Computer Modeling in Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2022.019965

ARTICLE

# Asymptotic Approximations of Apostol-Tangent Polynomials in Terms of Hyperbolic Functions

**Cristina B. Corcino**[1,2], **Wilson D. Castañeda Jr.**[3] **and Roberto B. Corcino**[1,2,*]

[1]Research Institute for Computational Mathematics and Physics, Cebu Normal University, Cebu City, 6000, Philippines

[2]Mathematics Department, Cebu Normal University, Cebu City, 6000, Philippines

[3]Department of Mathematics, Cebu Technological University, Cebu City, 6000, Philippines

[*]Corresponding Author: Roberto B. Corcino. Email: rcorcino@yahoo.com

Received: 27 October 2021 Accepted: 30 December 2021

## ABSTRACT

The tangent polynomials $T_n(z)$ are generalization of tangent numbers or the Euler zigzag numbers $T_n$. In particular, $T_n(0) = T_n$. These polynomials are closely related to Bernoulli, Euler and Genocchi polynomials. One of the extensions and analogues of special polynomials that attract the attention of several mathematicians is the Apostol-type polynomials. One of these Apostol-type polynomials is the Apostol-tangent polynomials $T_n(z, \lambda)$. When $\lambda = 1$, $T_n(z, 1) = T_n(z)$. The use of hyperbolic functions to derive asymptotic approximations of polynomials together with saddle point method was applied to the Bernoulli and Euler polynomials by Lopez and Temme. The same method was applied to the Genocchi polynomials by Corcino et al. The essential steps in applying the method are (1) to obtain the integral representation of the polynomials under study using their exponential generating functions and the Cauchy integral formula, and (2) to apply the saddle point method. It is found out that the method is applicable to Apostol-tangent polynomials. As a result, asymptotic approximation of Apostol-tangent polynomials in terms of hyperbolic functions are derived for large values of the parameter $n$ and uniform approximation with enlarged region of validity are also obtained. Moreover, higher-order Apostol-tangent polynomials are introduced. Using the same method, asymptotic approximation of higher-order Apostol-tangent polynomials in terms of hyperbolic functions are derived and uniform approximation with enlarged region of validity are also obtained. It is important to note that the consideration of Apostol-type polynomials and higher order Apostol-type polynomials were not done by Lopez and Temme. This part is first done in this paper. The accuracy of the approximations are illustrated by plotting the graphs of the exact values of the Apostol-tangent and higher-order Apostol-tangent polynomials and their corresponding approximate values for specific values of the parameters $n, \lambda$ and $m$.

## KEYWORDS

Apostol-tangent polynomials; tangent polynomials; Genocchi polynomials; Hermite polynomials; asymptotic approximation

## 1 Introduction

The Apostol-tangent polynomials denoted by $T_n(z;\lambda), \lambda \neq 0$ are defined by generating function (see [1])

$$\frac{2e^{zw}}{\lambda e^{2w}+1} = \sum_{n=0}^{\infty} T_n(z;\lambda)\frac{w^n}{n!}, \tag{1}$$

where $\lambda \epsilon \mathbb{C}$ and the validity of the series in Eq. (1) is given as follows:

$$|w| < \begin{cases} \frac{\pi}{2} & when \quad \lambda = 1 \\ \pi & when \quad \lambda \neq 1. \end{cases}$$

when $\lambda = 1$, the equation gives the generating function for the classical tangent polynomials $T_n(z)$ given by (see [2,3])

$$\frac{2e^{zw}}{e^{2w}+1} = \sum_{n=0}^{\infty} T_n(z)\frac{w^n}{n!}, \quad |w| < \frac{\pi}{2}. \tag{2}$$

Setting $z = 0$ in Eqs. (1) and (2), we obtain

$$T_n(0,\lambda) : = T_n(\lambda) \text{ and } T_n(0) : = T_n, \tag{3}$$

where $T_n(\lambda)$ and $T_n$ are called the Apostol-tangent numbers and classical tangent numbers, respectively (see [1,4]).

First few values of the Apostol-tangent polynomials are given below:

$$T_0(z;\lambda) = \frac{2}{1+\lambda}, T_1(z;\lambda) = \frac{2[z+(-2+z)\lambda]}{(1+\lambda)^2}, T_2(z;\lambda) = \frac{2\left[-4\lambda+(z+(-2+z)\lambda)^2\right]}{(1+\lambda)^3},$$

$$T_3\ (z;\lambda) = \frac{2}{(1+\lambda)^4}\Big[-6z^2\lambda(1+\lambda)^2\Big] + z^3(1+\lambda)^3 - 8\lambda(-4+\lambda)\lambda\Big) + 12\lambda\Big(-1+\lambda^2\Big)\Big],$$

$$T_4(z;\lambda) = \frac{2}{(1+\lambda)^5}\Big[24z^2(-1+\lambda)\lambda(1+\lambda)^2 - 8z^3\lambda(1+\lambda)^3 + z^4(1+\lambda)^4 + 16(-1+\lambda)\lambda$$
$$(1+(-10+\lambda)\lambda) - 32\lambda(1+\lambda)(1+(-4+\lambda)\lambda)].$$

The Apostol-tangent polynomials are extensions of the classical tangent polynomials. The latter have become an interesting area for many mathematicians for their extensions and analogues possess properties that are relevant in analytic number theory and physics (see [5–8]). In [1], the 2-variable q generalized tangent-Apostol type polynomials were introduced and investigated as a new class of q-hybrid special polynomials.

Asymptotic approximations for Bernoulli polynomials $B_n\left(nz+\frac{1}{2}\right)$ and Euler polynomials $E_n\left(nz+\frac{1}{2}\right)$ in terms of hyperbolic functions are established in [9]. In the study of Corcino et al. [10], the Genichi polynomials are expressed as

$$G_n\left(z+\frac{1}{2}\right)=\frac{n!}{2\pi i}\int_C \frac{we^{wz}}{\cosh(w/2)}\frac{dw}{w^{n+1}} \tag{4}$$

where the contour $C$ encircles the origin in the counterclockwise direction and contains no poles of $1/\cosh(w/2)$. With this, they have derived the asymptotic formulas for $G_n\left(z+\frac{1}{2}\right)$ in terms of hyperbolic functions. However, asymptotic approximations of Apostol-tangent polynomials parallel to the results obtained in [9] and [10], are not mentioned and found in those studies and other related literature.

In this study, the asymptotic approximations of the Apostol-tangent polynomials $T_n(z;\lambda)$ for large $n$ which are uniformly valid in some unbounded region of the complex variable $z$, are derived using saddle point method as used in [9] and [10]. Moreover, asymptotic expansion of higher-order Apostol-tangent polynomials $T_n^m(z;\lambda)$ is obtained. Corresponding asymptotic formulas of the tangent polynomials are given as corollaries.

## 2 Asymptotic Expansions of Apostol-Tangent Polynomials

**Theorem 2.1.** For $\lambda\epsilon\mathbb{C}-\{0\}$, and $z\epsilon\mathbb{C}$ such that $\left|Imz^{-1}\right|<\frac{\pi-Arg\lambda}{2}$ and $\left|z^{-1}\right|<\left|z^{-1}-\left(\frac{\pi i}{2}-\delta\right)\right|$ and $n\geq 1$,

$$T_n(nz+1;\lambda)=\frac{n^n z^n \operatorname{sech}\left(z^{-1}+\delta\right)}{\sqrt{\lambda}}\left\{1-\frac{1-2\operatorname{sech}^2\left(z^{-1}+\delta\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}, \tag{5}$$

where $\delta=(\log\lambda)/2$ and the logarithim is taken to be the principal branch.

