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ARTICLE

Degenerate s-Extended Complete and Incomplete Lah-Bell Polynomials

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ABSTRACT

Degenerate versions of special polynomials and numbers applied to social problems, physics, and applied mathematics have been studied variously in recent years. Moreover, the (s-)Lah numbers have many other interesting applications in analysis and combinatorics. In this paper, we divide two parts. We first introduce new types of both degenerate incomplete and complete s-Bell polynomials respectively and investigate some properties of them respectively. Second, we introduce the degenerate versions of complete and incomplete Lah-Bell polynomials as multivariate forms for a new type of degenerate s-extended Lah-Bell polynomials and numbers respectively. We investigate relations between these polynomials and degenerate incomplete and complete s-Bell polynomials, and derive explicit formulas for these polynomials.

KEYWORDS

Lah-Bell numbers and polynomials; s-extended Lah-Bell numbers and polynomials; complete s-Bell polynomials; incomplete s-Bell polynomials; s-Stirling numbers of second kind

Mathematics Subject Classification: 11F20; 11B68; 11B83

1 Introduction

For nonnegative integers $n,\ k,\ s$ such that $n \geq k$, the s-Lah number $L_s(n,k)$ counts the number of partitions of a set with $n+s$ elements into $k+s$ ordered blocks such that s distinguished elements have to be in distinct ordered blocks [1–5]. When $s=0$, the Lah numbers appears non-crossing partitions, Dyck paths as well as falling and rising factorials [6]. As multivariate forms for ordinary Bell polynomials and Stirling numbers of the second kind, respectively, both the complete and incomplete Bell polynomials play important role in combinatorics and number theory. Recently, many mathematicians have been studying various degenerate versions of special polynomials and numbers as well as enumerative combinatorics, probability theory, number theory, etc. [7–17]. In [7], as an example considering the psychological burden of baseball hitters, it well expresses the starting point of degenerate special polynomials and numbers being studied by many scholars. Also, both the complete and incomplete Bell polynomials are multivariate forms for Bell polynomials and Stirling numbers of the second kind, respectively. Beginning with

Bell [18], these polynomials have been studied by many mathematicians [1,16,19,20]. Recently, Kwon et al. [16] introduced the degenerate incomplete and complete s-Bell polynomials and Kim et al. [20] introduced the incomplete and complete s-extended Lah-Bell polynomials, respectively. With this in mind, we want to study the degenerate versions of complete and incomplete Lah-Bell polynomials as multivariate forms for a new type of degenerate s-extended Lah-Bell polynomials and numbers respectively. In Section 2, we consider new types of degenerate incomplete and complete s-Bell polynomials respectively different from those introduced in [16] for our goal. We study several properties and explicit formulas for them. In Section 3, we define both degenerate incomplete and complete s-extended Lah-Bell polynomials associated with a new type of degenerate s-extended Lah-Bell polynomials respectively, and derive relations between these polynomials and degenerate polynomials in the first part. We also investigate explicit formulas for degenerate complete and incomplete s-extended Lah-Bell polynomials, respectively.

First, we introduce some definitions and properties we needed in this paper.

For a nonnegative integer s, the s-Stirling numbers $S_2^{(s)}(n,k)$ of the second kind are given by the generating function

$$\frac{1}{k!}e^{st}(e^t-1)^k=\sum_{n=k}^{\infty}S_n^{(s)}(n+s,k+s)\frac{t^n}{n!},\quad \text{(see [15, 16])}. \tag{1}$$

When $s=0$, $S_2^{(0)}(n,k)=S_2(n,k)$ are the Stirling numbers of the second kind which are the number of ways to partition a set with n elements into k non-empty subsets.

From (8), it is to see that [15,16] the generating function of the s-Bell polynomials is

$$\sum_{n=0}^{\infty}bel_n^{(s)}(x)\frac{t^n}{n!}=e^{x(e^t-1)}e^{st}. \tag{2}$$

When $x=1$, $bel_n^{(s)}(1)=bel_n^{(s)}=\sum_{k=0}^{n}S_2(n+s,k+s)$ are called the s-bell numbers.

When $s=0$, $bel_n^{(0)}(x)=bel_n(x)$ are the ordinary Bell polynomials.

Furthermore, the incomplete s-Bell polynomials.

$B_{n+s,k+s}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)$ are given by the generating function

$$\frac{1}{k!}\left(\sum_{l=1}^{\infty}\beta_l\frac{t^l}{l!}\right)^k\left(\sum_{i=0}^{\infty}\nu_{i+1}\frac{t^i}{i!}\right)^s=\sum_{n=k}^{\infty}B_{n+s,k+s}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!},\quad \text{(see [16])}. \tag{3}$$

When $s=0$, $B_{n+0,k+0}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=B_{n,k}(\beta_1,\beta_2,\cdots,\beta_{n-k+1})$ are the incomplete Bell polynomials. From (10), we obtain immediately that [16]

$$B_{n+s,k+s}^{(s)}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=\sum_{\Lambda(n,k,s)}\left[\frac{n!}{k_1!,k_2!\cdots}\left(\frac{\beta_1}{1!}\right)^{k_1}\left(\frac{\beta_2}{2!}\right)^{k_2}\left(\frac{\beta_3}{3!}\right)^{k_3}\cdots\right] \tag{4}$$
$$\times\left[\frac{s!}{s_0!s_1!s_2!\cdots}\left(\frac{\nu_1}{0!}\right)^{s_0}\left(\frac{\nu_2}{1!}\right)^{s_1}\left(\frac{\nu_3}{2!}\right)^{s_3}\cdots\right],$$

where $\Lambda(n,k,s)$ denotes the set of all nonnegative integers $(k_i)_{i\geq 1}$ and $(s_i)_{i\geq 0}$ such that

$$\sum_{i\geq 1} k_i = k, \quad \sum_{i\geq 0} s_i = s \quad \text{and} \quad \sum_{i\geq 1} i(k_i+s_i) = n.$$

The combinatorial meaning of the incomplete s-Bell polynomials is in the reference [18].

