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ARTICLE

Determinantal Expressions and Recursive Relations for the Bessel Zeta Function and for a Sequence Originating from a Series Expansion of the Power of Modified Bessel Function of the First Kind

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Dedicated to Professor Dr. Mourad E. H. Ismail at University of Central Florida in USA

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ABSTRACT

In the paper, by virtue of a general formula for any derivative of the ratio of two differentiable functions, with the aid of a recursive property of the Hessenberg determinants, the authors establish determinantal expressions and recursive relations for the Bessel zeta function and for a sequence originating from a series expansion of the power of modified Bessel function of the first kind.

KEYWORDS

Determinantal representation; recursive relation; series expansion; first kind modified Bessel function; Bessel zeta function; Pochhammer symbol; gamma function; Hessenberg determinant

1 Introduction and Motivations

Recall from [1], and [2,3] that the classical Euler gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

Recall from [4], that the modified Bessel function of the first kind $I_\nu(z)$ is represented by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C}, \tag{1}$$

where $\nu \in \mathbb{C}\setminus\{-1,-2,\ldots\}$ is said to be the order of $I_\nu(z)$. Recall from [1], that the generalized combinatorial number (or say, generalized binomial coefficient) is denoted and defined by

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z, w, z-w \in \mathbb{C}\setminus\{-1,-2,\ldots\}; \\ 0, & z \in \mathbb{C}\setminus\{-1,-2,\ldots\}, w, z-w \in \{-1,-2,\ldots\}. \end{cases}$$

Concretely and explicitly, the power series expansion

$$[I_\nu(z)]^2 = \sum_{k=0}^{\infty} \frac{1}{[\Gamma(\nu+k+1)]^2} \binom{2k+2\nu}{k} \left(\frac{z}{2}\right)^{2k+2\nu}$$

was listed [5], For $\nu \in \mathbb{C}\setminus\{-1,-2,\ldots\}$ and $r, z \in \mathbb{C}$, the main result in [5], reads that

$$[I_\nu(z)]^r = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)[\Gamma(\nu+1)]^{r-1}} \frac{B_k(\nu,r)}{k!} \left(\frac{z}{2}\right)^{2k+r\nu}, \tag{2}$$

where

$$B_k(\nu,r) = \frac{r(\nu+k)}{\nu+1} B_{k-1}(\nu,r) + \frac{\Gamma(\nu+2)}{k} \sum_{j=2}^{k} \frac{b_j(\nu)}{\Gamma(\nu+j+2)} \binom{\nu+k}{j} B_{k-j}(\nu,r) \tag{3}$$

and

$$\sum_{k=0}^{\infty} \frac{b_{k+1}(\nu)}{\Gamma(\nu+k+2)} \frac{x^k}{k!} = \frac{1}{\Gamma(\nu+2)} \left[\frac{\sqrt{x}}{\nu+1} \frac{I_\nu(2\sqrt{x})}{I_{\nu+1}(2\sqrt{x})} - 2 \right] \tag{4}$$

with the convention that the sum is zero if the starting index exceeds the finishing index. By the way, in the paper [6], there are new conclusions and applications on series expansions of powers of several fundamental elementary functions. In [5], the first five expressions for $B_k(\nu, r)$ were listed as follows:

$$B_0(\nu,r) = 1, \quad B_1(\nu,r) = r, \quad B_2(\nu,r) = \frac{\nu+2}{\nu+1} r^2 - \frac{1}{\nu+1} r,$$

$$B_3(\nu,r) = \frac{(\nu+2)(\nu+3)}{(\nu+1)^2} r^3 - \frac{3(\nu+3)}{(\nu+1)^2} r^2 + \frac{4}{(\nu+1)^2} r,$$

$$B_4(\nu,r) = \frac{(\nu+2)(\nu+3)(\nu+4)}{(\nu+1)^3} r^4 - \frac{6(\nu+3)(\nu+4)}{(\nu+1)^3} r^3 + \frac{(\nu+4)(19\nu+41)}{(\nu+1)^3(\nu+2)} r^2 - \frac{6(5\nu+11)}{(\nu+1)^3(\nu+2)} r.$$

In [7], the recursive relation (3) was simplified as

$$B_k(\nu,r) = \frac{\Gamma(\nu+k+1)\Gamma(\nu+1)}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{j(r+1)-k}{\Gamma(\nu+j+1)\Gamma(\nu+k-j+1)} B_{k-j}(\nu,r)$$

in which the sequence $b_k(\nu)$ for $k \in \mathbb{N}$ is not involved.

In [8], Theorem 2.3, alternative recursive relations

$$B_k(\nu, r+1) = \frac{(r+1)\Gamma(\nu+k+1)\Gamma(\nu+1)}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{kB_{k-j}(\nu, r)}{\Gamma(\nu+j+1)\Gamma(\nu+k-j+1)}$$

and

$$B_k(\nu, r+1) = B_k(\nu, r) + \Gamma(\nu+k+1)\Gamma(\nu+1) \sum_{j=1}^{k} \binom{k}{j} \frac{B_{k-j}(\nu, r)}{\Gamma(\nu+j+1)\Gamma(\nu+k-j+1)}$$

were derived via a probabilistic interpretation of the series expansion of powers of a general series.

