



**ARTICLE**

## Study of Degenerate Poly-Bernoulli Polynomials by $\lambda$ -Umbral Calculus

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### ABSTRACT

Recently, degenerate poly-Bernoulli polynomials are defined in terms of degenerate polyexponential functions by Kim-Kim-Kwon-Lee. The aim of this paper is to further examine some properties of the degenerate poly-Bernoulli polynomials by using three formulas from the recently developed ' $\lambda$ -umbral calculus.' In more detail, we represent the degenerate poly-Bernoulli polynomials by Carlitz Bernoulli polynomials and degenerate Stirling numbers of the first kind, by fully degenerate Bell polynomials and degenerate Stirling numbers of the first kind, and by higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

### KEYWORDS

Degenerate poly-Bernoulli polynomials; degenerate polyexponential functions;  $\lambda$ -umbral calculus

## 1 Introduction

Carlitz investigated the degenerate Bernoulli and Euler polynomials and numbers in [1,2], as degenerate versions of the ordinary Bernoulli and Euler polynomials and numbers. In recent years, studying degenerate versions of some special numbers and polynomials has received increased attention by mathematicians with their interests not only in combinatorial and arithmetic properties but also in applications to differential equations, identities of symmetry and probability theory (see [3–9] and references therein). Quite a few different methods have been employed in investigating degenerate versions of special numbers and polynomials, which include combinatorial methods, generating functions, umbral calculus techniques,  $p$ -adic analysis, differential equations, special functions, probability theory and analytic number theory.

Gian-Carlo Rota laid a completely rigorous foundation for umbral calculus in the 1970s, which had been in a state of manipulating sequences by a symbolic technique. The Rota's theory is based on the modern concepts like linear functionals and differential operators. In addition, the central position in the theory is occupied by the Sheffer sequences whose generating functions are given in terms of the usual exponential function (see [10–12]). Thus one may say that umbral



calculus is the study of Sheffer sequences. The impetus for [4] started from the simple question, what if the usual exponential function is replaced by the degenerate exponential functions in (1). This question arises very naturally in light of the regained recent interests in degenerate special numbers and polynomials. As it turns out, it amounts to replacing the linear functionals by the family of  $\lambda$ -linear functionals in (11), and the differential operators by the family of  $\lambda$ -differential operators in (12). Furthermore, these replacements led to define  $\lambda$ -Sheffer sequences which are characterized by the desired generating functions in (18). Hence one may say that  $\lambda$ -umbral calculus is the study of  $\lambda$ -Sheffer sequences.

The motivation of the present research is to demonstrate its usefulness of the newly developed  $\lambda$ -umbral calculus in studying some degenerate special numbers and polynomials. Recently, degenerate polyexponential functions were introduced (see [13,14]) and degenerate poly-Bernoulli polynomials were defined by means of the degenerate polyexponential functions (see (2), (10)), and some properties of the degenerate poly-Bernoulli polynomials were investigated (see [8]). The aim of this paper is to further examine the degenerate poly-Bernoulli polynomials using the above-mentioned  $\lambda$ -linear functionals and  $\lambda$ -differential operators. In more detail, these polynomials are investigated by three different tools, namely a formula about representing a  $\lambda$ -Sheffer sequence by another (see (20)), a formula obtained from the generating functions of  $\lambda$ -Sheffer sequences (see Theorem 1) and a formula arising from the definitions for  $\lambda$ -Sheffer sequences (see Theorems 6, 8). Then, among other things, we represent the degenerate poly-Bernoulli polynomials by Carlitz Bernoulli polynomials and degenerate Stirling numbers of the first kind, by fully degenerate Bell polynomials and degenerate Stirling numbers of the first kind, and by higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

The rest of this section is devoted to recalling the necessary facts that are needed throughout the paper, which includes very brief review on  $\lambda$ -umbral calculus.

For any  $\lambda \in \mathbb{R}$ , the degenerate exponential functions are defined by

$$e_{\lambda}^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e_{\lambda}(t) = e_{\lambda}^1(t), \quad \text{see [3-9]}, \quad (1)$$

where  $(x)_{0,\lambda} = 1$ ,  $(x)_{n,\lambda} = x(x-\lambda)\dots(x-(n-1)\lambda)$ ,  $(n \geq 1)$ . Note that  $\lim_{\lambda \rightarrow 0} e_{\lambda}^x(t) = e^{xt}$ .

The degenerate polyexponential functions are defined by Kim-Kim as

$$\text{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)! n^k}, \quad (k \in \mathbb{Z}, |x| < 1), \quad \text{see [7]}. \quad (2)$$

From (1) and (2), we note that  $\text{Ei}_{1,\lambda}(x) = e_{\lambda}(x) - 1$ .