The complete s-Bell polynomials $B_n^{(s)}(x_1,x_2,\cdots : y_1,y_2,\cdots)$ are given by the generation function

$$exp\left(\sum_{l=1}^{\infty}\beta_l\frac{t^l}{l!}\right)\left(\sum_{i=0}^{\infty}\nu_{i+1}\frac{t^i}{i!}\right)^s = \sum_{n=k}^{\infty}B_n^{(s)}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!}, \quad \text{(see [18])}, \tag{5}$$

where $exp(t) = e^t$.

When $s=0$, $B_n^{(0)}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) = B_n(\beta_1,\beta_2,\cdots,\beta_n)$ are the complete Bell polynomials.

Let n, k, s be nonnegative integers with $n \geq k$. It is well known that [2] an explicit formula and the generating function of s-Lah Bell $L_s(n,k)$ are given by, respectively

$$L_s(n,k) = \frac{n!}{k!}\binom{n+2s-1}{k+2s-1},$$

and

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k\left(\frac{1}{1-t}\right)^{2s} = \sum_{n=k}^{\infty}L_s(n,k)\frac{t^n}{n!}. \tag{6}$$

When $s=0$, $L_0(n,k) = L(n,k)$ are the unsigned Lah-numbers.

Kim et al. [2] introduced the s-extended Lah-Bell polynomials $Lb_{n,s}(x)$ given by the generating function

$$e^{x\frac{t}{1-t}}\left(\frac{1}{1-t}\right)^{2s} = \sum_{n=0}^{\infty}Lb_{n,s}(x)\frac{t^n}{n!}. \tag{7}$$

When $x=1$, $Lb_{n,s}(1) = Lb_{n,s} = \sum_{k=0}^{n}L_s(n,k)$ $(n\geq 0)$ are called the s-extended Lah-Bell numbers. When $s=0$, $Lb_{n,0}(x) = Lb_n(x)$ are the Lah-Bell polynomials.

For $\lambda\in\mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \quad \text{and} \quad e_\lambda(t) = \sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}, \quad \text{(see [8-11])}. \tag{8}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

The degenerate fully Bell polynomials are given by

$$e_\lambda(x(e_\lambda(t)-1)) = \sum_{n=0}^{\infty}Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [21])}. \tag{9}$$

When $\lambda\to 0$, $Bel_{n,\lambda}(x) = bel_n(x)$.

In addition, the partially degenerate Bell polynomials are given by

$$e^x(e_\lambda(t)-1)) = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{10}$$

When $\lambda \to 0$, $Bel_{n,\lambda}(x) = bel_n(x)$.

2 A New Type of Degenerate Complete and Incomplete s-Bell Polynomials

In this section, we introduce new types of degenerate complete s-Bell polynomials and degenerate incomplete s-Bell polynomials different from (9) and (10), respectively. We also give some identities and explicit formulas for these polynomials.

For our goal, we introduce a new type of the degenerate extended s-Bell polynomials defined by

$$e_\lambda(x(e^t-1))e_\lambda^s(t) = \sum_{n=0}^{\infty} bel_{n,s,\lambda}(x)\frac{t^n}{n!}. \tag{11}$$

When $x=1$, $bel_{n,s,\lambda} = bel_{n,s,\lambda}(1)$ are called the degenerate extended s-Bell numbers.

When $\lim_{\lambda\to 0} e_\lambda(x(e^t-1))e_\lambda^s(t) = \exp(x(e^t-1))\exp(t) = \sum_{n=0}^{\infty} bel_{n,s}(x)\frac{t^n}{n!}$.

When $s=0$, $bel_{n,0,\lambda}(x) = bel_{n,\lambda}(x)$ are the degenerate Bell polynomials.

Theorem 2.1. For $n,\ s \in \mathbb{N}\cup 0$, we have

$$bel_{n,s,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(1)_{k,\lambda}(s)_{n-m,\lambda}S_2(m,k)x^k.$$

Proof. From (1), (8) and (11), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} bel_{n,s,\lambda}(x)\frac{t^n}{n!} &= e_\lambda(x(e^t-1))e_\lambda^s(t) \\ &= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\frac{1}{k!}(e^t-1)^k e_\lambda^s(t) \\ &= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\sum_{m=k}^{\infty}S_2(m,k)\frac{t^m}{m!}\sum_{j=0}^{\infty}(s)_{j,\lambda}\frac{t^j}{j!} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}(1)_{k,\lambda}x^kS_2(m,k)\frac{t^m}{m!}\sum_{j=0}^{\infty}(s)_{j,\lambda}\frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(1)_{k,\lambda}(s)_{n-m,\lambda}S_2(m,k)x^k\right)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

By comparing with the coefficients of both side of (12), we get the desired result.