In [9], the complete Bell polynomials, denoted by $\mathrm{B}_n(a_1, a_2, \ldots, a_n)$, are defined by

$$\exp\left(\sum_{k=1}^{\infty} \frac{a_k}{k!} z^k\right) = \sum_{n=0}^{\infty} \mathrm{B}_n(a_1, a_2, \ldots, a_n) \frac{z^n}{n!}.$$

By the way, in the article [10], some new results on the Bell polynomials of the second kind are surveyed and reviewed. Let $j_{\nu,n}$ for $n \in \mathbb{N}$ denote the zeros of $\frac{J_\nu(z)}{z^\nu}$, where $J_\nu(z)$ is the first kind Bessel function which can be represented as

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\nu+n+1)}, \quad z \in \mathbb{C},$$

where $\nu \in \mathbb{C} \setminus \{-1, -2, \ldots\}$ is called the order of $J_\nu(z)$. The Bessel zeta function

$$\zeta_\nu(q) = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^q} \tag{5}$$

for $q > 1$ was originally introduced and studied in [11–14]. In [8], there are the following special values:

$$\begin{aligned} &\zeta_\nu(2) = \frac{1}{4(\nu+1)}, \quad \zeta_\nu(4) = \frac{1}{16(\nu+1)^3}, \quad \zeta_\nu(6) = \frac{1}{16(\nu+1)^4(2\nu+3)}, \\ &\zeta_\nu(8) = \frac{10\nu+11}{256(\nu+1)^6\left(2\nu^2+7\nu+6\right)}. \end{aligned} \tag{6}$$

Theorems 3.1 and 3.2 in [8] read that

$$B_k(\nu, r) = 2^{2k} \frac{\Gamma(\nu+k+1)}{\Gamma(\nu+1)} B_k\left(0!\, r\zeta_\nu(2), -1!\, r\zeta_\nu(4), \ldots, (-1)^{k-1}(k-1)!\, r\zeta_\nu(2k)\right)$$

and

$$\frac{B_k(\nu, r)}{\Gamma(\nu+k+1)} = r \sum_{j=0}^{k-1} (-1)^k k!\, 2^{2j+2} \zeta_\nu(2j+2) \binom{k-1}{j} \frac{B_{k-j-1}(\nu, r)}{\Gamma(\nu+k-j)}.$$

Corollary 4.2 in [8] confirms that $B_k(\nu, r)$ is a polynomial in r of degree k.

One of the reasons why ones investigated the series expansion (2) and the sequence $B_k(\nu, r)$ is that the products of the first kind Bessel functions and of the first kind modified Bessel functions appear frequently in problems of statistical mechanics and plasma physics considered in [15–17]. This reason has been mentioned in [5,7].

In the papers [8,18] and in Entry A131490 of The On-Line Encyclopedia of Integer Sequences, the sequence $b_{k+1}(\nu)$ generated in (4) has been studied. In [8], there are two concrete values

$$b_1(\nu) = -1 \quad \text{and} \quad b_2(\nu) = \frac{1}{\nu+1}. \tag{7}$$

Theorem 5.1 in [8] reads that

$$(-1)^{k+1} b_{k+1}(0) = k!\,(k+1)!\,2^{2k}\zeta_1(2k).$$

Corollary 5.2 in [8] asserts that the number $b_{k+1}(0)$ is an integer. Theorem 5.4 in [8] reads that

$$b_k(\nu) = (-1)^k \frac{(k-1)!\,\Gamma(\nu+k+1)}{(\nu+1)\,\Gamma(\nu+2)} 2^{2k-2}\zeta_{\nu+1}(2k-2), \quad k \ge 2 \tag{8}$$

and confirms that, due to the second value in (7), the sequence $b_k(\nu)$ for $k \in \mathbb{N}$ is not an integer sequence.

In this paper, we will establish determinantal expressions and recursive relations of the sequences $b_{k+1}(\nu)$, $b_{k+1}(0)$, and $\zeta_\nu(2k)$ for $k \in \mathbb{N}$. It is clear that, if all elements in determinants are closed forms, determinantal expressions are also closed forms.

2 Determinantal Representations via Ratios of Gamma Functions

We are now in a position to establish determinantal expressions of the sequences $b_{k+1}(\nu)$, $b_{k+1}(0)$, and $\zeta_\nu(2k)$ for $k \in \mathbb{N}$.

Theorem 2.1. For $k \in \mathbb{N}$ and $\nu \in \mathbb{C}\setminus\{-1, -2, \ldots\}$, the sequence $b_{k+1}(\nu)$ can be determinantally represented as

$$b_{k+1}(\nu) = \frac{(2k)!!}{(2k-1)!!}\binom{\nu+k+1}{k}\frac{[\Gamma(\nu+2)]^{2k+1}}{\nu+1}\left|P_{2k+1,1}(\nu) \quad Q_{2k+1,2k}(\nu)\right|_{(2k+1)\times(2k+1)}, \tag{9}$$

where

$$P_{2k+1,1}(\nu) = \left(\frac{1-(-1)^i}{2}\frac{(i-2)!!}{2^{(i-1)/2}}\frac{1}{\Gamma\left(\nu+\frac{i-1}{2}+1\right)}\right)_{1\le i\le 2k+1} \tag{10}$$

with $(-1)!! = 1$ is a $(2k+1)\times 1$ matrix and, with the convention $\binom{n}{m} = 0$ for $n < m$,

$$Q_{2k+1,2k}(\nu) = \left(\frac{1+(-1)^{i-j}}{2}\binom{i-1}{j-1}\frac{(i-j-1)!!}{2^{(i-j)/2}}\frac{1}{\Gamma\left(\nu+\frac{|i-j|}{2}+2\right)}\right)_{1\le i\le 2k+1, 1\le j\le 2k} \tag{11}$$

is a $(2k+1)\times(2k)$ matrix.