*Proof*. Applying the Cauchy Integral Formula [11] to Eq. (1), we have

$$T_n(z;\lambda)=\frac{n!}{2\pi i}\int_C \frac{2e^{wz}}{e^{2\delta+2w}+1}\frac{dw}{w^{n+1}}, \tag{6}$$

where $C$ is a circle about the origin with radius $<\left|\frac{\pi i}{2}-\delta\right|$. With $2e^{(w+\delta)}\cosh\left(w+\delta=e^{(2w+2\delta)}\right)+1$, it follows from Eq. (6) that

$$T_n(z+1;\lambda)=\frac{n!}{2\pi i\sqrt{\lambda}}\int_C f(w)\,e^{zw}\frac{w}{w^{n+1}}, \tag{7}$$

where $\sqrt{\lambda}=e^{(\log\lambda)/2}=e^{\delta}$ *and* $f(w)=1/\cosh(w+\delta)$. The function $f(w)$ is meromorphic function with simple poles at the zeros of $\cosh(w+\delta)$ which are given by $w_j=(2j+1)\frac{\pi i}{2}-\delta$, $j=0,\pm1,\pm2,\ldots$

Now take $z\longmapsto nz$ and let $nz\longmapsto\infty$ with $z$ fixed. It follows from Eq. (7) that

$$T_n(nz+1;\lambda)=\frac{n!}{2\pi i\sqrt{\lambda}}\int_C f(w)\,e^{n(zw-\log w)}\frac{dw}{w}. \tag{8}$$

The main contribution of the integrand above to the integral occurs at the saddle point of the argument of the exponential [12]. This saddle point is at the point $w=1/z=z^{-1}, z\neq 0$.

Assume that $z^{-1}$ is not a pole of $f(w)$. Then approximations of $T_n(nz+1;\lambda)$ can be obtained by expanding $f(w)$ around the saddle point [13–16]. Let

$$f(w)=\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)}{k!}\left(w-z^{-1}\right)^{k},\left|w-z^{-1}\right|<r \tag{9}$$

where $r$ is the distance from $z^{-1}$ to the nearest singularity of $f(w)$. For $w$ is the circle $C$, the above series is absolutely convergent if the saddle point $z^{-1}$ is closer to the origin than to any of the singularities $w_j$. That is, if $z^{-1}$ is in the strip $\left|Imz^{-1}\right|<\frac{\pi-Arg\lambda}{2}$ and $\left|z^{-1}\right|<\left|z^{-1}-w_j\right|$ for $j=0,\pm1,\pm2,\ldots$ It follows from Lemma 1, Lemma 2 and Theorem 1 of [16] that

$$T_n(nz+1;\lambda)=\frac{(nz)^n}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)}{k!}\frac{p_k(n)}{(nz)^k}, \tag{10}$$

where

$$p_0(n)=1,p_1(n)=0,p_2(n)=-n,p_3(n)=2n \tag{11}$$

$$p_k(n)=(1-k)\,p_{k-1}(n)+np_{k-2}(n)\,,k\geq 3. \tag{12}$$

Computing the derivatives $f^{(k)}\left(z^{-1}\right)$ for $k=0,1,2$ give

$$f\left(z^{-1}\right)=\operatorname{sech}\left(z^{-1}+\delta\right), \tag{13}$$

$$f^{(1)}\left(z^{-1}\right)=-\tanh\left(z^{-1}+\delta\right)\operatorname{sech}\left(z^{-1}+\delta\right), \tag{14}$$

$$f^{(2)}\left(z^{-1}\right)=\operatorname{sech}\left(z^{-1}+\delta\right)\left(1-2\operatorname{sech}^2\left(z^{-1}+\delta\right)\right). \tag{15}$$

Expanding the sum in Eq. (10) and keeping only the first three terms yield

$$\begin{aligned}T_n(nz+1;\lambda)&=\frac{(nz)^n}{\sqrt{\lambda}}\left[\frac{u\left(z^{-1}\right)}{0!}+\frac{u^{(1)}\left(z^{-1}\right)}{1!}\frac{p_1(n)}{nz}+\frac{u^{(2)}\left(z^{-1}\right)}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right]\\&=\frac{n^nz^z}{\sqrt{\lambda}}\left\{\operatorname{sech}\left(\frac{1}{z}+\delta\right)-\frac{\operatorname{sech}\left(\frac{1}{z}+\delta\right)\left(1-2\operatorname{sech}^2\left(\frac{1}{z}+\delta\right)\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}\\&=\frac{n^nz^n\left(\operatorname{sech}\left(\frac{1}{z}+\delta\right)\right)}{\sqrt{\lambda}}\left\{1-\frac{1-2\operatorname{sech}^2\left(\frac{1}{z}+\delta\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}.\end{aligned}$$

The accuracy of the asymptotic formula obtained in Eq. (5) is shown in Fig. 1.

To enlarge the region of validity of Eq. (5) and obtain an asymptotic expansion valid in a larger region, the following theorem will be utilized.

**Theorem 2.2.** [9] The polynomials

$$P_n(z) = \frac{n!}{2\pi i}\int_C f(w)\, e^{wz}\frac{dw}{w^{n+1}}, \tag{16}$$

where $f(w)$ is analytic at the origin with simple poles $w_1, w_2, \cdots$ (and respective residues $r_1, r_2, \cdots$), can be represented, for each integer $m > 0$, as

$$P_n(nz) = -\sum_{k=1}^{m}\frac{r_k e^{w_k nz}}{w_k^{n+1}}\Gamma(n+1, w_k nz)$$
$$+ (nz)^n \sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right) + h_m^{(k)}\left(z^{-1}\right)}{k!}\frac{p_k(n)}{(nz)^k}, \tag{17}$$

that is valid for $z \in C, \left|z^{-1}\right| < \left|z^{-1} - w_j\right|$, for all $j = m+1, m+2, \cdots$, where the polynomials $p_k(n)$ are given in Eq. (12) and $h_m^{(k)}$ is the kth derivative of the function

$$h_m(w) = -\sum_{l=1}^{m}\frac{r_l}{w - w_l},$$

where the residues $w_l$ are ordered by increasing modulus $|w_l| \le \left|w_{l+1}\right|$. Each term of the finite sum in the above equation equals $n!\, r_k / w_k^{n+1}$ multiplied by the Taylor polynomial of degree n in $z = 0$ of $e^{w_k nz}$.

The second asymptotic formula for $T_n(nz+1;\lambda)$ with enlarged region of validity is given in the following theorem.

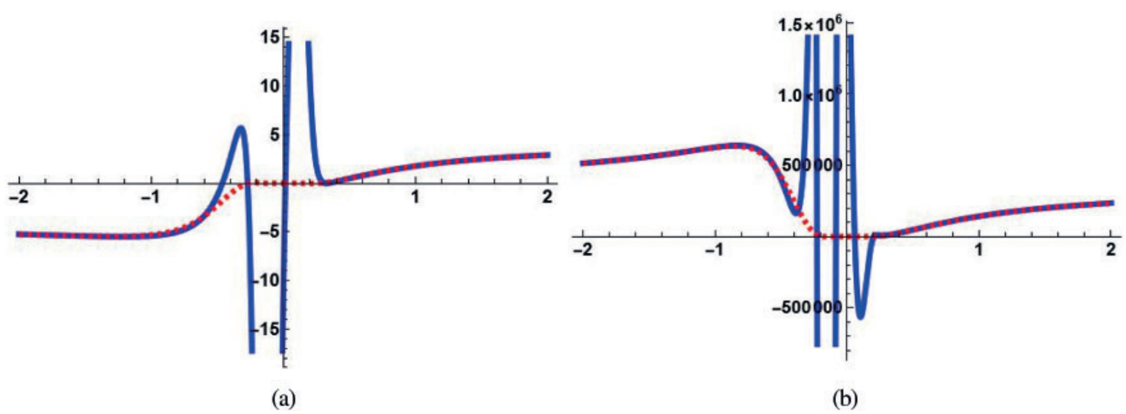



**Figure 1:** Solid lines represent $T_n(nx+1;\lambda)$ for several values of $n$, whereas dashed lines represent the right-hand side of (5) with $z = x$, both normalized by the factor $\left(1 + \left|\frac{x}{\alpha}\right|^n\right)^{-1}$ where we choose $\alpha = 0.2$ (a) $n = 7$ and $\lambda = 4$ (b) $n = 14$ and $\lambda = 9$