Theorem 2.2. For $n,\ s \in \mathbb{N} \cup 0$, we have

$$bel_{n,s,\lambda}(x) = \sum_{k=0}^{\infty}\sum_{h=0}^{n}\sum_{m=0}^{k}\binom{n}{h}\binom{k}{m}m^{n-h}(s)_{h,\lambda}\frac{(1)_{k,\lambda}(-1)^{k-m}}{k!}x^k.$$

Proof. From (8) and (11), we observe that

$$\sum_{n=0}^{\infty} bel_{n,s,\lambda}(x)\frac{t^n}{n!} = e_\lambda(x(e^t-1))e_\lambda^s(t) \tag{13}$$

$$= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\frac{1}{k!}(e^t-1)^k e_\lambda^s(t)$$

$$= \sum_{k=0}^{\infty}(1)_{k,\lambda}x^k\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\sum_{j=0}^{\infty}m^j\frac{t^j}{j!}\sum_{h=0}^{\infty}(s)_{h,\lambda}\frac{t^h}{h!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{h=0}^{n}\sum_{m=0}^{k}\binom{n}{h}\binom{k}{m}m^{n-h}(s)_{h,\lambda}\frac{(1)_{k,\lambda}(-1)^{k-m}}{k!}x^k\right)\frac{t^n}{n!}.$$

By comparing with the coefficients of both side of (13), we get the desired result.

First, we define a new type of the degenerate complete Bell polynomials $B_n^\lambda(\beta_1, \beta_2, \cdots, \beta_n)$ associated with the degenerate Bell polynomials $bel_{n,\lambda}(x)$ by

$$e_\lambda\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right) = \sum_{n=0}^{\infty}W_n^\lambda(\beta_1, \beta_2, \cdots, \beta_n)\frac{t^n}{n!}, \tag{14}$$

and a new type of the degenerate incomplete Bell polynomials $W_{n,k}^\lambda(\beta_1, \beta_2, \cdots, \beta_{n-k+1})$ associated with some degenerate Stirling numbers defined by

$$\frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^k = \sum_{n=k}^{\infty}W_{n,k}^\lambda(\beta_1, \beta_2, \cdots, \beta_{n-k+1})\frac{t^n}{n!}, \quad (n \geq k \geq 0). \tag{15}$$

Theorem 2.3. For $n \geq k \geq 0$, we have

$$W_0^\lambda(\beta_1, \beta_2, \cdots, \beta_n) = 1$$

$$W_n^\lambda(\beta_1, \beta_2, \cdots, \beta_n) = \sum_{k=1}^{n}(1)_{k,\lambda}B_{n,k}(\beta_1, \beta_2, \cdots, \beta_{n-k+1}) \quad \text{if } n \geq 1.$$

In particular, we have $W_n^\lambda(x, x, \cdots, x) = bel_{n,\lambda}(x)$.

Proof. From (3), (4) and (14), we have

$$\sum_{n=0}^{\infty} W_n^{\lambda}(\beta_1,\beta_2,\cdots,\beta_n)\frac{t^n}{n!} = e_{\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right) = 1+\sum_{k=1}^{\infty}(1)_{k,\lambda}\frac{1}{k!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^k \tag{16}$$

$$= 1+\sum_{k=1}^{\infty}(1)_{k,\lambda}\sum_{n=k}^{\infty}B_{n,k}(\beta_1,\beta_2,\cdots,\beta_{n-k+1})\frac{t^n}{n!}$$

$$= 1+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1)_{k,\lambda}B_{n,k}(\beta_1,\beta_2,\cdots,\beta_{n-k+1})\right)\frac{t^n}{n!}.$$

Therefore, by comparing with coefficients of both sides of (16), we have the desired result.

In particular, from (16), we have

$$\sum_{n=0}^{\infty} W_n^{\lambda}(x,x,\cdots,x)\frac{t^n}{n!} = e_{\lambda}\left(x\sum_{h=1}^{\infty}\frac{t^h}{h!}\right) = e_{\lambda}(x(e^t-1)) = \sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{17}$$

Thus, by comparing with coefficients of both sides of (17), we have

$W_n^{\lambda}(x,x,\ldots,x) = bel_{n,\lambda}(x)$.

In next theorem, we obtain a new type of degenerate Stirling numbers of second kind $(1)_{k,\lambda}S_2(n,k)$.

Theorem 2.4. For $n \geq k \geq 0$, we have

$W_0^{\lambda}(\beta_1,\beta_2,\ldots,\beta_n) = 1$

$$W_n^{\lambda}(\beta_1,\beta_2,\ldots,\beta_n) = \sum_{k=1}^{n} W_{n,k}^{\lambda}(\beta_1,\beta_2,\cdots,\beta_{n-k+1}) \quad \text{if } n \geq 1.$$

In particular, we get $W_{n,k}^{\lambda}(1,1,\ldots,1) = (1)_{k,\lambda}S_2(n,k)$.

Proof. From (8), (14) and (15), we observe that

$$\sum_{n=0}^{\infty} W_n^{\lambda}(\beta_1,\beta_2,\cdots,\beta_n)\frac{t^n}{n!} = e_{\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^k \tag{18}$$

$$= 1+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} W_{n,k}^{\lambda}(\beta_1,\beta_2,\cdots,\beta_{n-k+1})\frac{t^n}{n!}$$

$$= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n} W_{n,k}^{\lambda}(\beta_1,\beta_2,\cdots,\beta_{n-k+1})\frac{t^n}{n!}.$$

Therefore, by comparing with coefficients of both sides of (18), we have what we want.