Proof. Replacing $2\sqrt{x}$ by t in the power series expansion (4) yields

$$\sum_{k=0}^{\infty}\frac{b_{k+1}(\nu)}{\Gamma(\nu+k+2)}\frac{1}{k!}\left(\frac{t}{2}\right)^{2k}=\frac{1}{\Gamma(\nu+2)}\left[\frac{t}{2(\nu+1)}\frac{I_\nu(t)}{I_{\nu+1}(t)}-2\right],$$

$$\frac{tI_\nu(t)}{I_{\nu+1}(t)}=4(\nu+1)+2(\nu+1)\Gamma(\nu+2)\sum_{k=0}^{\infty}\frac{b_{k+1}(\nu)}{\Gamma(\nu+k+2)}\frac{1}{k!}\left(\frac{t}{2}\right)^{2k},$$

$$\frac{tI_\nu(t)}{I_{\nu+1}(t)}=2(\nu+1)[2+b_1(\nu)]+2(\nu+1)\Gamma(\nu+2)\sum_{k=1}^{\infty}\frac{b_{k+1}(\nu)}{\Gamma(\nu+k+2)}\frac{1}{k!}\left(\frac{t}{2}\right)^{2k}.$$

This implies that

$$b_1(\nu)=\frac{1}{2(\nu+1)}\lim_{t\to0}\frac{tI_\nu(t)}{I_{\nu+1}(t)}-2$$

and

$$b_{k+1}(\nu)=\frac{2^{k-1}}{(2k-1)!!}\frac{\Gamma(\nu+k+2)}{(\nu+1)\Gamma(\nu+2)}\lim_{t\to0}\left[\frac{tI_\nu(t)}{I_{\nu+1}(t)}\right]^{(2k)} \tag{12}$$

for $k\in\mathbb{N}$. From (1), it follows that

$$\frac{tI_\nu(t)}{I_{\nu+1}(t)}=\frac{2\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{t}{2}\right)^{2k}}{\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+2)}\left(\frac{t}{2}\right)^{2k}}\to2(\nu+1),\quad t\to0.$$

Hence, we obtain $b_1(\nu)=-1$, which confirms the first value in (7).

Let

$$\phi_\nu(t)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{t}{2}\right)^{2k}$$

and

$$\varphi_\nu(t)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+2)}\left(\frac{t}{2}\right)^{2k}.$$

Then

$$[\phi_\nu(t)]^{(\ell)}=\frac{1}{2^\ell}\sum_{k=\frac{\ell}{2}+\frac{1-(-1)^\ell}{2}}^{\infty}\frac{(2k)!}{(2k-\ell)!k!\Gamma(\nu+k+1)}\left(\frac{t}{2}\right)^{2k-\ell}\to\begin{cases}\dfrac{(2m-1)!!}{2^m\Gamma(\nu+m+1)}, & \ell=2m\\ 0, & \ell=2m+1\end{cases}$$

and

$$[\varphi_\nu(t)]^{(\ell)} = \frac{1}{2^\ell} \sum_{k=\frac{\ell}{2}+\frac{1-(-1)^\ell}{2}}^{\infty} \frac{(2k)!}{(2k-\ell)!\,k!\,\Gamma(\nu+k+2)} \left(\frac{t}{2}\right)^{2k-\ell} \to \begin{cases} \dfrac{(2m-1)!!}{2^m\Gamma(\nu+m+2)}, & \ell = 2m \\ 0, & \ell = 2m+1 \end{cases}$$

as $t \to 0$, where $m \in \{0\} \cup \mathbb{N}$ and $(-1)!! = 1$.

In [19], there exists a general formula

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{p(t)}{q(t)}\right] = \frac{(-1)^k}{q^{k+1}(t)} \begin{vmatrix} p(t) & q(t) & 0 & \cdots & 0 & 0 \\ p'(t) & q'(t) & q(t) & \cdots & 0 & 0 \\ p''(t) & q''(t) & \binom{2}{1} q'(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p^{(k-2)}(t) & q^{(k-2)}(t) & \binom{k-2}{1} q^{(k-3)}(t) & \cdots & q(t) & 0 \\ p^{(k-1)}(t) & q^{(k-1)}(t) & \binom{k-1}{1} q^{(k-2)}(t) & \cdots & \binom{k-1}{k-2} q'(t) & q(t) \\ p^{(k)}(t) & q^{(k)}(t) & \binom{k}{1} q^{(k-1)}(t) & \cdots & \binom{k}{k-2} q''(t) & \binom{k}{k-1} q'(t) \end{vmatrix} \tag{13}$$

for $k \geq 0$. By the way, this formula has been extensively applied in recent years, see [20,21] and closely related references therein. Applying $p(t)$ and $q(t)$ in (13) to $\phi_\nu(t)$ and $\varphi_\nu(t)$ results in