**Theorem 2.3.** Let $z \in \mathbb{C}\backslash\{0\}$ such that $\left|z - \frac{2}{\pm(2k+1)\pi i - 2\delta}\right| > \frac{2}{(2k+1)\pi + 2\delta}$ for $k = 0, 1, 2, \cdots, m-1$ and $\lambda \in \mathbb{C}\{0\}$. Then, as $n \to \infty$,

$$T_n(nz+1;\lambda) = \frac{-2^{n+1}i}{\left(\sqrt{\lambda}\right)^{nz+1}} \sum_{k=0}^{m-1} (-1)^k \left[ -\frac{e^{\frac{(2k+1)\pi inz}{2}}}{[(2k+1)\pi i - 2\delta]^{n+1}} \Gamma\left(n+1, \left[\frac{(2k+1)\pi i}{2} - \delta\right] nz\right) \right.$$
$$\left. + \frac{e^{\frac{-(2k+1)\pi inz}{2}}}{[-(2k+1)\pi i - 2\delta]^{n+1}} \Gamma\left(n+1, \left[\frac{-(2k+1)\pi i}{2} - \delta\right] nz\right) \right]$$
$$+ \frac{(nz)^n}{\sqrt{\lambda}} \left\{ \operatorname{sech}\left(\frac{1}{z} + \delta\right) \right.$$
$$+ \sum_{k=0}^{m-1} \frac{(-1)^{k+1} 4(2k+1)\pi}{4\left(\frac{1}{z}+\delta\right)^2 + (2k+1)^2\pi^2} - \frac{\operatorname{sech}\left(\frac{1}{z}+\delta\right)\left(1 - 2\operatorname{sech}^2\left(\frac{1}{z}+\delta\right)\right)}{2nz^2}$$
$$\left. - \sum_{k=0}^{m-1} \frac{(-1)^k 16(2k+1)\pi \left[(2k+1)^2\pi^2 - 12\left(\frac{1}{z}+\delta\right)^2\right]}{nz^2\left\{4\left(\frac{1}{z}+\delta\right)^2 + (2k+1)^2\pi^2\right\}^3} + O\left(\frac{1}{n^2}\right) \right\}. \tag{18}$$

*Proof*. We start by computing the residues $r_l$ for the Apostol-tangent polynomials. Observe that the case is the function $f(w) = \sec h(w+\delta) = \frac{1}{\cos h(w+\delta)} = \frac{p(w)}{q(w)}$ which has simple poles at $w_l = \frac{\pm(2l+1)\pi i}{2} - \delta, l = 0, 1, 2, \ldots, m-1$. Thus, the corresponding residues are

$$r_l = \frac{p(w_l)}{q'(w_l)} = \frac{1}{\sin h(w_l+\delta)}, \tag{19}$$

where

$$\sin h(w_l+\delta) = \sin h\left(\left(l+\frac{1}{2}\right)\pi i\right) = i \sin h\left(l\pi + \frac{\pi}{2}\right) = (-1)^{l_i}. \tag{20}$$

On the other hand, for $w_{-l} = \frac{-(2l+1)\pi i}{2} - \delta, l = 0, 1, 2, \ldots, m-1$,

$$\sin h\left(w_{-l}+\delta\right) = -\sin h\left(\left(l+\frac{1}{2}\right)\pi i\right) = -i \sin h\left(l\pi + \frac{\pi}{2}\right) = (-1)^{l+1}. \tag{21}$$

Thus, the residues $r_l, l = 0, 1, 2, \ldots m-1$ of the function $f(w)$ are

$$r_l = \frac{1}{(-1)^{l_i}} \textit{and } r_l = \frac{1}{(-1)^{l+1_i}} = (-1)^l. \tag{22}$$

Next, the derivatives of $h_m(w)$ at the saddle point $z^{-1}$ will be computed. With the simple poles $w_l = \frac{(2l+1)\pi i}{2} - \delta$ and $w_{-1} = \frac{-(2l+1)\pi i}{2} - \delta$ of the function $f(w)$, an expression for $h_m(w)$ is obtained as follows:

$$\begin{aligned}
h_m(w) &= -\sum_{l=0}^{m-1} \frac{r_l}{w - w_l} - \sum_{l=0}^{m-1} \frac{r_l}{w - w_{-l}} \\
&= -\sum_{l=0}^{m-1} \frac{(-1)^{l+1_i}}{w - \left[\frac{(2l+1)\pi i}{2} - \delta\right]} - \sum_{l=0}^{m-1} \frac{(-1)^{l_i}}{w - \left[\frac{-(2l+1)\pi i}{2} - \delta\right]} \\
&= -\sum_{l=0}^{m-1} (-1)^{l_i} \left( \frac{-1}{[w+\delta] - \frac{(2l+1)\pi i}{2})} + \frac{1}{[w+\delta] + \frac{(2l+1)\pi i}{2})} \right) \\
&= -\sum_{l=0}^{m-1} (-1)^{l_i} \left( \frac{-(2l+1)\pi i}{[w+\delta]^2 + \frac{(2l+1)^2\pi^2}{4}} \right) \\
&= \sum_{l=0}^{m-1} \frac{(-1)^{l+1} 4(2l+1)\pi}{4(w+\delta)^2 + (2l+1)^2\pi^2}.
\end{aligned}$$

Computing the derivatives yields

$$h_m^{(1)}(w) = \sum_{l=0}^{m-1} \frac{(-1)^l 32(2l+1)\pi [w+\delta]}{\left\{4(w+\delta)^2 + (2l+1)^2\pi^2\right\}^2}, \tag{23}$$

$$h_m^{(2)}(w) = \sum_{l=0}^{m-1} \frac{(-1)^l 32(2l+1)\pi \left[(2l+1)^2\pi^2 - 12(w+\delta)^2\right]}{\left\{4(w+\delta)^2 + (2l+1)^2\pi^2\right\}^3}. \tag{24}$$

At the saddle point $z^{-1}$,

$$h_m^{(0)}\left(z^{-1}\right) = \sum_{l=0}^{m-1} \frac{(-1)^{l+1} 4(2l+1)\pi}{4\left(\frac{1}{z}+\delta\right)^2 + (2l+1)^2\pi^2}, \tag{25}$$

$$h_m^{(1)}\left(z^{-1}\right) = \sum_{l=0}^{m-1} \frac{(-1)^l 32(2l+1)\pi \left(\frac{1}{z}+\delta\right)}{\left\{4\left(\frac{1}{z}+\delta\right)^2 + (2l+1)^2\pi^2\right\}^2}, \tag{26}$$