In addition, from (18), we get

$$\sum_{n=k}^{\infty} W_{n,k}^{\lambda}(1,1,\cdots,1) = (1)_{k,\lambda}\frac{1}{k!}(e^t-1)^k = (1)_{k,\lambda}\sum_{n=k}^{\infty} S_2(n,k).$$

Thus, we have $W_{n,k}^{\lambda}(1,1,\cdots,1) = (1)_{k,\lambda}S_2(n,k)$.

Next, for $\lambda \in \mathbb{R}$, we consider a new type of degenerate incomplete s-Bell polynomials defined by

$$W_{n+s,k+s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) = \frac{1}{k!}\left(\sum_{h=1}^{\infty}(1)_{k,\lambda}\beta_h\frac{t^h}{h!}\right)^k\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s. \tag{19}$$

From (4) and (19), we have the following explicit formula

For $n \geq k \geq 0$, we have

$$\begin{aligned} &W_{n+s,k+s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) \\ &=\sum_{n=k}^{\infty} B_{n+s,k+s}((1)_{k,\lambda}\beta_1,(1)_{k,\lambda}\beta_2,\cdots:(1)_{0,\lambda}\nu_1,(1)_{1,\lambda}\nu_2,\cdots)\frac{t^n}{n!} \\ &= \sum_{\Lambda(n,k,\lambda)}\left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{(1)_{k,\lambda}\beta_1}{1!}\right)^{k_1}\left(\frac{(1)_{k,\lambda}\beta_1}{2!}\right)^{k_2}\cdots\right] \\ &\times\left[\frac{r!}{s_0!s_1!\cdots}\left(\frac{(1)_{0,\lambda}\nu_1}{0!}\right)^{s_0}\left(\frac{(1)_{1,\lambda}\nu_2}{1!}\right)^{s_1}\cdots\right], \end{aligned} \tag{20}$$

where $\Lambda(n,k,s)$ denote the set of all nonnegative integers $(k_i)_{i\geq 1}$ and $(s_i)_{i\geq 1}$ such that

$$\sum_{i\geq 1} k_i = k, \quad \sum_{i\geq 0} s_i = s \quad \text{and} \quad \sum_{i\geq 1} i(k_i+s_i) = n$$

Naturally, we define a new type of the degenerate complete s-Bell polynomials by

$$W_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) = e_{\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s, \tag{21}$$

where $\lambda \in \mathbb{R}$ and $n,\ k \in \mathbb{N}$ with $n \geq k$.

We note that

$$\lim_{\lambda\to\infty} W_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) = B_n^{(s)}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots).$$

From (19) and (21), we note that
$W_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots) = \sum_{k=0}^{n} W_{n+s,k+s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)$.

Theorem 2.5. For $n,\ k,\ s \geq 0$ with $n \geq k$, we have
$W_{n+s,k+s}^{\lambda}\big(1,1,\cdots:1,1,\cdots\big) = (1)_{k,\lambda}S_2(n,k)e_{\lambda}^{s}(t)$.

Proof. From (1), (8) and (19), we have

$$\sum_{n=k}^{\infty} W_{n+s,k+s}^{\lambda}(1,1,\cdots:1,1,\cdots)\frac{t^n}{n!} \tag{22}$$

$$=\frac{1}{k!}\left(\sum_{h=1}^{\infty}(1)_{k,\lambda}\frac{t^h}{h!}\right)^k\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)^s$$

$$=(1)_{k,\lambda}\frac{1}{k!}(e^t-1)e_{\lambda}^s(t)=\sum_{n=k}^{\infty}(1)_{k,\lambda}S_2(n,k)e_{\lambda}^s(t).$$

Therefore, by comparing with coefficients of both side of (22), we obtain the desired result.
In Theorem 2.5, we obtain a new type of degenerate s-extended Stirling number of second.

Theorem 2.6. For $n \geq k \geq 0$, we have

$$W_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=n!\left(\sum_{k=0}^{n}\sum_{b_1+2b_2+\cdots+kb_k=k}\sum_{c_1+c_2+\cdots+c_s=n-k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}k!}{b_1!\,b_2!\cdots b_k!}\left(\frac{\beta_1}{1!}\right)^{b_1}\left(\frac{\beta_2}{2!}\right)^{b_2}\cdots\left(\frac{\beta_k}{k!}\right)^{b_k}\frac{\prod_{i=1}^{s}(1)_{c_i,\lambda}\nu_{c_{i+1}}}{c_1!\,c_2!\cdots c_s!}\right).$$

