binom{\ell/2}{n}\binom{n+1}{\ell+1}=(-1)^n\frac{(2n-1)!!}{(2n)!!}, \tag{43}$$

$$\sum_{k=0}^{n}k!\,[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}=(2n-1)!!, \tag{44}$$

and

$$\sum_{k=1}^{n}\binom{2n-k-1}{n-1}k(k+1)2^k=n2^{2n}. \tag{45}$$

Proof. From (29), we conclude that

$$\sum_{k=0}^{n}(-1)^k k!\,\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})=\langle-\alpha\rangle_n. \tag{46}$$

for $n\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}$.

Substituting (21) into (46) gives

$$\sum_{k=0}^{n}(-1)^k k!\binom{0}{n-k}=\langle-1\rangle_n=(-1)^n n!\,.$$

The identity (39) is thus proved.

Substituting (23) into (46) results in

$$\sum_{k=0}^{n}(-1)^k\binom{k}{n-k}2^{2k}=\frac{2^n\langle-2\rangle_n}{n!}=(-1)^n2^n(n+1).$$

The identity (40) is verified.

Utilizing the relations (2) and (5), we can reformulate the identity (26) as

$$\mathrm{B}_{n,k}(\langle-1\rangle_1,\langle-1\rangle_2,\dots,\langle-1\rangle_{n-k+1})=(-1)^n\frac{n!}{k!}\binom{n-1}{k-1}.$$

Substituting this equality into (46) arrives at

$$\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}=(-1)^n\frac{\langle1\rangle_n}{n!}=\begin{cases}1, & n=0;\\ -1, & n=1;\\ 0, & n\ge 2.\end{cases}$$

The formula (41) follows.

Utilizing the relations (2) and (5), we can reformulate the identity (27) as

$$\mathrm{B}_{n,k}(\langle -2\rangle_1, \langle -2\rangle_2, \dots, \langle -2\rangle_{n-k+1}) = (-1)^n \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \binom{n+2\ell-1}{n}.$$

Substituting this equality into (46) and employing (31) reveal

$$\begin{aligned}(-1)^n \frac{\langle 2\rangle_n}{n!} &= \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{n+2\ell-1}{n}\\ &= \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n+2\ell-1}{n} \sum_{k=\ell}^{n} \binom{k}{\ell}\\ &= \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n+2\ell-1}{n} \binom{n+1}{\ell+1}\end{aligned}$$

and

$$(-1)^n \frac{\langle 2\rangle_n}{n!} = \begin{cases} 1, & n=0;\\ -2, & n=1;\\ 1, & n=2;\\ 0, & n\ge 3.\end{cases}$$

The fourth equality (42) in Theorem 4.2 is thus proved.

Employing (31), we can rearrange the identity (43) as

$$(-1)^n \frac{(2n-1)!!}{(2n)!!} = \sum_{\ell=0}^{n} (-1)^{\ell} \binom{\ell/2}{n} \sum_{k=\ell}^{n} \binom{k}{\ell} = \sum_{\ell=0}^{n} (-1)^{\ell} \binom{\ell/2}{n} \binom{n+1}{\ell+1}.$$

The equality (43) is deduced.

In Theorem 3.2 of [2], on page 5 in [15], and in Theorem 4.2 of [19], there is the equality

$$\sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n = (-1)^n k! \frac{[2(n-k)-1]!!}{2^n} \binom{2n-k-1}{2(n-k)}$$

for $n \ge k \ge 0$. Hence, we obtain

$$\begin{aligned}\frac{(-1)^n}{2^n} \sum_{k=0}^{n} k! [2(n-k)-1]!! \binom{2n-k-1}{2(n-k)} &= \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n\\ &= \sum_{\ell=0}^{n} (-1)^{\ell} \left[\sum_{k=\ell}^{n} \binom{k}{\ell}\right] \left\langle \frac{\ell}{2} \right\rangle_n\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n+1}{\ell+1} \sum_{m=0}^{n} s(n,m) \left(\frac{\ell}{2}\right)^{m} \\
&= \sum_{m=0}^{n} \frac{s(n,m)}{2^m} \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n+1}{\ell+1} \ell^{m} \\
&= \sum_{m=0}^{n} s(n,m) \left(-\frac{1}{2}\right)^{m} \\
&= \left\langle -\frac{1}{2} \right\rangle_{n} \\
&= (-1)^n \frac{(2n-1)!!}{2^n}
\end{aligned}
$$

for $n \in \mathbb{N}$, where we assumed $0^0 = 1$ and used (7), (31), and (32). Hence, we acquire (44) and

$$
\sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_{n} = (-1)^n \frac{(2n-1)!!}{2^n}.
$$

Combining the last one with the relation

$$
\sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_{n} = n! \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{\ell/2}{n},
$$

which is obtained by applying $\alpha = \frac{1}{2}$ to the second equality in (29), yields the identity (43) again.