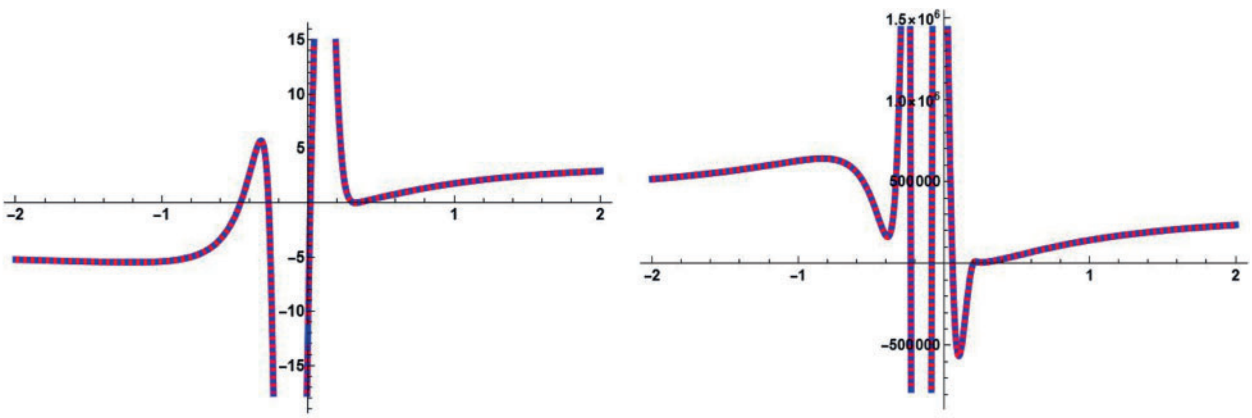
$$h_m^{(2)}\left(z^{-1}\right) = \sum_{l=0}^{m-1} \frac{(-1)^l 32(2l+1)\pi \left((2l+1)^2\pi^2 - 12\left(\frac{1}{z}+\delta\right)^2\right)}{\left\{4\left(\frac{1}{z}+\delta\right)^2 + (2l+1)^2\pi^2\right\}^3}. \tag{27}$$

From Theorem 2.2,

$$T_n\left(nz+1;\lambda\right)=-\left(\lambda\right)^{-\frac{1}{2}}\sum\frac{r_{k}e^{w_{k}nz}}{w_k^{n+1}}\Gamma\left(n+1,w_{k}nz\right)+\frac{(nz)^n}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)+h_m^{(k)}\left(z^{-1}\right)}{k!}\frac{P_k\left(n\right)}{(nz)^k}. \tag{28}$$

Keeping only the first three terms of the infinite sum in (28) and using $P_k(n)$ in Eq. (11), $f^{(k)}\left(z^{-1}\right)$ given in Eqs. (13)–(15) and $h_m^{(k)}\left(z^{-1}\right)$ given Eqs. (25)–(27) with $w_k=\frac{\pm(2k+1)\pi i}{2}-\delta, r_k=\frac{(-1)^{k+1}i}{\sqrt{\lambda}}$ and $\frac{(-1)^k i}{\sqrt{\lambda}}, k=0,1,\ldots,m-1$, the desired asymptotic formula is obtained.

The accuracy of the asymptotic formula obtained in Eq. (18) is shown in Fig. 2. The accuracy of the approximation in the oscillatory region is better that that the of the formula in Eq. (5).




(a) (b)

**Figure 2:** Solid lines represent $T_n(nx+1;\lambda)$ for several values of $n$, whereas dashed lines represent the right-hand side of Eq. (18) with $z\equiv x$, both normalized by the factor $\left(1+\left|\frac{x}{\alpha}\right|^n\right)^{-1}$ where we choose $\alpha=0.2$. (a) $n=7$ *and* $\lambda=4$ (b) $n=14$ *and* $\lambda=9$

**Remark 2.4.** Taking $\lambda=1$, Theorem 2.1 and Theorem 2.3, respectively, will give uniform approximationformula and an asypmtotic expansion with enlarged region of validity which are same formulas as those obtained in [17] for the tangent polynomials.

## 3 Approximation of Higher-Order Apostol-Tangent Polynomials

Higher-order Apostol-tangent polynomials are defined by the generating function

$$\left(\frac{2}{\lambda e^{2w}+1}\right)^m e^{zw}=\sum_{n=0}^{\infty}T_n^m\left(z;\lambda\right)\frac{w^n}{n!},|w|<\frac{\pi}{2}\mathit{when}\ \lambda=1$$

$$\text{and } |w|<\pi \text{ when } \lambda\neq 1:\lambda\epsilon\mathbb{C}\setminus\{0\} \tag{29}$$

In this section, it is shown that the method in Section 2 can be extended to obtain asymptotic expansion of the Apostol-tangent polynomials of order $m$.

**Theorem 3.1** For $\lambda\epsilon\mathbb{C}\backslash\{0\}$, and $z\epsilon\mathbb{C}\backslash\{0\}$ such that $\left|Imz^{-1}\right| < \frac{\pi-Arg\lambda}{2}$ and $\left|z^{-1}\right| < \left|z^{-1}-\left(\frac{\pi i}{2}-\delta\right)\right|$ and $n,m\geq 1$, the Apostol-tangent polynomials of order $m$ satisfy

$$T_n^m(nz+m;\lambda)=\frac{n^n z^n sech^m\left(z^{-1}+\delta\right)}{\left(\sqrt{\lambda}\right)^m}\left\{1-\frac{m\left(m-(m+1)\,sech^2\left(z^{-1}+\delta\right)\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}, \tag{30}$$

when $\delta=(\log\lambda)/2$ and the logarithm is taken to be the principal branch.

*Proof*. Applying the Cauchy Integral Formula to Eq. (29),

$$T_n^m(z;\lambda)=\frac{n!}{2\pi i}\int_C\frac{2^m e^{zw}}{\left(e^{In(\lambda)+2w}+1\right)^m}\frac{dw}{w^{n+1}}, \tag{31}$$

where $C$ is a circle about 0 with radius less than $\frac{|\pi-In(\lambda)|}{2}$. With $\left(2e^{(\delta+w)}\right)^m(cosh\,(w+\delta))^m=\left(e^{2\delta+2w}+1\right)^m$, it follows from Eq. (31) that

$$T_n^m(z;\lambda)=\frac{n!}{2\pi i\left(\sqrt{\lambda}\right)^m}\int_C f(w)\frac{e^{zw}}{e^{wm}}\frac{dw}{w^{n+1}} \tag{32}$$

where $\lambda^m=\left(e^{\log(\lambda)/2}\right)^m=e^{\delta m}$ and $f(w)=\frac{1}{cosh^m(w+\delta)}$. The function $f(w)$ is a meromorphic function with poles of order $m$ at the zeros of $cosh^m(w+\delta)$ which are given by $w_j=(2j+1)\frac{\pi i}{2}-\delta, j=0,\pm1,\pm2,\cdots$. It follows that by taking $z\longmapsto nz$ an letting $nz\to\infty$ with fixed $z$,

$$T_n^m(nz+m;\lambda)=\frac{n!}{2\pi i\left(\sqrt{\lambda}\right)^m}\int_C f(w)\,e^{n(zw-\log w)}\frac{dw}{w^n}. \tag{33}$$

Likewise, the approximations of $T_n^m(nz+m;\lambda)$ can be obtained by expanding $f(w)$ around the saddle point $w=z^{-1}$. Using Lemma 1, Lemma 2, and Theorem 1 of [9],

$$T_n^m(nz+m;\lambda)=\frac{(nz)^n}{\left(\sqrt{\lambda}\right)^m}\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{P_k(n)}{(nz)^k}, \tag{34}$$

where $P_k(n)$ are the polynomials given in Eqs. (11) and (12). The derivative of $f^{(k)}\left(z^{-1}\right)$ for $k=0,1,2$ are given by

$$f^{(0)}\left(z^{-1}\right)=f(z^{-1})=sech^m\left(z^{-1}+\delta\right), \tag{35}$$

$$f^{(1)}\left(z^{-1}\right)=-m\tanh\left(z^{-1}+\delta\right)sech^m\left(z^{-1}+\delta\right), \tag{36}$$

$$f^{(2)}\left(z^{-1}\right)=m\;sech^m\left(z^{-1}+\delta\right)\left(m-(m+1)\,sech^2\left(z^{-1}+\delta\right)\right). \tag{37}$$