$$\begin{aligned} \lim_{t\to 0}\left[\frac{tI_\nu(t)}{I_{\nu+1}(t)}\right]^{(2k)} &= 2\lim_{t\to 0}\left[\frac{\phi_\nu(t)}{\varphi_\nu(t)}\right]^{(2k)} \\ &= 2\lim_{t\to 0}\frac{(-1)^{2k}}{\varphi_\nu^{2k+1}(t)} \begin{vmatrix} \phi_\nu(t) & \varphi_\nu(t) & 0 & \cdots & 0 & 0 \\ \phi_\nu'(t) & \varphi_\nu'(t) & \varphi_\nu(t) & \cdots & 0 & 0 \\ \phi_\nu''(t) & \varphi_\nu''(t) & \binom{2}{1}\varphi_\nu'(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_\nu^{(2k-2)}(t) & \varphi_\nu^{(2k-2)}(t) & \binom{2k-2}{1}\varphi_\nu^{(2k-3)}(t) & \cdots & \varphi_\nu(t) & 0 \\ \phi_\nu^{(2k-1)}(t) & \varphi_\nu^{(2k-1)}(t) & \binom{2k-1}{1}\varphi_\nu^{(2k-2)}(t) & \cdots & \binom{2k-1}{2k-2}\varphi_\nu'(t) & \varphi_\nu(t) \\ \phi_\nu^{(2k)}(t) & \varphi_\nu^{(2k)}(t) & \binom{2k}{1}\varphi_\nu^{(2k-1)}(t) & \cdots & \binom{2k}{2k-2}\varphi_\nu''(t) & \binom{2k}{2k-1}\varphi_\nu'(t) \end{vmatrix} \\ &= \frac{2}{\varphi_\nu^{2k+1}(0)} \begin{vmatrix} \phi_\nu(0) & \varphi_\nu(0) & 0 & \cdots & 0 & 0 \\ \phi_\nu'(0) & \varphi_\nu'(0) & \varphi_\nu(0) & \cdots & 0 & 0 \\ \phi_\nu''(0) & \varphi_\nu''(0) & \binom{2}{1}\varphi_\nu'(0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_\nu^{(2k-2)}(0) & \varphi_\nu^{(2k-2)}(0) & \binom{2k-2}{1}\varphi_\nu^{(2k-3)}(0) & \cdots & \varphi_\nu(0) & 0 \\ \phi_\nu^{(2k-1)}(0) & \varphi_\nu^{(2k-1)}(0) & \binom{2k-1}{1}\varphi_\nu^{(2k-2)}(0) & \cdots & \binom{2k-1}{2k-2}\varphi_\nu'(0) & \varphi_\nu(0) \\ \phi_\nu^{(2k)}(0) & \varphi_\nu^{(2k)}(0) & \binom{2k}{1}\varphi_\nu^{(2k-1)}(0) & \cdots & \binom{2k}{2k-2}\varphi_\nu''(0) & \binom{2k}{2k-1}\varphi_\nu'(0) \end{vmatrix} \end{aligned}$$

$$
=2[\Gamma(\nu+2)]^{2k+1}\begin{vmatrix}
\frac{1}{\Gamma(\nu+1)} & \frac{1}{\Gamma(\nu+2)} & 0 & \cdots & 0 & 0\\
0 & 0 & \frac{1}{\Gamma(\nu+2)} & \cdots & 0 & 0\\
\frac{1}{2\Gamma(\nu+2)} & \frac{1}{2\Gamma(\nu+3)} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2k-3)!!}{2^{k-1}\Gamma(\nu+k)} & \frac{(2k-3)!!}{2^{k-1}\Gamma(\nu+k+1)} & 0 & \cdots & \frac{1}{\Gamma(\nu+2)} & 0\\
0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}\Gamma(\nu+k+1)} & \cdots & 0 & \frac{1}{\Gamma(\nu+2)}\\
\frac{(2k-1)!!}{2^{k}\Gamma(\nu+k+1)} & \frac{(2k-1)!!}{2^{k}\Gamma(\nu+k+2)} & 0 & \cdots & \binom{2k}{2k-2}\frac{1}{2\Gamma(\nu+3)} & 0
\end{vmatrix}.
$$

Accordingly, we acquire

$$
b_{k+1}(\nu)=\frac{(2k)!!}{(2k-1)!!}\binom{\nu+k+1}{\nu+1}\frac{[\Gamma(\nu+2)]^{2k+1}}{\nu+1}
$$

$$
\times\begin{vmatrix}
\frac{(-1)!!}{2^0}\frac{1}{\Gamma(\nu+1)} & \frac{(-1)!!}{2^0}\frac{1}{\Gamma(\nu+2)} & 0 & \cdots & 0 & 0\\
0 & 0 & \frac{(-1)!!}{2^0}\frac{1}{\Gamma(\nu+2)} & \cdots & 0 & 0\\
\frac{1!!}{2^1}\frac{1}{\Gamma(\nu+2)} & \frac{1!!}{2^1}\frac{1}{\Gamma(\nu+3)} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2k-3)!!}{2^{k-1}}\frac{1}{\Gamma(\nu+k)} & \frac{(2k-3)!!}{2^{k-1}}\frac{1}{\Gamma(\nu+k+1)} & 0 & \cdots & \frac{(-1)!!}{2^0}\frac{1}{\Gamma(\nu+2)} & 0\\
0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}}\frac{1}{\Gamma(\nu+k+1)} & \cdots & 0 & \frac{(-1)!!}{2^0}\frac{1}{\Gamma(\nu+2)}\\
\frac{(2k-1)!!}{2^{k}}\frac{1}{\Gamma(\nu+k+1)} & \frac{(2k-1)!!}{2^{k}}\frac{1}{\Gamma(\nu+k+2)} & 0 & \cdots & \binom{2k}{2k-2}\frac{1!!}{2^1}\frac{1}{\Gamma(\nu+3)} & 0
\end{vmatrix}
$$

which can be rearranged as the form in (9). The proof of Theorem 2.1 is complete.