Proof. By using (8), we observe that

$$e_{\lambda}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)=\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{1}{k!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^k \tag{23}$$

$$=1+(1)_{1,\lambda}\frac{1}{1!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)+(1)_{2,\lambda}\frac{1}{2!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^2+(1)_{3,\lambda}\frac{1}{3!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^3+\cdots$$

$$=1+(1)_{1,\lambda}\frac{1}{1!}\frac{\beta_1}{1!}t+\left((1)_{1,\lambda}\frac{1}{1!}\frac{\beta_2}{2!}+(1)_{2,\lambda}\frac{1}{2!}\frac{\beta_1^2}{1!\,1!}\right)t^2$$

$$+\left((1)_{1,\lambda}\frac{1}{1!}\frac{\beta_3}{3!}+(1)_{2,\lambda}\frac{1}{2!}\frac{2\beta_1^2\beta_2}{1!\,2!}+(1)_{3,\lambda}\frac{1}{3!}\frac{\beta_1^3}{1!\,1!\,1!}\right)t^3+\cdots$$

$$=\sum_{k=0}^{\infty}\sum_{b_1+2b_2+\cdots+kb_k=k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}k!}{b_1!\,b_2!\cdots b_k!}\left(\frac{\beta_1}{1!}\right)^{b_1}\left(\frac{\beta_2}{2!}\right)^{b_2}\cdots\left(\frac{\beta_k}{k!}\right)^{b_k}\frac{t^k}{k!},$$

and

$$\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s=\sum_{j=0}^{\infty}\sum_{c_1+c_2+\cdots+c_s=j}\frac{j!}{c_1!\,c_2!\cdots c_s!}\left(\prod_{i=1}^{s}(1)_{c_i,\lambda}\nu_{c_{i+1}}\right)\frac{t^j}{j!}. \tag{24}$$

From (23) and (24), we get

$$e_\lambda\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)\sum_{m=0}^{\infty}\left((1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s \tag{25}$$

$$=n!\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{b_1+2b_2+\cdots+kb_k=k}\sum_{c_1+c_2+\cdots+c_s=n-k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}k!}{b_1!b_2!\cdots b_k!}\left(\frac{\beta_1}{1!}\right)^{b_1}\left(\frac{\beta_2}{2!}\right)^{b_2}\cdots\cdots\left(\frac{\beta_k}{k!}\right)^{b_k}\cdot\frac{\prod_{i=1}^{s}(1)_{c_i,\lambda}\nu_{c_{i+1}}}{c_1!c_2!\cdots c_s!}\right)\frac{t^n}{n!}.$$

On the other hand, from (21), we have

$$e_\lambda\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s=\sum_{n=0}^{\infty}W_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!}. \tag{26}$$

Thus, by comparing with coefficients of (25) and (26), we have what we want.

Next, we consider the extended degenerate complete s-Bell polynomials defined by the generating function

$$e_\lambda\left(z\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s=\sum_{n=0}^{\infty}W_n^{(s),\lambda}(z|\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!}. \tag{27}$$

Theorem 2.7. For $n\geq k\geq 0$, we have

$$W_n^{(s),\lambda}(z\ |\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=\sum_{k=0}^{n}z^kW_{n+s,k+s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots).$$

When $z=1$, we get

$$W_n^{(s),\lambda}(1|\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=\sum_{k=0}^{n}W_{n+s,k+s}^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots).$$

Proof. From (19) and (27), we have

$$\sum_{n=0}^{\infty}W_n^{(s),\lambda}(z\ |\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!} \tag{28}$$

$$=\sum_{k=0}^{\infty}z^k(1)_{k,\lambda}\frac{1}{k!}\left(\sum_{h=1}^{\infty}\beta_h\frac{t^h}{h!}\right)^k\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}\frac{t^m}{m!}\right)^s$$

$$=\sum_{k=0}^{\infty}z^k\sum_{n=k}^{\infty}W_{n+s,k+s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} z^k W_{n+s,k+s}^{(s),\lambda}(\beta_1, \beta_2, \cdots : \nu_1, \nu_2, \cdots)\right)\frac{t^n}{n!}.$$

Thus, by comparing with coefficients of both sides of (28), we have what we want.

Theorem 2.8. For $n, \ k, \ s \geq 0$ with $n \geq k$, we have

$$W_n^{(s),\lambda}(z \mid 1, 1, \cdots : 1, 1, \cdots) = bel_{n,s,\lambda}(z).$$

Proof. From (27), we observe that

$$e_\lambda\left(z\sum_{l=1}^{\infty}\frac{t^h}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)^s \tag{29}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} z^k W_{n+s,k+s}^{(s)}\big((1)_{k,\lambda}, (1)_{k,\lambda}, \cdots : (1)_{0,\lambda}, (1)_{1,\lambda}, \cdots\big)\frac{t^n}{n!}.$$

On the other hand, from (11), we get

$$e_\lambda\left(z \sum_{l=1}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)^s = e_\lambda\big(z(e^t - 1)\big)e_\lambda^s(t) = \sum_{n=0}^{\infty} bel_{n,s,\lambda}(z)\frac{t^n}{n!}. \tag{30}$$

Thus, from (29) and (30), we get the desired result.

3 Degenerate *s*-Extended Complete and Incomplete Lah-Bell Polynomials

In this section, we introduce a new type of the degenerate Lah-Bell polynomials different from Kim-Kim's in [8] and define both the *s*-extended complete and incomplete degenerate Lah-Bell polynomials associated with a new type of the degenerate Lah-Bell polynomials. We also demonstrate some interesting properties related to these polynomials and explicit formulas for them.

We consider a new type of the degenerate Lah-Bell polynomials $Lb_{n,\lambda}(x)$ given by the generating function

$$e_\lambda\left(x\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} Lb_{n,\lambda}(x)\frac{t^n}{n!}. \tag{31}$$

When $x = 1, \ Lb_{n,\lambda}(1) = Lb_{n,\lambda} \quad (n \geq 0)$ are called the degenerate Lah-Bell numbers (see Figs. 1 and 2).