Expanding the sum in (34) and keeping only the first three terms give

$$T_n^m(nz+m;\lambda)=\frac{(nz)^n}{\left(\sqrt{\lambda}\right)^m}\left[\frac{v\left(z^{-1}\right)}{0!}+\frac{v^{(1)}\left(z^{-1}\right)}{1!}\frac{p_1(n)}{nz}+\frac{v^{(2)}\left(z^{-1}\right)}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right]$$

$$=\frac{n^nz^n}{\left(\sqrt{\lambda}\right)^m}\left\{sech^m\left(\frac{1}{z}+\delta\right)-\frac{msech^m\left(\frac{1}{z}+\delta\right)\left(m-(m+1)\,sech^2\left(\frac{1}{z}+\delta\right)\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}$$

$$=\frac{n^nz^n\left(sech^m\left(\frac{1}{z}+\delta\right)\right)}{\left(\sqrt{\lambda}\right)^m}\left\{1-\frac{m\left(m-(m+1)\,sech^2\left(\frac{1}{z}+\delta\right)\right)}{2nz^2}+O\left(\frac{1}{n^2}\right)\right\}.$$

The accuracy of the asymptotic formula obtained in Eq. (30) is shown in Fig. 3.

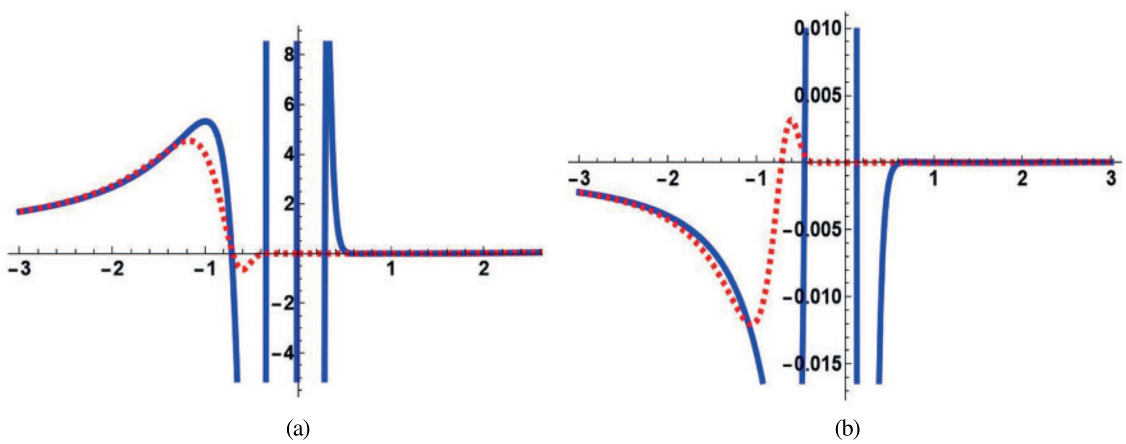


Figure 3: Solid lines represent $T_n^m(nx+m;\lambda)$ for several values of $n$ and $m$, whereas dashed lines represent the right-hand side of Eq. (30) with $z\equiv x$, both normalized by the factor $\left(1+\left|\frac{x}{\alpha}\right|^n\right)^{-1}$ where we choose $\alpha=0.2$ (a) $m=7, n=10$ *and* $\lambda=5$ (b) $m=8, n=7$ *and* $\lambda=6$

**Corollary 3.2.** *For* $z\in\mathbb{C}\setminus\{0\}$ *such that* $\left|Imz^{-1}\right|<\frac{\pi}{2}, \left|z^{-1}\right|<\left|z^{-1}-\frac{\pi i}{2}\right|$ *and* $n,m\geq 1$,

$$T_n^m(nz+m)=n^nz^nsech^m\left(z^{-1}\right)\left\{1-\frac{m(m-(m+1)\,sech^2\left(z^{-1}+\delta\right))}{2nz^2}+O\left(1/n^2\right)\right\}. \tag{38}$$

*Proof*. This follows from Theorem 3.1 by taking $\lambda=1$. To enlarge the region of validity of Eq. (30) and obtain an asymptotic expansion valid in a larger region the following theorem will be used.

**Theorem 3.3.** For $\lambda \in \mathbb{C}\setminus\{0\}, m \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ such that $\left|z^{-1}\right| < \left|z^{-1} - w_k\right|$ for all $k = l+1, l+2, \cdots$, the Apostol-tangent polynomials of order $m$ satisfy

$$T_n^m(nz+m;\lambda) = \lambda^{-\frac{m}{2}} \left\{ \sum_{k=1}^{l} \sum_{j-1}^{m} e^{w_k n z r_{k_j}} \left[ \sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1\rangle_s (w_k)^{-(j-1+s)} \left( \frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k n z)}{w_k^{n-s+1}} \right) + \frac{(-1)^j \langle j\rangle n}{w_k^{j+n}} \right] + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}\left(z^{-1}\right) - h_l^{(k)}\left(z^{-1}\right)}{k!} \frac{Pk(n)}{(nz)^k} \right\}, \tag{39}$$

where the polynomisals $p_k(n)$ are given in Eq. (12) $h_l^{(k)}$ is the kth derivative of the function $h_l(w)$ given by Eq. (49) and

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w-w_k)^j}$$

Are the given principal parts of the Laurent series corresponding to the poles $w_k$, where the entire function $h(z)$ is determined by $f(z)$.

*Proof*. With $f(w) = \cosh^{-m}(w+\delta)$, it follows from Mittag-Leffler's Theorem (see [18,19]) that

$$\begin{aligned} f(w) &= \sum_{k=1}^{l} \left[ \sum_{j=1}^{m} \frac{r_{k_j}}{(w-w_k)^j} + q_k(w) \right] + g(w) \\ &= \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w-w_k)^j} + \sum_{k=1}^{l} q_k(w) + g(w) \\ &= \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w-w_k)^j} + f_l(w), \end{aligned} \tag{40}$$

where

$$f_l(w) = \sum_{k=1}^{l} q_k(w) + g(w),$$

$q_k(w)$ is a polynomial of $w, r_{kj}$ are residues of $f(w)$ at $w_k, k = 1, 2, \ldots, l$. Note that inside the disk $|w| < |w_{m+1}|, f_l(w)$ has no poles.