Theorem 2.2. For $k\in\mathbb{N}$, the sequence $b_{k+1}(0)$ can be determinantally represented as

$$
\frac{b_{k+1}(0)}{k+1}=\frac{(2k)!!}{(2k-1)!!}\left|P_{2k+1,1}(0)\quad Q_{2k+1,2k}(0)\right|_{(2k+1)\times(2k+1)}, \tag{14}
$$

where

$$
P_{2k+1,1}(0)=\left(\frac{1-(-1)^i}{2}\frac{(i-2)!!}{2^{(i-1)/2}}\frac{1}{\Gamma\left(\frac{i+1}{2}\right)}\right)_{1\le i\le 2k+1}
$$

and

$$
Q_{2k+1,2k}(0)=\left(\frac{1+(-1)^{i-j}}{2}\binom{i-1}{j-1}\frac{(i-j-1)!!}{2^{(i-j)/2}}\frac{1}{\Gamma\left(2+\frac{|i-j|}{2}\right)}\right)_{1\le i\le 2k+1,1\le j\le 2k}
$$

with $(-1)!!=1$ and the convention $\binom{n}{m}=0$ for $n<m$.

Proof. This can be deduced from taking $\nu=0$ in Theorem 2.1 and reformulating it for intuitive and visual beauty.

Theorem 2.3. For $k \in \mathbb{N}$ and $\nu \in \mathbb{C}\setminus\{-1,-2,\dots\}$, the values at $q = 2k$ of the Bessel zeta function $\zeta_{\nu+1}(q)$ can be determinantally represented as

$$\zeta_{\nu+1}(2k) = (-1)^{k+1}\frac{[\Gamma(\nu+2)]^{2k+1}}{(2k)!}\left|P_{2k+1,1}(\nu) \quad Q_{2k+1,2k}(\nu)\right|_{(2k+1)\times(2k+1)}, \tag{15}$$

where the matrices $P_{2k+1,1}(\nu)$ and $Q_{2k+1,2k}(\nu)$ are defined by (10) and (11), respectively.

Proof. Combining (8) with (9) in Theorem 2.1 results in

$$\begin{aligned} b_{k+1}(\nu) &= (-1)^{k+1}\frac{k!\,\Gamma(\nu+k+2)}{(\nu+1)\Gamma(\nu+2)}2^{2k}\zeta_{\nu+1}(2k) \\ &= \frac{(2k)!!}{(2k-1)!!}\binom{\nu+k+1}{k}\frac{[\Gamma(\nu+2)]^{2k+1}}{\nu+1}\left|P_{2k+1,1}(\nu) \quad Q_{2k+1,2k}(\nu)\right|_{(2k+1)\times(2k+1)}. \end{aligned}$$

Further simplifying gives (15). The proof of Theorem 2.3 is complete.

3 Determinantal Representations via the Pochhammer Symbols

For $z \in \mathbb{C}$ and $n \in \{0\} \cup \mathbb{N}$, the Pochhammer symbol $(z)_n$, or say, the rising factorial $(z)_n$, is defined in [6,22] and [1], by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \prod_{\ell=0}^{n-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \ge 1; \\ 1, & n = 0. \end{cases} \tag{16}$$

In terms of the Pochhammer symbol $(z)_n$ defined by (16), we can rewrite Theorems 2.1–2.3 for intuitive and visual beauty respectively as follows:

Theorem 3.1. For $k \in \mathbb{N}$ and $\nu \in \mathbb{C}\setminus\{-1,-2,\dots\}$, the sequence $b_{k+1}(\nu)$ can be determinantally represented as

$$b_{k+1}(\nu) = \frac{2^k(\nu+2)_k}{(2k-1)!!} \times \begin{vmatrix} \frac{1}{(\nu+1)_0} & \frac{1}{(\nu+2)_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{(\nu+2)_0} & \cdots & 0 & 0 \\ \frac{1}{2(\nu+1)_1} & \frac{1}{2(\nu+2)_1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(2k-3)!!}{2^{k-1}(\nu+1)_{k-1}} & \frac{(2k-3)!!}{2^{k-1}(\nu+2)_{k-1}} & 0 & \cdots & \frac{1}{(\nu+2)_0} & 0 \\ 0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}(\nu+2)_{k-1}} & \cdots & 0 & \frac{1}{(\nu+2)_0} \\ \frac{(2k-1)!!}{2^k(\nu+1)_k} & \frac{(2k-1)!!}{2^k(\nu+2)_k} & 0 & \cdots & \binom{2k}{2k-2}\frac{1}{2(\nu+2)_1} & 0 \end{vmatrix}. \tag{17}$$

Proof. In the proof of Theorem 2.1, we can write

$$\lim_{t\to 0}\left[\frac{tI_\nu(t)}{I_{\nu+1}(t)}\right]^{(2k)}=2(\nu+1)$$
$$\times\begin{vmatrix}
\frac{(-1)!!}{2^0}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} & \frac{(-1)!!}{2^0}\frac{\Gamma(\nu+2)}{\Gamma(\nu+2)} & 0 & \cdots & 0 & 0\\
0 & 0 & \frac{(-1)!!}{2^0}\frac{\Gamma(\nu+2)}{\Gamma(\nu+2)} & \cdots & 0 & 0\\
\frac{1!!}{2^1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} & \frac{1!!}{2^1}\frac{\Gamma(\nu+2)}{\Gamma(\nu+3)} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2k-3)!!}{2^{k-1}}\frac{\Gamma(\nu+1)}{\Gamma(\nu+k)} & \frac{(2k-3)!!}{2^{k-1}}\frac{\Gamma(\nu+2)}{\Gamma(\nu+k+1)} & 0 & \cdots & \frac{(-1)!!}{2^0}\frac{\Gamma(\nu+2)}{\Gamma(\nu+2)} & 0\\
0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}}\frac{\Gamma(\nu+2)}{\Gamma(\nu+k+1)} & \cdots & 0 & \frac{(-1)!!}{2^0}\frac{\Gamma(\nu+2)}{\Gamma(\nu+2)}\\
\frac{(2k-1)!!}{2^k}\frac{\Gamma(\nu+1)}{\Gamma(\nu+k+1)} & \frac{(2k-1)!!}{2^k}\frac{\Gamma(\nu+2)}{\Gamma(\nu+k+2)} & 0 & \cdots & \binom{2k}{2k-2}\frac{1!!}{2^1}\frac{\Gamma(\nu+2)}{\Gamma(\nu+3)} & 0
\end{vmatrix}.$$

Substituting this equation into (12) and considering the definition in (16) lead to (17). The proof of Theorem 3.1 is complete.