Here we note that the polyexponential function was first considered by Hardy in [15,16], which are given by

$$e(x, a | s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}, \quad (\text{Re}(a) > 0).$$

Also, a slightly different special case of Hardy’s polyexponential function is considered, which is given by

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}, \quad \text{see [7,14].}$$

Note that  $xe(x, 1 | k) = Ei_k(x)$ . Let  $\log_{\lambda}(t)$  be the compositional inverse function of  $e_{\lambda}(t)$ . Then we have

$$\log_{\lambda}(1+t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n!} t^n, \quad \text{see [5].} \tag{3}$$

Note that  $\lim_{\lambda \rightarrow 0} \log_{\lambda}(1+t) = \log(1+t)$ . Kim-Kim considered the degenerate Stirling numbers of the second kind  $S_{2,\lambda}(n, k)$ ,  $(n, k \geq 0)$ , which are given by

$$(x)_{n,\lambda} = \sum_{k=0}^n S_{2,\lambda}(n, k)(x)_k, \quad (n \geq 0), \quad \text{see [5].} \tag{4}$$

As the inversion formula of (4), they also considered the degenerate Stirling numbers of the first kind given by

$$(x)_n = \sum_{k=0}^n S_{1,\lambda}(n, k)(x)_{k,\lambda}, \quad (n \geq 0), \quad \text{see [5].} \tag{5}$$

From (4) and (5), we can derive the following equations:

$$\frac{1}{k!} (e^{\lambda}(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}, \quad \text{see [4-6],} \tag{6}$$

and

$$\frac{1}{k!} (\log_{\lambda}(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!}, \quad \text{see [4,5,8].} \tag{7}$$

For  $r \in \mathbb{N}$ , Carlitz introduced the higher-order degenerate Bernoulli polynomials given by

$$\left( \frac{t}{e_{\lambda}(t) - 1} \right)^r e_{\lambda}^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad \text{see [1,2].} \tag{8}$$

When  $x = 0$ ,  $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$  are called the higher-order degenerate Bernoulli numbers.

In particular, for  $r = 1$ ,  $\beta_{n,\lambda}(x) = \beta_{n,\lambda}^{(1)}(x)$  are the Carlitz degenerate Bernoulli polynomials. From (8), we easily get  $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x)$ , where  $B_n^{(r)}(x)$  are the ordinary higher-order Bernoulli polynomials given by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{see [17–19].} \quad (9)$$

In [8], the degenerate poly-Bernoulli polynomials  $B_{n,\lambda}^{(k)}(x)$ , ( $n \geq 0$ ), are defined in terms of the degenerate polyexponential function by

$$\frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t) - 1} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \quad (10)$$

For  $x = 0$ ,  $B_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)}(0)$  are called the degenerate poly-Bernoulli numbers. Note here that  $B_{n,\lambda}^{(1)}(x) = \beta_{n,\lambda}(x)$ , ( $n \geq 0$ ).

For the rest of this section, we will briefly go over ‘ $\lambda$ -umbral calculus’ that includes  $\lambda$ -linear functionals,  $\lambda$ -differential operators and  $\lambda$ -Sheffer sequences and so on, the details of which can be found in the recent paper [4]. Let  $\mathbb{C}$  be the field of complex numbers, and let

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\},$$

be the algebra of all formal power series in  $t$  with coefficients in  $\mathbb{C}$ . Let  $\mathbb{P} = \mathbb{C}[x]$  be the ring of all polynomials in  $x$  with coefficients in  $\mathbb{C}$ , and let  $\mathbb{P}^*$  denote the vector space of all linear function as on  $\mathbb{P}$ .

For  $f(t) \in \mathcal{F}$ , with  $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ , each  $\lambda \in \mathbb{R}$  gives rise to the  $\lambda$ -linear functional  $\langle f(t) \mid \cdot \rangle_{\lambda}$

on  $\mathbb{P}$ , which is defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_{\lambda} = a_n, \quad (n \geq 0), \quad \text{see [4],} \quad (11)$$

and by linear extension. In particular, by (11) we have

$$\langle t^k \mid (x)_{n,\lambda} \rangle_{\lambda} = n! \delta_{n,k}, \quad (n, k \geq 0), \quad \text{see [4,6],}$$

where  $\delta_{n,k}$  is the Kronecker’s symbol. For any  $f(t) \in \mathcal{F}$ , and any  $p(x) \in \mathbb{P}$ , we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid (x)_{k,\lambda} \rangle_{\lambda}}{k!} t^k,$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x) \rangle_{\lambda}}{k!} (x)_{k,\lambda}, \quad \text{see [4].}$$