Recall from Eq. (33),

$$T_n^m(nz+m;\lambda) = \frac{1}{\lambda^{\frac{m}{2}}} \frac{n!}{2\pi i} \int_C f(w) e^{wnz} \frac{dw}{w^{n+1}}, \tag{41}$$

where $f(w) = 1/cosh^m(w+\delta) = \operatorname{sech}^m(w+\delta)$. Substituting Eq. (40) to Eq. (41) gives

$$T_n^m(nz+m;\lambda) = \frac{1}{\lambda^{\frac{m}{2}}} \frac{n!}{2\pi i} \int_C \left( \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w-w_k)^j} + f_l(w) \right) e^{wnz} \frac{dw}{w^{n+1}}$$

$$= \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w - w_k)^j} e^{wnz} \frac{dw}{w^{n+1}} + \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C f_l(w) e^{wnz} \frac{dw}{w^{n+1}}. \Bigg| \tag{42}$$

Let

$$X_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C f_l(w) e^{wnz} \frac{dw}{w^{n+1}}, \tag{43}$$

$$Y_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w - w_k)^j} e^{wnz} \frac{dw}{w^{n+1}}$$

$$= \lambda^{-\frac{m}{2}} \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{n!}{2\pi i} \int_C \frac{r_{kj}}{(w - w_k)^j} e^{wnz} \frac{dw}{w^{n+1}}. \tag{44}$$

Repeating the process to prove Theorem 3.1 where $f(w)$ there is replaced by $f_l(w)$, we have

$$X_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C f_l(w) e^{n(wz - \log w)} \frac{dw}{w}. \tag{45}$$

Assume that $z^{-1}$ is not a pole of $f_l(w)$. We can expand $f_l(w)$ around the saddle point. That is

$$f_l(w) = \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1})}{k!} \left(w - z^{-1}\right)^k, \left|w - z^{-1}\right| < r \tag{46}$$

where $r$ is the distance from $z^{-1}$ to the nearest singularity of $f_l(w)$. Substitute Eq. (46) to Eq. (43)

$$X_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \frac{n!}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1})}{k!} \left(w - z^{-1}\right)^k e^{wnz} \frac{dw}{w^{n+1}}$$

$$= \lambda^{-\frac{m}{2}} (nz)^n \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1})}{k!} \frac{1}{(nz)^n} \frac{n!}{2\pi i} \int_C \left(w - z^{-1}\right)^k e^{wnz} \frac{dw}{w^{n+1}}$$

$$= \lambda^{-\frac{m}{2}} (nz)^n \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1})}{k!} u_k(n, z),$$

where

$$u_k(n, z) = \frac{1}{(nz)^n} \frac{n!}{2\pi i} \int_C \left(w - z^{-1}\right)^k e^{wnz} \frac{dw}{w^{n+1}}.$$

It follows from Lemma 1 [9] that

$$u_k(n, z) = \frac{p_k(n)}{(nz)^k}, \tag{47}$$

where $p_k(n)$ are the polynomials in Eqs. (11) and (12). Thus,

$$X_l^{n,m}(z) = \lambda^{-\frac{m}{2}} (nz)^n \sum_{k=0}^{\infty} \frac{f_l^{(k)}\left(z^{-1}\right)}{k!} \frac{p_k(n)}{(nz)^k}, \tag{48}$$

valid for $m \in \mathbb{Z}^+, z \in \mathbb{C} \setminus \{0\}$ such that $\left|z^{-1}\right| < \left|z^{-1} - w_j\right|$ for $j = l+1, l+2, \ldots$ given the first $2l$ poles of $f(w)$. From Eq. (40),

$$f_l(w) = f(w) - \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w - w_k)^j}.$$

This gives

$$f_l^{(k)}(w) = f^k(w) - h_l^{(k)}(w),$$

where

$$h_l(w) = -\sum_{k=1}^{l} \sum_{j=1}^{m} \frac{r_{kj}}{(w - w_k)^j}. \tag{49}$$

The expansion of $X_l^{n,m}(z)$ in Eq. (48) becomes

$$X_l^{n,m}(z) = \lambda^{-\frac{m}{2}} (nz)^n \sum_{k=0}^{\infty} \frac{f_l^{(k)}\left(z^{-1}\right) - h_l^{(k)}\left(z^{-1}\right)}{k!} \frac{p_k(n)}{(nz)^k}, \Bigg| \tag{50}$$

valid for $\left|z^{-1}\right| < \left|z^{-1} - w_j\right|, j = l+1, l+2, \ldots$ and $z \neq 0$. This range of validity is larger than that in Theorem 2.1 and Theorem 3.1.

On the other hand, to obtain an expansion for $Y_l^{n,m}(z)$, shift the integration contour in Eq. (44) by $w = w_k + t$. Then $dw = dt$ and

$$\begin{aligned} Y_l^{n,m}(z) &= \lambda^{-\frac{m}{2}} \sum_{k=1}^{l} \sum_{j=1}^{m} \frac{n!}{2\pi i} \int_{C'} \frac{r_{kj}}{t^j} e^{(wk+t)nz} \frac{dt}{(w_k + t)^{n+1}} \\ &= \lambda^{-\frac{m}{2}} \sum_{k=1}^{l} \sum_{j=1}^{m} e^{w_k nz} r_{kj} \frac{n!}{2\pi i} \int_{C'} \frac{e^{tnz}}{t^j} \frac{dt}{(w_k + t)^{n+1}}, \end{aligned} \tag{51}$$

where $C' : t = -w_k + Re^{i\theta}, -\pi < \theta \leq \pi$ is a circle with radius $R$ and center at $-w_k$. Note that 0 is not on the $w_k' s$. This $C'$ is the image of $C : w = Re^{i\theta}$ through the shift $w = w_k + t$. Note that

$$\int_0^z e^{tx} dx = \frac{e^{tx}}{t} \bigg|_0^z = \frac{e^{tz}}{t} - \frac{1}{t},$$

giving

$$\frac{e^{tz}}{t} = \int_0^z e^{tx} dx + \frac{1}{t}.$$

Similarly,

$$\int_0^2 \frac{e^{tx}}{t^{j-1}}dx = \frac{e^{tx}}{t^j}\bigg|_0^z = \frac{e^{tz}}{t^j} - \frac{1}{t}.$$

so that

$$\frac{e^{tz}}{t^j} = \int_0^z \frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j}.$$

It follows that

$$\frac{e^{tnz}}{t^j} = \int_0^{nz} \frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j}.$$

Then Eq. (51) becomes

$$Y_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j} \frac{n!}{2\pi i}\int_{C'}\left(\int_0^{nz}\frac{e^{tx}}{t^{j-1}}dx + \frac{1}{t^j}\right)\frac{dt}{(w_k+t)^{n+1}}. \tag{52}$$

First, we compute

$$\begin{aligned}\frac{n!}{2\pi i}\int_{C'}\frac{e^{tx}}{t^{j-1}}\frac{dt}{(w_k+t)^{n+1}} &= \frac{n!}{2\pi i}\int_{C'} e^{tx} t^{-(j-1)}\frac{dt}{(w_k+t)^{n+1}} \\ &= \frac{d^n}{dt^n}\left(e^{tx}t^{-(j-1)}\right)\bigg|_{t=-w_k}.\end{aligned} \tag{53}$$

Note that when $j=1$, the RHS of Eq. (53) is $x^n$. For $j \geq 1$, we use the Leibniz rule for differentiation.