Theorem 3.2. For $k\in\mathbb{N}$, the sequence $b_{k+1}(0)$ can be determinantally represented as

$$b_{k+1}(0)=\frac{1}{2}\frac{[2(k+1)]!!}{(2k-1)!!}$$
$$\times\begin{vmatrix}
\frac{(-1)!!}{0!!} & \frac{(-1)!!}{1\times 0!!} & 0 & \cdots & 0 & 0\\
0 & 0 & \frac{(-1)!!}{1\times 0!!} & \cdots & 0 & 0\\
\frac{1!!}{2!!} & \frac{1!!}{2\times 2!!} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2k-3)!!}{(2k-2)!!} & \frac{(2k-3)!!}{k(2k-2)!!} & 0 & \cdots & \frac{(-1)!!}{1\times 0!!} & 0\\
0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{k(2k-2)!!} & \cdots & 0 & \frac{(-1)!!}{1\times 0!!}\\
\frac{(2k-1)!!}{(2k)!!} & \frac{(2k-1)!!}{(k+1)(2k)!!} & 0 & \cdots & \binom{2k}{2k-2}\frac{1!!}{2\times 2!!} & 0
\end{vmatrix}, \tag{18}$$

where $(-1)!! = 1$.

Proof. This can be deduced from letting $\nu = 0$ in Theorem 3.1 and reformulating it for intuitive and visual beauty.

Theorem 3.3. For $k\in\mathbb{N}$ and $\nu\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$, the sequence $\zeta_\nu(2k)$ can be determinantally represented as

$$\zeta_\nu(2k)=\frac{(-1)^{k+1}\nu}{(2k)!}$$

$$\times\begin{vmatrix} \frac{1}{(\nu)_0} & \frac{1}{(\nu+1)_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{(\nu+1)_0} & \cdots & 0 & 0 \\ \frac{1}{2(\nu)_1} & \frac{1}{2(\nu+1)_1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(2k-3)!!}{2^{k-1}(\nu)_{k-1}} & \frac{(2k-3)!!}{2^{k-1}(\nu+1)_{k-1}} & 0 & \cdots & \frac{1}{(\nu+1)_0} & 0 \\ 0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}(\nu+1)_{k-1}} & \cdots & 0 & \frac{1}{(\nu+1)_0} \\ \frac{(2k-1)!!}{2^k(\nu)_k} & \frac{(2k-1)!!}{2^k(\nu+1)_k} & 0 & \cdots & \binom{2k}{2k-2}\frac{1}{2(\nu+1)_1} & 0 \end{vmatrix}. \tag{19}$$

Proof. Combining (8) with (17) in Theorem 3.1 results in

$$b_{k+1}(\nu) = (-1)^{k+1}\frac{k!\,\Gamma(\nu+k+2)}{(\nu+1)\,\Gamma(\nu+2)}2^{2k}\zeta_{\nu+1}(2k) = \frac{2^k(\nu+2)_k}{(2k-1)!!}$$

$$\times\begin{vmatrix} \frac{1}{(\nu+1)_0} & \frac{1}{(\nu+2)_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{(\nu+2)_0} & \cdots & 0 & 0 \\ \frac{1}{2(\nu+1)_1} & \frac{1}{2(\nu+2)_1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(2k-3)!!}{2^{k-1}(\nu+1)_{k-1}} & \frac{(2k-3)!!}{2^{k-1}(\nu+2)_{k-1}} & 0 & \cdots & \frac{1}{(\nu+2)_0} & 0 \\ 0 & 0 & \binom{2k-1}{1}\frac{(2k-3)!!}{2^{k-1}(\nu+2)_{k-1}} & \cdots & 0 & \frac{1}{(\nu+2)_0} \\ \frac{(2k-1)!!}{2^k(\nu+1)_k} & \frac{(2k-1)!!}{2^k(\nu+2)_k} & 0 & \cdots & \binom{2k}{2k-2}\frac{1}{2(\nu+2)_1} & 0 \end{vmatrix}.$$

Further simplifying gives (19). The proof of Theorem 3.3 is complete.

4 Recursive Relations

In this section, we establish recursive relations of the sequences $b_{k+1}(\nu)$ and $(-1)^{k+1}2^k\zeta_\nu(2k)$ for $k\in\mathbb{N}$.