Substituting (24) into (46) leads to

$$
\sum_{k=0}^{n} (-1)^k k! \, (-1)^{n+k} \frac{[2(n-k)-1]!!}{2^n} \binom{2n-k-1}{2(n-k)} = \left\langle -\frac{1}{2} \right\rangle_{n} = \frac{(2n-1)!!}{2^n}
$$

which is a recovery of the formula (44).

For $k \in \mathbb{N}$, let s_k and S_k be two sequences independent of n such that $n \ge k \in \mathbb{N}$. Theorem 4.4 on page 528 in [20] reads that

$$
s_n = \sum_{k=1}^{n} \binom{k}{n-k} S_k \quad \text{if and only if} \quad (-1)^n n S_n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} (-1)^k k s_k. \tag{47}
$$

Letting $S_n = (-1)^n 2^{2n}$ and $s_n = (-1)^n 2^n (n+1)$, considering (40), applying the inversion theorem expressed by (47), and simplifying figure out the identity (45).

Remark 4.2. The formula (44) is also alternatively established in the proof of Theorem 3.2 in [18] and in Remark 5.3 of [21].

Remark 4.3. The identity (34) established in Lemma 2.2 of [18] and recovered in Theorem 4.1, the identity (36) in Theorem 4.1, and the formula (43) in Theorem 4.2 were announced at https://math.stackexchange.com/a/4268339 and https://math.stackexchange.com/a/4268341 online.

Remark 4.4. In Remark 3.4 of [18], applying the inversion theorem expressed by (47), we obtained

$$\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\binom{2k-1}{k}=(-1)^n2^{n-1}, \quad n\in\mathbb{N}$$

and

$$\sum_{k=1}^{n}(-1)^k\binom{2n-k-1}{n-1}2^{(k+1)/2}k\sin\frac{3(k+1)\pi}{4}=2^n n, \quad n\in\mathbb{N}.$$

5 Several Problems and Numerical Demonstrations

Can one find out simpler closed-form formulas like those in Theorem 3.1 for the quantities

$$\sum_{j=k}^{n}S(n,j)(-1)^j s(j,k), \quad \sum_{j=k}^{n}S(n,j)(\pm 2)^j s(j,k), \quad \sum_{j=k}^{n}S(n,j)\left(\pm\frac{1}{2}\right)^j s(j,k)$$

for $n\ge k\in\mathbb{N}_0$?

By the methods used in this paper, can one find out more combinatorial identities like those in Theorems (4.1) and 4.2?

In general, can one find explicit and closed-form formulas of the quantities

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1}), \quad \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n, \quad \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\alpha\ell}{n},$$

$$\sum_{\ell=k}^{n}s(n,\ell)\alpha^\ell S(\ell,k), \quad \sum_{\ell=k}^{n}S(n,\ell)\alpha^\ell s(\ell,k)$$

for some special values $\alpha\in\mathbb{C}\setminus\left\{0,\pm1,\pm2,\pm\frac{1}{2}\right\}$?

For better understanding the above problems, by the Wolfram Mathematica 12, we numerically compute the quantity

$$Q(k,n;\alpha)=\sum_{\ell=k}^{n}S(n,\ell)\alpha^\ell s(\ell,k)$$

for $0\le k\le n\le 9$ and list their values as

$Q(0,0;\alpha)=1, \quad Q(0,n;\alpha)=0, \quad 1\le n\le 9;$

$Q(1,1;\alpha)=\alpha, \quad Q(1,2;\alpha)=-(\alpha-1)\alpha, \quad Q(1,3;\alpha)=(\alpha-1)\alpha(2\alpha-1),$

$Q(1,4;\alpha)=-(\alpha-1)\alpha\left(6\alpha^2-6\alpha+1\right), \quad Q(1,5;\alpha)=(\alpha-1)\alpha(2\alpha-1)\left(12\alpha^2-12\alpha+1\right),$

$Q(1,6;\alpha)=-(\alpha-1)\alpha\left(120\alpha^4-240\alpha^3+150\alpha^2-30\alpha+1\right),$