This gives

$$\frac{d^n}{dt^n}\left(e^{tx}t^{-(j-1)}\right) = \sum_{s=0}^{n}\binom{n}{s} x^{n-s} e^{tx}\frac{d^s}{dt^s}t^{-(j-1)}\Bigg|_{t=-w_k}. \tag{54}$$

It can be computed that

$$\begin{aligned}\frac{d^s}{dt^s}t^{-(j-1)} &= (-1)^s (j-1)\, j\, (j+1)\ldots(j-1+(s-1))\, t^{-(j-1+s)} \\ &= (-1)^s \langle j-1\rangle_s\, t^{-(j-1+s)},\end{aligned}$$

where $\langle j-1\rangle_s$ denotes the rising factorial of $j-1$ with increment $s$. Then Eq. (54) becomes

$$\begin{aligned}\frac{d^n}{dt^n}\left(e^{tx}t^{-(j-1)}\right) &= \sum_{s=0}^{n}\binom{n}{s} x^{n-s} e^{-w_k x}(-1)^s \langle j-1\rangle_s (-w_k)^{-(j-1+s)} \\ &= \sum_{s=0}^{n}\binom{n}{s} x^{n-s} e^{-w_k x}(-1)^{(j-1)} \langle j-1\rangle_s (w_k)^{-(j-1+s)}.\end{aligned}$$

Thus, Eq. (53) can be written

$$\frac{n!}{2\pi i}\int_{C'}\frac{e^{tx}}{t^{j-1}}\frac{dt}{(w_k+t)^{n+1}}=\sum_{s=0}^{n}\binom{n}{s}x^{n-s}e^{-w_kx}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}, \tag{55}$$

while

$$\begin{aligned}\frac{n!}{2\pi i}\int_{C'}t^{-j}\frac{dt}{(w_k+t)^{n+1}}&=\frac{d^n}{dt^n}\left(t^{-j}\right)\Bigg|_{t=-w_k}\\&=(-1)^n\langle j\rangle_n(-w_k)^{-j-n}\\&=(-1)^j\langle j\rangle_n(w_k)^{-(j+n)}\\&=\frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}.\end{aligned} \tag{56}$$

Note also that

$$\int_0^{nz}x^{n-s}e^{-w_kx}dx=\int_0^{nz}t^{n-s}e^{-w_kx}dt.$$

Now the incomplete gamma function

$$\Gamma(a,z)=\int_z^{\infty}e^{-t}t^{a-1}dt,$$

gives

$$\Gamma(n-s+1,w_kz)=\int_{w_kz}^{\infty}e^{-t}t^{n-s}dt.$$

Let $\eta=\frac{t}{w_k}$. Then $t=\eta w_k$ and $w_kd\eta=dt$. Moreover, $t=\infty\Longleftrightarrow\eta=\infty; t=w_kz\Longleftrightarrow\eta=z$.

Consequently,

$$\Gamma(n-s+1,w_kz)=\int_z^{\infty}e^{-w_k\eta}(w_k\eta)^{\eta-s}w_kd\eta$$

$$\begin{aligned}\frac{\Gamma(n-s+1,w_kz)}{w_k^{n-s+1}}&=\int_0^{\infty}e^{-w_k\eta}\eta^{n-s}d\eta\\&=\int_0^{\infty}e^{-w_k\eta}\eta^{n-s}dn-\int_0^{z}e^{-w_k\eta}\eta^{n-s}d\eta\end{aligned}$$

or

$$\int_0^{z}e^{-w_k\eta}\eta^{n-s}d\eta=\int_0^{\infty}e^{-w_k\eta}\eta^{n-s}d\eta-\frac{\Gamma(n-s+1.w_kz)}{w_k^{n-s+1}}.$$

Take note $z \longmapsto nz$. Then

$$\int_0^{nz} e^{-w_k\eta}\eta^{n-s}d\eta = \int_0^{\infty} e^{-w_k\eta}\eta^{n-s}d\eta - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}. \tag{57}$$

Substituting Eqs. (55) and (56) to Eq. (52) yields

$$Y_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j}\left[\left(\left(\int_0^{nz}\sum_{s=0}^{n}\binom{n}{s} x^{n-s}e^{-w_k x}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\right)dx\right.\right.$$
$$\left. + \frac{(-1)^j\langle j\rangle n}{w_k^{j+n}}\right]$$
$$= \lambda^{-\frac{m}{2}} \sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\left(\int_0^{nz} x^{n-s}e^{-w_k x}\right)dx\right]$$
$$+ \frac{(-1)^j\langle \mathrm{j}\rangle_n}{w_k^{j+n}}]. \tag{58}$$

Using Eq. (57) into Eq. (58) we have

$$Y_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle \mathrm{j}-1\rangle_s (w_k)^{-(j-1+s)}\left(\int_0^{\infty} e^{-w_k t}t^{n-s}dt\right.\right.$$
$$\left.\left. - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle \mathrm{j}\rangle_n}{w_k^{j+n}}\right]. \tag{59}$$

Since

$$\int_0^{\infty} t^{n-s}e^{-w_k t}dt = \frac{(n-s)!}{w_k^{n-s+1}}, \quad n \geq s, \tag{60}$$

we can write Eq. (59) as follows:

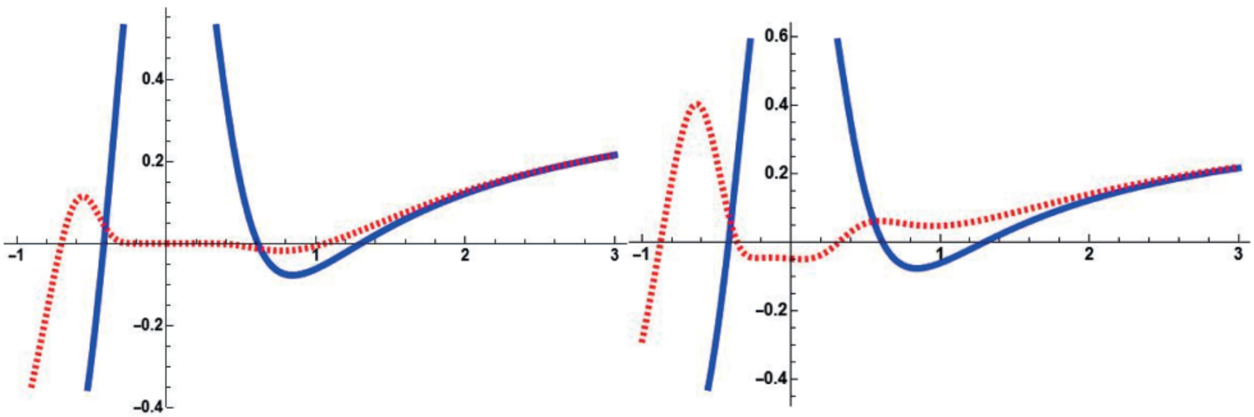
$$Y_l^{n,m}(z) = \lambda^{-\frac{m}{2}} \sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle \mathrm{j}-1\rangle_s (w_k)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}}\right.\right.$$
$$\left.\left. - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle \mathrm{j}\rangle_n}{w_k^{j+n}}\right]. \tag{61}$$

Substituting Eqs. (52) and (61) into Eq. (44) we have

$$T_n^m(nz+m;\lambda) = \lambda^{-\frac{m}{2}}\left\{\sum_{k=1}^{l}\sum_{j=1}^{m} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle \mathrm{j}-1\rangle_s (w_k)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}}\right.\right.\right.$$

$$-\frac{\Gamma\left(n-s+1,w_k nz\right)}{w_k^{n-s+1}}\Bigg)+\frac{(-1)^j\left\langle \mathrm{j}\right\rangle_n}{w_k^{j+n}}\Bigg]+(nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)-h_l^{(k)}\left(z^{-1}\right)}{k!}\frac{p_k\left(n\right)}{(nz)^k}\Bigg\}$$

The comparison of the accuracy of the asymptotic formula obtained in Eq. (30) and Eq. (39) is shown in Fig. 4.



(a) (b)

**Figure 4:** Solid lines in (a) and (b) represent $T_m^n(nx+m;A)$ for $n=3,m=3$, whereas dashed lines in (a) and (b) represent the right hand side of Eqs. (30) and (39), respectively, with $z\equiv x$, both normalized by the factor $\left(1+\left|\frac{x}{a}\right|^n\right)^{-1}$ where we choose $a=0.5$ (a) $n=3,m=3$ *and* $\lambda=3$ (b) $n=3,m=3$ *and* $\lambda=3$

valid for $m\in\mathbb{Z}^+,z\in\mathbb{C}\backslash\{0\}$ such that $|z^{-1}|<\left|z^{-1}-w_k\right|$ for all $k=l+1,l+2,\ldots,$ where the polynomials $p_k(n)$ are given in Eq. (12) and $h_l^{(k)}$ is the $kth$ derivative of $h_l(w)$ given by Eq. (49).