When $\lambda \to 0, \ Lb_{n,\lambda}(x) = Lb_n(x)$.

In view of the ordinary Bell polynomials, the degenerate 2*s*-extended Lah-Bell polynomials are defined by the generating function

$$\sum_{n=k}^{\infty} Lb_{n,2s,\lambda}(x) = e_\lambda^x\left(\frac{t}{1-t}\right)e_\lambda^{2s}(t). \tag{32}$$

When $x = 1$, $Lb_{n,2s,\lambda} = Lb_{n,2s,\lambda}(1)$ are called the degenerate extended $2s$-extended Lah-Bell numbers numbers. When $s = 0$, the degenerate complete s-extended Lah-Bell polynomials are the degenerate Lah-Bell polynomials.

Next, we introduce the degenerate complete Lah-Bell polynomials $LW_n^{\lambda}(x_1, x_2, \cdots, x_n)$ defined by the generating function

$$e_{\lambda}\left(\sum_{h=1}^{\infty}\beta_h t^h\right) = \sum_{n=0}^{\infty} LW_n^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n)\frac{t^n}{n!}. \tag{33}$$

We note that

$$\sum_{n=0}^{\infty} LW_n^{\lambda}(x, x, \cdots, x)\frac{t^n}{n!} = e_{\lambda}\left(x\sum_{h=1}^{\infty}t^h\right) = e_{\lambda}\left(x\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} Lb_{n,\lambda}(x)\frac{t^n}{n!}. \tag{34}$$

From (31), we have $LW_n^{\lambda}(x, \cdots, x) = Lb_{n,\lambda}(x)$ and $LW_n^{\lambda}(1, 1, \cdots, 1) = Lb_{n,\lambda}$.

From (20) and (34), we get

$$\begin{aligned}\sum_{n=0}^{\infty} LW_n^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n)\frac{t^n}{n!} \\ = e_{\lambda}\left(\sum_{h=1}^{\infty} h!\,\beta_h\frac{t^h}{h!}\right) = \sum_{n=0}^{\infty} W_n^{\lambda}(1!\,\beta_1, 2!\,\beta_2, \cdots, n!\,\beta_n)\frac{t^n}{n!}.\end{aligned} \tag{35}$$

By (14), (15), (35) and Theorem 2.3, we obtain the following theorem.

Theorem 3.1. For $n \geq k \geq 0$, we have

$$LW_0^{\lambda}(\beta_1, \beta_2, \cdots, x_n) = 1 \quad \textit{and}$$

$$\begin{aligned}LW_n^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n) &= W_n^{\lambda}(1!\,\beta_1, 2!\,\beta_2, \cdots, n!\,\beta_n) \\ &= \sum_{k=1}^{n}(1)_{k,\lambda}B_{n,k}(1!\,\beta_1, 2!\,\beta_2, \cdots, (n-k+1)!\,\beta_{n-k+1}).\end{aligned}$$

Naturally, we can define a new type of the degenerate incomplete Lah-Bell polynomials $LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1})$ are defined by the generating function

$$\frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty}\beta_h t^h\right)^k = \sum_{n=k}^{\infty} LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1})\frac{t^n}{n!}, \quad (n \geq k \geq 0). \tag{36}$$

Note that when $\lambda \to 0$, $LW_{n,k}^{\lambda}(1, 1, \cdots, 1) = Lb_n$ are the Lah-bell numbers.

From (15) and (36), we observe that

$$LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1}) = W_{n,k}^{\lambda}(1!\,\beta_1, 2!\,\beta_2, \cdots, (n-k+1)!\,\beta_{n-k+1}). \tag{37}$$

Theorem 3.2. For $n \geq k \geq 0$, we have

$$LW_0^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n) = 1,$$

$$LW_n^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n) = \sum_{k=1}^{n} LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1}) \quad \text{if} \quad n \geq 1.$$

Proof.

From (8), (36) and (37), we observe that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} LW_n^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n)\frac{t^n}{n!} = e_{\lambda}\left(\sum_{h=1}^{\infty} \beta_h t^h\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty} \beta_h t^h\right)^k \\ &= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1})\frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} LW_{n,k}^{\lambda}(\beta_1, \beta_2, \cdots, \beta_{n-k+1})\frac{t^n}{n!}\right). \end{aligned} \tag{38}$$

Therefore, by comparing with coefficients of both side of (38), we get the desired identity.

We define the degenerate s-extended incomplete Lah-Bell polynomials

$LW_{n+2s,k+2s}^{\lambda}(\beta_1, \beta_2, \cdots : \nu_1, \nu_2, \cdots)$ by the generating function

$$\frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty} \beta_h t^h\right)^k \left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}t^m\right)^{2s} = \sum_{n=k}^{\infty} LW_{n+2s,k+2s}^{\lambda}(\beta_1, \beta_2, \cdots : \nu_1, \nu_2, \cdots)\frac{t^n}{n!}. \tag{39}$$

When $s = 0$, the degenerate incomplete s-extended Lah-Bell polynomials are the degenerate incomplete Lah-Bell polynomials.

From (19) and Theorem 2.5, we get easily the following explicit formula.