$Q(1,7;\alpha)=(\alpha-1)\alpha(2\alpha-1)\left(360\alpha^4-720\alpha^3+420\alpha^2-60\alpha+1\right),$

$Q(1,8;\alpha)=-(\alpha-1)\alpha\left(5040\alpha^6-15120\alpha^5+16800\alpha^4-8400\alpha^3+1806\alpha^2-126\alpha+1\right),$

$$Q(1,9;\alpha)=(\alpha-1)\alpha(2\alpha-1)\big(20160\alpha^6-60480\alpha^5+65520\alpha^4-30240\alpha^3+5292\alpha^2-252\alpha+1\big);$$

$$Q(2,2;\alpha)=\alpha^2,\quad Q(2,3;\alpha)=-3(\alpha-1)\alpha^2,\quad Q(2,4;\alpha)=(\alpha-1)\alpha^2(11\alpha-7),$$

$$Q(2,5;\alpha)=-5(\alpha-1)\alpha^2\big(10\alpha^2-12\alpha+3\big),\quad Q(2,6;\alpha)=(\alpha-1)\alpha^2\big(274\alpha^3-476\alpha^2+239\alpha-31\big),$$

$$Q(2,7;\alpha)=-7(\alpha-1)\alpha^2\big(252\alpha^4-570\alpha^3+430\alpha^2-120\alpha+9\big),$$

$$Q(2,8;\alpha)=(\alpha-1)\alpha^2\big(13068\alpha^5-36324\alpha^4+36560\alpha^3-15940\alpha^2+2771\alpha-127\big),$$

$$Q(2,9;\alpha)=-3(\alpha-1)\alpha^2\big(36528\alpha^6-120288\alpha^5+151368\alpha^4-90300\alpha^3+25550\alpha^2-2940\alpha+85\big);$$

$$Q(3,3;\alpha)=\alpha^3,\quad Q(3,4;\alpha)=-6(\alpha-1)\alpha^3,\quad Q(3,5;\alpha)=5(\alpha-1)\alpha^3(7\alpha-5),$$

$$Q(3,6;\alpha)=-15(\alpha-1)\alpha^3\big(15\alpha^2-20\alpha+6\big),$$

$$Q(3,7;\alpha)=7(\alpha-1)\alpha^3\big(232\alpha^3-443\alpha^2+257\alpha-43\big),$$

$$Q(3,8;\alpha)=-14(\alpha-1)\alpha^3\big(938\alpha^4-2310\alpha^3+1965\alpha^2-660\alpha+69\big),$$

$$Q(3,9;\alpha)=(\alpha-1)\alpha^3\big(118124\alpha^5-354628\alpha^4+395660\alpha^3-199690\alpha^2+43595\alpha-3025\big);$$

$$Q(4,4;\alpha)=\alpha^4,\quad Q(4,5;\alpha)=-10(\alpha-1)\alpha^4,\quad Q(4,6;\alpha)=5(\alpha-1)\alpha^4(17\alpha-13),$$

$$Q(4,7;\alpha)=-35(\alpha-1)\alpha^4\big(21\alpha^2-30\alpha+10\big),$$

$$Q(4,8;\alpha)=7(\alpha-1)\alpha^4\big(967\alpha^3-1973\alpha^2+1257\alpha-243\big),$$

$$Q(4,9;\alpha)=-42(\alpha-1)\alpha^4\big(1602\alpha^4-4200\alpha^3+3885\alpha^2-1470\alpha+185\big);$$

$$Q(5,5;\alpha)=\alpha^5,\quad Q(5,6;\alpha)=-15(\alpha-1)\alpha^5,\quad Q(5,7;\alpha)=35(\alpha-1)\alpha^5(5\alpha-4),$$

$$Q(5,8;\alpha)=-70(\alpha-1)\alpha^5\big(28\alpha^2-42\alpha+15\big),$$

$$Q(5,9;\alpha)=21(\alpha-1)\alpha^5\big(1069\alpha^3-2291\alpha^2+1559\alpha-331\big);$$

$$Q(6,6;\alpha)=\alpha^6,\quad Q(6,7;\alpha)=-21(\alpha-1)\alpha^6,\quad Q(6,8;\alpha)=14(\alpha-1)\alpha^6(23\alpha-19),$$

$$Q(6,9;\alpha)=-126(\alpha-1)\alpha^6\big(36\alpha^2-56\alpha+21\big);$$

$$Q(7,7;\alpha)=\alpha^7,\quad Q(7,8;\alpha)=-28(\alpha-1)\alpha^7,\quad Q(7,9;\alpha)=42(\alpha-1)\alpha^7(13\alpha-11);$$

$$Q(8,8;\alpha)=\alpha^8,\quad Q(8,9;\alpha)=-36(\alpha-1)\alpha^8,\quad Q(9,9;\alpha)=\alpha^9.$$

If fixing $k=4,5$ and $n=7,8$ and regarding α as a real variable on the interval $[-9,9]$, then the graphs plotted by the Wolfram Mathematica 12 are showed in Figs. 1 and 2.