Note that if $m=1$, Eq. (61) reduces to

$$Y_l^n(z)=\lambda^{-\frac{1}{2}}\left(-\sum_{k=1}^{l}\frac{e^{w_k nz}r_{k_j}}{w_k^{n+1}}\Gamma\left(n+1,w_k nz\right)\right),$$

since $\langle 1\rangle_0=1$ *and* $\langle 0\rangle_1=0$. This is exactly the first term Eq. (28).

**Corollary 3.4.** For $z\in\mathbb{C}\backslash\{0\}$ such that $|z^{-1}|<\left|z^{-1}-w_k\right|$ for all $k=l+1,l+2,\ldots,m,n\in\mathbb{Z}^+$, the tangent polynomials of order $m$ satisfy,

$$T_n^m(nz+m)=\sum_{k=1}^{l}\sum_{j=1}^{m}e^{w_k nz}r_{k_j}\Bigg[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\left\langle \mathrm{j}-1\right\rangle_s\left(w_k\right)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}}-\frac{\Gamma\left(n-s+1,w_k nz\right)}{w_k^{n-s+1}}\right)$$
$$+\frac{(-1)^j\left\langle \mathrm{j}\right\rangle_n}{w_k^{j+n}}\Bigg]+(nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)-h_l^{(k)}\left(z^{-1}\right)}{k!}\frac{p_k\left(n\right)}{(nz)^k},\tag{62}$$

where $w_k = (2k+1)\frac{\pi i}{2}$, the polynomials $p_k(n)$ are given in Eq. (12), $h_l^{(k)}$ is the $k$th derivative of the function $h_l(w)$ given by Eq. (49) and

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w-w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k$.

*Proof*. This follows from Theorem 3.3 by taking $\lambda = 1$.

## 4 Conclusion

The saddle-point method and the use of hyperbolic functions are shown to give good approximations to the Apostol-tangent polynomials. Uniform approximations of the Apostol-tangent polynomials and of higher-order Apostol-tangent polynomials are derived. Moreover, approximation formulas with larger region of validity are obtained. The computations to derive the approximation formulas with larger region of validity for the case of Apostol-tangent polynomials of order m are quite tedious and the formulas obtained are original. Corollaries are being stated to explicitly give the corresponding formulas for the special case $\lambda = 1$ and can be used as check formulas of the general case. It will be interesting also to investigate if the methods used in the paper will be applicable to the Apostol-tangent-Bernoulli polynomials and Apostol-tangent-Genocchi polynomials of higher order.

For future research work, one may try to investigate more properties of Apostol-tangent and higher order Apostol-tangent polynomials and establish $q$-analogues of these polynomials (see [20–22]).

**Funding Statement:** This research is funded by Cebu Normal University through its Research Institute for Computational Mathematics and Physics (RICMP).

**Conflicts of Interest:** The authors declare that they have no conflicts of interest to report regarding the present study.

## References

1. Yasmin, G., Muhyi, A. (2020). Certain results of 2-variable q-generalized tangent-Apostol type polynomials. *Journal of Mathematics and Computer Science, 22(3),* 238–251. DOI 10.22436/jmcs.022.03.04.
2. Ryoo, C. S. (2013). A note on the tangent numbers and polynomials. *Advance Studies in Theoretical Physics, 7(9),* 447–454. DOI 10.12988/astp.2013.13042.
3. Ryoo, C. S. (2014). A numerical investigation on the zeros of the tangent polynomials. *Journal of Applied Mathematics and Informatics, 32(3–4),* 315–322. DOI 10.14317/jami.2014.315.
4. Ryoo, C. S. (2016). Differential equations associated with tangent numbers. *Journal of Applied Mathematics and Informatics, 34(5–6),* 487–494. DOI 10.14317/jami.2016.487.
5. Bildirici, C., Acikgoz, M., Araci, S. (2014). A note on analogues of tangent polynomials. *Journal for Algebra and Number Theory Academica, 4(1),* 21.
6. Ryoo, C. S. (2013). Generalized tangent numbers and polynomials associated with p-adic integral on Zp. *Applied Mathematical Sciences, 7,* 4929–4934. DOI 10.12988/ams.2013.36260.
7. Ryoo, C. S. (2013). A note on the symmetric properties for the tangent polynomials. *International Journal of Mathematical Analysis, 7(52),* 2575–2581. DOI 10.12988/ijma.2013.38195.
8. Yasmin, G., Ryoo, C. S., Islahi, H. (2020). A numerical computation of zeros of q-generalized tangent-Appell polynomials. *Mathematics, 8(3),* 383. DOI 10.3390/math8030383.

9. López, J. L., Temme, N. M. (1999). Uniform approximations of Bernoulli and Euler polynomials in terms of hyperbolic functions. *Studies in Applied Mathematics, 103(3),* 241–258. DOI 10.1111/1467-9590.00126.
10. Corcino, C. B., Corcino, R. B., Ontolan, J. M., Castañeda, W. D. (2019). Approximations of Genocchi polynomials in terms of hyperbolic functions. *Journal of Mathematical Analysis, 10(3),* 76–88.
11. Wong, M. W. (2008). *Complex analysis: Series on analysis, applications and computation*. Singapore: World Scientific Publishing Co. Pvt, Ltd.
12. Wong, R. (1989). *Asymptotic approximations of integrals*. New York: Academic Press.
13. Temme, N. M. (1983). Uniform asymptotic expansions of Laplace integrals. *Analysis, 3(1–4),* 221–250. DOI 10.1524/anly.1983.3.14.221.
14. Temme, N. M. (1985). Laplace type integrals: transformation to standard form and uniform asymptotic expansions. *Quarterly of Applied Mathematics, 43(1),* 103–123. DOI 10.1090/qam/782260.
15. Temme, N. M. (1990). Uniform asymptotic expansions of a class of integrals in terms of modified Bessel functions, with application to confluent hypergeometric functions. *SIAM Journal on Mathematical Analysis, 21(1),* 241–261. DOI 10.1137/0521013.
16. Temme, N. M. (1995). Uniform asymptotic expansions of integrals: A selection of problems. *Journal of Computational and Applied Mathematics, 65(1–3),* 395–417. DOI 10.1016/0377-0427(95)00127-1.
17. Corcino, C. B., Corcino, R. B., Ontolan, J. M. (2021). Approximations of tangent polynomials, tangent-bernoulli and tangent-genocchi polynomials in terms of hyperbolic functions. *Journal of Applied Mathematics, 2021(2),* 10. DOI 10.1155/2021/8244000.
18. Korn, G. A., Korn, T. M. (2013). *Mathematical handbook for scientists and engineers*. New York: Dover Publications, Inc.
19. Solomentsev, E. D. (2017). *Encyclopedia of mathematics: Mittag-Leffler theorem*. http://encyclopediaofmath.org/index.php?title=Mittag-Leffler_theorem&oldid=41565.
20. Ryoo, C. S. (2013). On the analogues of Tangent numbers and polynomials associated with p-adic integral on Zp. *Applied Mathematical Sciences, 7(64),* 3177–3183. DOI 10.12988/ams.2013.13277.
21. Ryoo, C. S. (2013). On the twisted q-tangent numbers and polynomials. *Applied Mathematical Sciences, 7(99),* 4935–4941. DOI 10.12988/ams.2013.37386.
22. Ryoo, C. S. (2018). Explicit identities for the generalized tangent polynomials. *Nonlinear Analysis and Differential Equations, 6(1),* 43–51. DOI 10.12988/nade.