Theorem 3.3. For $n \geq k \geq 0$, we have

$$\begin{aligned} &LW_{n+2s,k+2s}^{\lambda}(\beta_1, \beta_2, \cdots : \nu_1, \nu_2, \cdots) \\ &\quad = W_{n+2s,k+2s}^{\lambda}(1!\,\beta_1, 2!\,\beta_2, \cdots : 0!\,\nu_1, 1!\,\nu_2, \cdots) \\ &\quad = \sum_{\Lambda(n,k,2r)}\left[\frac{n!}{k_1!, k_2!\cdots}{\beta_1}^{k_1}{\beta_2}^{k_2}\cdots\right]\left[\frac{(2s)!}{s_0!\,s_1!\cdots}{\nu_1}^{s_0}{\nu_2}^{s_1}\cdots\right], \end{aligned}$$

where $\Lambda(n,k,2s)$ denote the set of all nonnegative integers $\{k_i\}_{i\geq 1}$ and $\{s_i\}_{i\geq 0}$ such that

$$\sum_{i\geq 1} k_i = k, \quad \sum_{i\geq 0} s_i = 2s \quad \text{and} \quad \sum_{i\geq 1} i(k_i + s_i) = n.$$

We also define the degenerate s-extended complete Lah-Bell polynomials

$LW_n^{(2s),\lambda}(z|\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)$ by the generating function

$$e_\lambda\left(z\sum_{h=1}^{\infty}\beta_h t^h\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}t^m\right)^{2s}=\sum_{n=0}^{\infty}LW_n^{(2s),\lambda}(z|\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!}. \tag{40}$$

Theorem 3.4. For $n\geq k\geq 0$, we have

$$LW_{n+2s,k+2s}^{\lambda}(x,x,x,\cdots:\frac{1}{0!}\frac{1}{1!}\frac{1}{2!}\cdots)=Lb_{n,2s,\lambda}(x).$$

When $x=1$, we have

$$LW_{n+2s,k+2s}^{\lambda}(1,1,1,\cdots:\frac{1}{0!}\frac{1}{1!}\frac{1}{2!}\cdots)=Lb_{n,2s,\lambda}.$$

Proof. From (32) and (39), we have

$$\sum_{n=k}^{\infty}LW_{n+2s,k+2s}^{\lambda}(x,x,x,\cdots:\frac{1}{0!}\frac{1}{1!}\frac{1}{2!}\cdots)\frac{t^n}{n!}=\frac{1}{k!}(1)_{k,\lambda}\left(\sum_{h=1}^{\infty}t^h\right)^k\left(x^k\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)^{2s} \tag{41}$$

$$=e_\lambda^x\left(\frac{t}{1-t}\right)e_\lambda^{2s}(t)=\sum_{n=k}^{\infty}Lb_{n,2s,\lambda}(x).$$

Therefore, by comparing with coefficients of both side of (41), we get the desired result.

From (19), (39) and (40), we note that

$$LW_n^{(2s),\lambda}(z|\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)=\sum_{k=0}^{n}z^k LW_{n+2s,k+2s}^{\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)$$

$$=\sum_{k=0}^{n}z^k W_{n+2s,k+2s}^{\lambda}(1!\,\beta_1,2!\,\beta_2,\cdots:0!\,\nu_1,1!\,\nu_2,\cdots),$$

for $n\geq k\geq 0$.

Theorem 3.5. For $n\geq k\geq 0$ and $s\geq 0$, we have

$$LW_n^{(s),\lambda}(\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)$$

$$=n!\left(\sum_{k=0}^{n}\sum_{b_1+2b_2+\cdots+kb_k=k}\sum_{c_1+c_2+\cdots+c_{2s}=n-k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}(\beta_1)^{b_1}(\beta_2)^{b_2}\cdots(\beta_k)^{b_k}}{b_1!\,b_2!\cdots b_k!}\prod_{i=1}^{2s}(1)_{c_i,\lambda}\nu_{c_{i+1}}\right).$$

Proof. From (17), we have

$$e_\lambda\left(\sum_{h=1}^{\infty}\beta_h t^h\right)=\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{1}{k!}\left(\sum_{h=1}^{\infty}\beta_h t^h\right)^k \tag{42}$$

$$= 1 + (1)_{1,\lambda}\frac{1}{1!}\left(\sum_{h=1}^{\infty}\beta_h t^h\right) + (1)_{2,\lambda}\frac{1}{2!}\left(\sum_{h=1}^{\infty}\beta_h t^h\right)^2 + (1)_{3,\lambda}\frac{1}{3!}\left(\sum_{h=1}^{\infty}\beta_h t^h\right)^3 + \cdots$$

$$= 1 + \left((1)_{1,\lambda}\frac{1}{1!}\beta_1\right)t + \left((1)_{1,\lambda}\frac{1}{1!}\beta_2 + (1)_{2,\lambda}\frac{1}{2!}\beta_1^2\right)t^2$$

$$+ \left((1)_{1,\lambda}\frac{1}{1!}\beta_3 + (1)_{2,\lambda}\frac{1}{2!}2\beta_1\beta_2 + (1)_{3,\lambda}\frac{1}{3!}\beta_1^3\right)t^3 + \cdots$$

$$= \sum_{k=0}^{\infty}\sum_{b_1+2b_2+\cdots+kb_k=k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}}{b_1!\,b_2!\cdots b_k!}(\beta_1)^{b_1}(\beta_2)^{b_2}\cdots(\beta_k)^{b_k}t^k,$$

and

$$\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}t^m\right)^{2s} = \sum_{j=0}^{\infty}\sum_{c_1+c_2+\cdots+c_{2s}=j}\left(\prod_{i=1}^{2s}(1)_{c_i,\lambda}\nu_{c_{i+1}}\right)t^j. \tag{43}$$