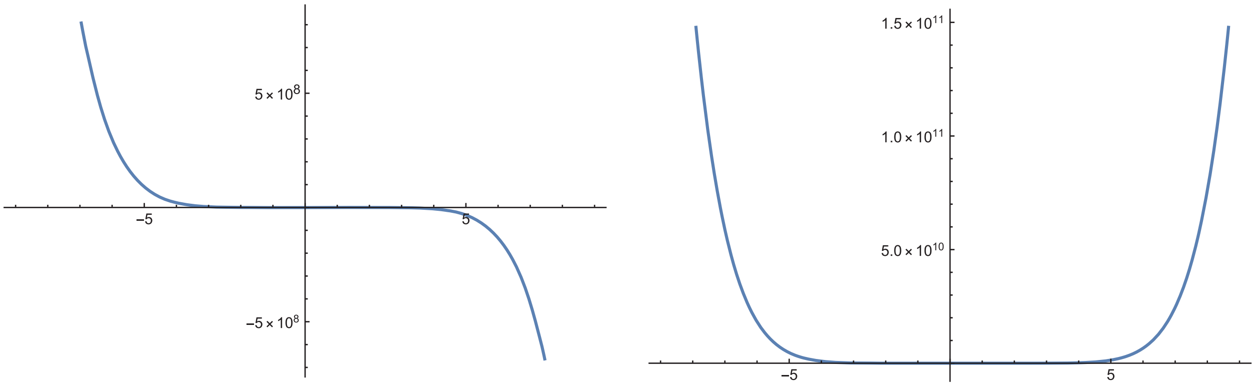

Figure 1: The graphs of $Q(4,7;\alpha)$ and $Q(4,8;\alpha)$ for $\alpha \in [-9,9]$

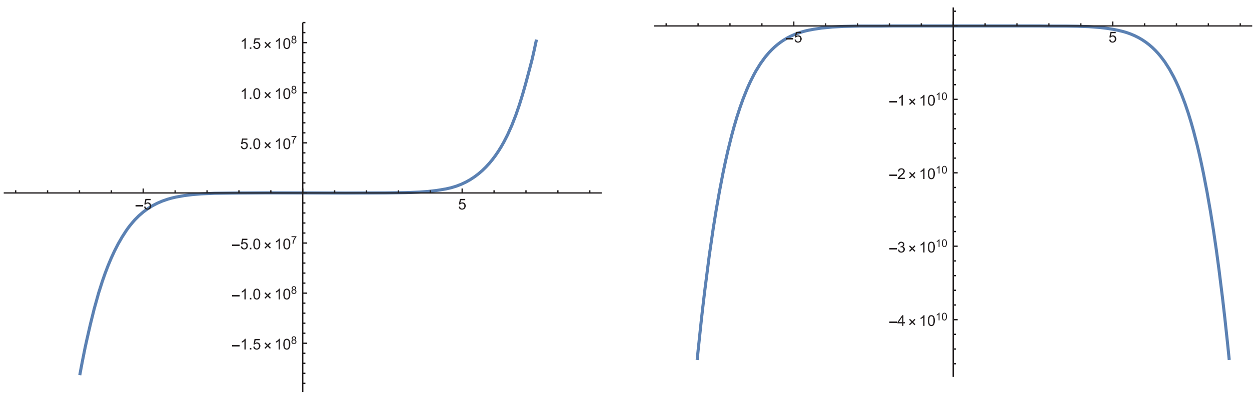

Figure 2: The graphs of $Q(5,7;\alpha)$ and $Q(5,8;\alpha)$ for $\alpha \in [-9,9]$

6 Conclusions

In this paper, we collected, discussed, and found out significant connections, equivalences, closed-form formulas, and combinatorial identities concerning partial Bell polynomials $\mathrm{B}_{n,k}$, falling factorials $\langle z\rangle_n$, rising factorials $(z)_n$, extended binomial coefficients $\binom{z}{w}$, and the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$. These results are new, interesting, important, useful, and applicable in combinatorial number theory and other areas, as done in the papers [22–27] and closely related references therein.

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