Theorem 4.1. For $k\ge 2$ and $\nu\in\mathbb{C}\setminus\{-1,-2,\dots\}$, the sequence $b_{k+1}(\nu)$ has the recursive relation

$$b_{k+1}(\nu) = \frac{k}{\nu+1} - \frac{\Gamma(\nu+k+2)\,\Gamma(\nu+2)}{(2k-1)!!}\sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!\,(2\ell-3)!!}{\Gamma(\nu+k-\ell+3)\,\Gamma(\nu+\ell+1)}b_\ell(\nu) \tag{20}$$

Consequently, the sequence $(-1)^{k+1}2^k\zeta_\nu(2k)$ for $k\ge 2$ and $\nu\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$ satisfies the recursive relation

$$(-1)^{k+1}2^k\zeta_\nu(2k) = \frac{k}{(2k)!!}\frac{\Gamma(\nu+1)}{\Gamma(\nu+k+1)} - \Gamma(\nu+1)\sum_{\ell=2}^{k}\frac{(-1)^\ell 2^{\ell-1}\zeta_\nu(2\ell-2)}{(2k-2\ell+2)!!\,\Gamma(\nu+k-\ell+2)}. \tag{21}$$

Proof. Let $D_0 = 1$ and

$$D_n = \begin{vmatrix} e_{1,1} & e_{1,2} & 0 & \dots & 0 & 0 \\ e_{2,1} & e_{2,2} & e_{2,3} & \dots & 0 & 0 \\ e_{3,1} & e_{3,2} & e_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{n-2,1} & e_{n-2,2} & e_{n-2,3} & \dots & e_{n-2,n-1} & 0 \\ e_{n-1,1} & e_{n-1,2} & e_{n-1,3} & \dots & e_{n-1,n-1} & e_{n-1,n} \\ e_{n,1} & e_{n,2} & e_{n,3} & \dots & e_{n,n-1} & e_{n,n} \end{vmatrix} \tag{22}$$

for $n \in \mathbb{N}$. In [23], it was proved that the Hessenberg determinant D_n for $n \geq 0$ satisfies $D_1 = e_{1,1}$ and

$$D_n = \sum_{r=1}^{n} (-1)^{n-r} e_{n,r} \left(\prod_{j=r}^{n-1} e_{j,j+1} \right) D_{r-1} \tag{23}$$

for $n \geq 1$, where the product is 1 if the starting index exceeds the finishing index. Replacing n by $2k+1$ for $k \geq 0$, letting

$$e_{i,1} = \frac{1-(-1)^i}{2} \frac{(i-2)!!}{2^{(i-1)/2}} \frac{1}{\Gamma\left(\nu + \frac{i-1}{2} + 1\right)}$$

for $1 \leq i \leq 2k+1$, and taking

$$e_{i,j+1} = \frac{1+(-1)^{i-j}}{2} \binom{i-1}{j-1} \frac{(i-j-1)!!}{2^{(i-j)/2}} \frac{1}{\Gamma\left(\nu + \frac{|i-j|}{2} + 2\right)}$$

for $1 \leq i \leq 2k+1$ and $1 \leq j \leq 2k$ in (23) yield

$$\begin{aligned} D_{2k+1} &= \sum_{r=1}^{2k+1} (-1)^{2k-r+1} e_{2k+1,r} \left(\prod_{j=r}^{2k} e_{j,j+1} \right) D_{r-1} \\ &= (-1)^{2k} e_{2k+1,1} \left(\prod_{j=1}^{2k} e_{j,j+1} \right) D_0 + \sum_{r=2}^{2k+1} (-1)^{2k-r+1} e_{2k+1,r} \left(\prod_{j=r}^{2k} e_{j,j+1} \right) D_{r-1} \\ &= \frac{(2k-1)!!}{2^k} \frac{1}{\Gamma(\nu+k+1)} \prod_{j=1}^{2k} \frac{1}{\Gamma(\nu+2)} \\ &\quad - \sum_{r=2}^{2k+1} \frac{1+(-1)^r}{2} \binom{2k}{r-2} \frac{(2k-r+1)!!}{2^{(2k-r+2)/2}} \frac{1}{\Gamma\left(\nu + \frac{|2k-r+2|}{2} + 2\right)} \left[\prod_{j=r}^{2k} \frac{1}{\Gamma(\nu+2)} \right] D_{r-1} \\ &= \frac{(2k-1)!!}{2^k \Gamma(\nu+k+1) [\Gamma(\nu+2)]^{2k}} - \sum_{\ell=1}^{k} \binom{2k}{2\ell-2} \frac{(2k-2\ell+1)!!}{2^{k-\ell+1} \Gamma(\nu+k-\ell+3) [\Gamma(\nu+2)]^{2k-2\ell+1}} D_{2\ell-1} \end{aligned}$$

$$= \frac{(2k-1)!!}{2^k\Gamma(\nu+k+1)[\Gamma(\nu+2)]^{2k}} - \frac{(2k-1)!!}{2^k\Gamma(\nu+k+2)[\Gamma(\nu+2)]^{2k-1}}D_1$$
$$-\sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!}{2^{k-\ell+1}\Gamma(\nu+k-\ell+3)[\Gamma(\nu+2)]^{2k-2\ell+1}}D_{2\ell-1}$$
$$= \frac{(2k-1)!!}{2^k\Gamma(\nu+k+1)[\Gamma(\nu+2)]^{2k}} - \frac{(2k-1)!!}{2^k\Gamma(\nu+k+2)[\Gamma(\nu+2)]^{2k-1}\Gamma(\nu+1)}$$
$$-\sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!}{2^{k-\ell+1}\Gamma(\nu+k-\ell+3)[\Gamma(\nu+2)]^{2k-2\ell+1}}D_{2\ell-1}$$
$$= \frac{(2k-1)!!k}{2^k\Gamma(\nu+k+2)[\Gamma(\nu+2)]^{2k}} - \sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!}{2^{k-\ell+1}\Gamma(\nu+k-\ell+3)[\Gamma(\nu+2)]^{2k-2\ell+1}}D_{2\ell-1}$$