By (42) and (43), we have

$$\sum_{n=0}^{\infty}LW_n^{(2s),\lambda}(1\ |\beta_1,\beta_2,\cdots:\nu_1,\nu_2,\cdots)\frac{t^n}{n!} = e_\lambda\left(\sum_{h=1}^{\infty}\beta_h t^h\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\nu_{m+1}t^m\right)^{2s} \tag{44}$$

$$= n!\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{b_1+2b_2+\cdots+kb_k=k}\sum_{c_1+c_2+\cdots+c_{2s}=n-k}\frac{(1)_{b_1+b_2+\cdots+b_k,\lambda}(\beta_1)^{b_1}(\beta_2)^{b_2}\cdots(\beta_k)^{b_k}}{b_1!\,b_2!\cdots b_k!}\prod_{i=1}^{2s}(1)_{c_i,\lambda}\nu_{c_{i+1}}\right)\frac{t^n}{n!}.$$

Therefore, by comparing with coefficients of both side of (44), we get the desired result.

Remark. We recall the degenerate Lah-Bell numbers $Lb_{n,\lambda}$ as follows (31):

$$e_\lambda\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty}Lb_{n,\lambda}\frac{t^n}{n!}.$$

In the following figures (x-axis = t, y-axis = $e_\lambda(\frac{t}{1-t})$) in which the simulation program uses Fortran language, We can see the change $Lb_{n,\lambda}$ depending on λ.

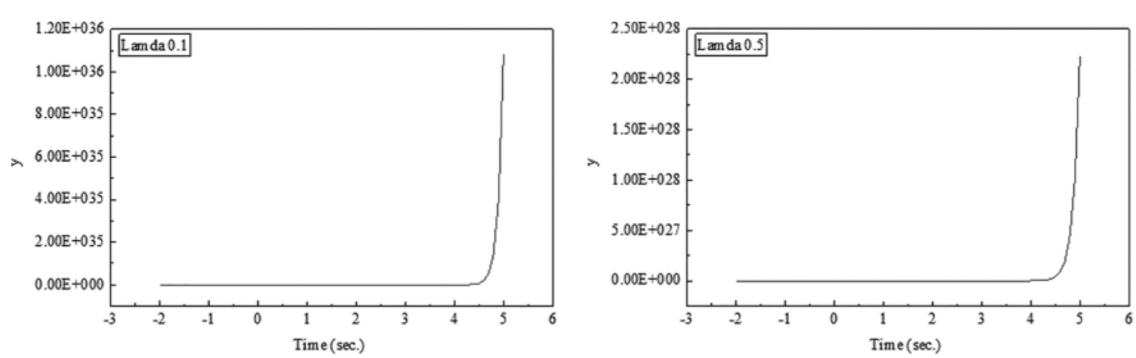

Figure 1: Degenerate Lah-Bell numbers when $\lambda = 0.1$ and $\lambda = 0.5$, respectively

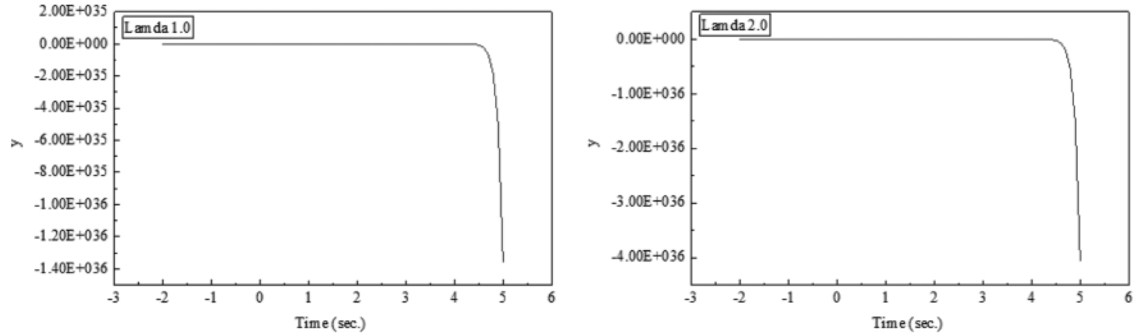

Figure 2: Degenerate Lah-Bell numbers when $\lambda = 1.0$ and $\lambda = 2.0$, respectively

4 Conclusion

In this paper, we introduced both the degenerate s-extended incomplete and complete Lah-Bell polynomials associated with a new type of degenerate s-extended Lah-Bell polynomials. We demonstrated some combinatorial identities between these polynomials and polynomials introduced in Section 2, and explicit formulas for them respectively. In addition, we obtained new types of the degenerate Stirling numbers and s-extended Stirling numbers of the second kind in Theorem 2.4 and 2.5, respectively.

Special polynomials have been applied not only in mathematics and physics, but also in various fields of application [1,3,6,9,17,18,22–27]. In recent years, one of our research areas has been to explore some special numbers and polynomials and their degenerate versions, and to discover their arithmetical and combinatorial properties and some of their applications. We intend to study various degenerate polynomial and numbers using several means such as function generation, combinatorial methods, umbral calculus, differential equations, and probability theory.

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