for $k \geq 2$. Further setting

$$D_{2k+1} = \frac{(2k-1)!!}{(2k)!!}\frac{1}{\binom{\nu+k+1}{k}}\frac{\nu+1}{[\Gamma(\nu+2)]^{2k+1}}b_{k+1}(\nu)$$

for $k \in \mathbb{N}$ produces

$$\frac{(2k-1)!!}{(2k)!!}\frac{1}{\binom{\nu+k+1}{k}}\frac{\nu+1}{[\Gamma(\nu+2)]^{2k+1}}b_{k+1}(\nu) = \frac{(2k-1)!!k}{2^k\Gamma(\nu+k+2)[\Gamma(\nu+2)]^{2k}}$$
$$-\sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!}{2^{k-\ell+1}\Gamma(\nu+k-\ell+3)[\Gamma(\nu+2)]^{2k-2\ell+1}}\frac{(2\ell-3)!!}{(2\ell-2)!!}\frac{1}{\binom{\nu+\ell}{\ell-1}}\frac{\nu+1}{[\Gamma(\nu+2)]^{2\ell-1}}b_\ell(\nu)$$

which can be simplified as (20).

Substituting (8) into (20) produces

$$(-1)^{k+1}\frac{k!\,\Gamma(\nu+k+2)}{(\nu+1)\Gamma(\nu+2)}2^{2k}\zeta_{\nu+1}(2k)$$
$$= \frac{k}{\nu+1} - \frac{\Gamma(\nu+k+2)}{(2k-1)!!}\sum_{\ell=2}^{k}\binom{2k}{2\ell-2}\frac{(2k-2\ell+1)!!(2\ell-3)!!}{\Gamma(\nu+k-\ell+3)}(-1)^\ell\frac{(\ell-1)!}{\nu+1}2^{2\ell-2}\zeta_{\nu+1}(2\ell-2)$$

which can be rearranged as

$$(-1)^{k+1}2^k\zeta_{\nu+1}(2k) = \frac{k}{(2k)!!}\frac{\Gamma(\nu+2)}{\Gamma(\nu+k+2)} - \Gamma(\nu+2)\sum_{\ell=2}^{k}\frac{(-1)^\ell 2^{\ell-1}\zeta_{\nu+1}(2\ell-2)}{(2k-2\ell+2)!!\,\Gamma(\nu+k-\ell+3)}.$$

The recursive relation (21) is thus proved. The proof of Theorem 4.1 is complete.

5 More Numerical Computation of the First Few Values

Via newly-established determinantal expressions (9), (14), (15), (17)–(19), with the aid of the famous software Mathematica version 12.0, we numerically compute more special values of the sequences $b_{k+1}(\nu)$, $b_{k+1}(0)$, and $\zeta_\nu(2k)$ for $k \in \mathbb{N}$, which are supplements of those listed in (6) and (7), as follows:

$$\zeta_\nu(4) = \frac{1}{16(\nu+1)^2(\nu+2)}, \quad \zeta_\nu(6) = \frac{1}{32(\nu+1)^3(\nu+2)(\nu+3)},$$
$$\zeta_\nu(8) = \frac{5\nu+11}{256(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}, \quad b_3(\nu) = -\frac{2}{(\nu+1)(\nu+2)},$$
$$b_4(\nu) = \frac{12}{(\nu+1)(\nu+2)^2}, \quad b_5(\nu) = -\frac{24(5\nu+16)}{(\nu+1)(\nu+2)^3(\nu+3)}.$$

We notice that the numerical computation of $\zeta_\nu(4)$, $\zeta_\nu(6)$, and $\zeta_\nu(8)$ here correct corresponding ones listed in (6).

Using the famous software Mathematica version 12.0, we plotted graphs of $\zeta_\nu(2k)$ for $1 \le k \le 4$ on the interval $(-1, 9)$ in Fig. 1.

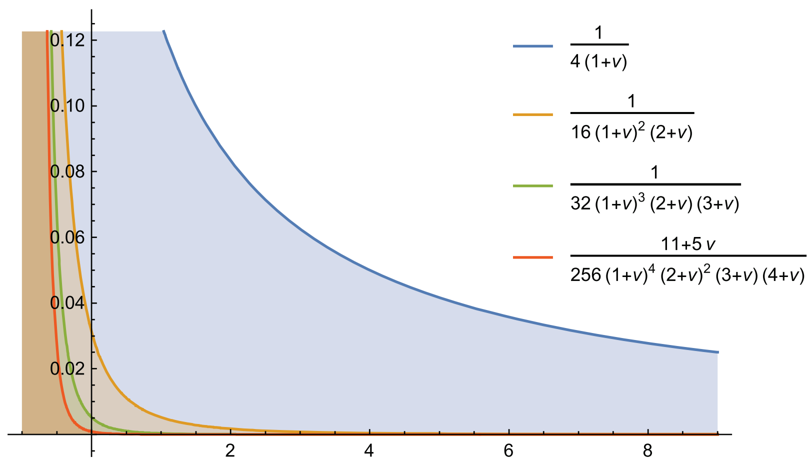

Figure 1: Graphs of $\zeta_\nu(2k)$ for $1 \le k \le 4$ on the interval $(-1, 9)$

6 Conclusions

In this paper, by virtue of a general formula (13) for derivatives of the ratio of two differentiable functions and with the aid of a recursive property (23) of the Hessenberg determinants (22), we establish six determinantal expressions (9), (14), (15), (17)–(19), find two recursive relations (20) and (21) for the sequence $b_{k+1}(\nu)$ defined by (4) and for the Bessel zeta function $\zeta_\nu(2k)$ defined by (5).

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