A power series  $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$  yields the  $\lambda$ -differential operator  $(f(t))_\lambda$  on  $\mathbb{P}$ , which is defined by

$$(f(t))_\lambda(x)_{n,\lambda} = \sum_{k=0}^n \binom{n}{k} a_k(x)_{n-k,\lambda}, \quad (n \geq 0), \tag{12}$$

and by linear extension. In particular, for each  $\lambda \in \mathbb{R}$ , and each nonnegative integer  $k$ , we have

$$(t^k)_\lambda(x)_{n,\lambda} = \begin{cases} (n)_k(x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases} \tag{13}$$

where  $(x)_0 = 1$ ,  $(x)_n = x(x-1)\dots(x-(n-1))$ ,  $(n \geq 1)$ . We note here that, for any  $p(x) \in \mathbb{P}$ ,  $(e_\lambda^y(t))_\lambda p(x) = p(x+y)$ ,  $\langle e_\lambda^y(t) | p(x) \rangle_\lambda = p(y)$ .

Let  $f_1(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$ ,  $f_2(t) = \sum_{k=0}^{\infty} \frac{b_k}{k!} t^k \in \mathcal{F}$ . Then we have

$$(f_1(t)f_2(t))_\lambda(x)_{n,\lambda} = (f_1(t))_\lambda((f_2(t))_\lambda(x)_{n,\lambda}). \tag{14}$$

In other words, this says that  $(f_1(t)f_2(t))_\lambda = (f_1(t))_\lambda(f_2(t))_\lambda$ . For  $f(t), g(t) \in \mathcal{F}$ , and  $p(x) \in \mathbb{P}$ , we have

$$\langle f(t)g(t) | p(x) \rangle_\lambda = \langle g(t) | (f(t))_\lambda p(x) \rangle_\lambda = \langle f(t) | (g(t))_\lambda p(x) \rangle_\lambda. \tag{15}$$

The order  $o(f(t))$  of the power series  $f(t) (\neq 0)$  is the smallest integer  $k$  for which  $a_k$  does not vanish. If  $o(f(t)) = 0$ , then  $f(t)$  is said to be an invertible series; if  $o(f(t)) = 1$ , then  $f(t)$  is called a delta series. Let  $f(t)$  be a delta series and let  $g(t)$  be an invertible series. Then there exists a unique  $s_{n,\lambda}(x)$  ( $\deg s_{n,\lambda}(x) = n$ ) of polynomials satisfying the orthogonality conditions

$$\left\langle g(t) (f(t))^k \Big|_{s_{n,\lambda}(x)} \right\rangle_\lambda = n! \delta_{n,k}, \quad (n, k \geq 0), \quad \text{see [4,6]}. \tag{16}$$

Such a sequence  $s_{n,\lambda}(x)$  is called the  $\lambda$ -Sheffer sequence for  $(g(t), f(t))$ , which is denoted by  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ . Here we remark that, if  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ , then we have

$$(g(t))_\lambda s_{n,\lambda}(x) \sim (1, f(t))_\lambda. \tag{17}$$

The sequence  $s_{n,\lambda}(x)$  is the  $\lambda$ -Sheffer sequence for  $(g(t), f(t))$  if and only if

$$\frac{1}{g(\bar{f}(t))} e_\lambda^y(\bar{f}(t)) = \sum_{n=0}^{\infty} s_{n,\lambda}(y) \frac{t^n}{n!}, \quad \text{for all } y \in \mathbb{C}, \quad \text{see [4,6]}, \tag{18}$$

where  $\bar{f}(t)$  is the compositional inverse function of  $f(t)$  such that  $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ .

Let  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ . Then we have

$$f(t)s_{n,\lambda}(x) = ns_{n-1,\lambda}(x), \quad (n \geq 1), \quad \text{see [4]}. \tag{19}$$

For  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ ,  $r_{n,\lambda}(x) \sim (h(t), l(t))_\lambda$ , we have

$$s_{n,\lambda}(x) = \sum_{k=0}^n c_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \quad \text{see [4],}$$

where

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (l(\bar{f}(t)))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda. \quad (20)$$

## 2 Degenerate Poly-Bernoulli Polynomials Arising from Degenerate Polyexponential Function

Let  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ ,  $(n \geq 0)$ . Then by (11) and (18), we get

$$\left\langle \frac{1}{g(\bar{f}(t))} e_\lambda^y(\bar{f}(t)) \middle| (x)_{n,\lambda} \right\rangle_\lambda = \left\langle \sum_{k=0}^{\infty} s_{k,\lambda}(y) \frac{t^k}{k!} \middle| (x)_{n,\lambda} \right\rangle_\lambda = s_{n,\lambda}(y), \quad (n \geq 0). \quad (21)$$

On the other hand, by (1) and (21), we get

$$\begin{aligned} s_{n,\lambda}(y) &= \left\langle \frac{1}{g(\bar{f}(t))} e_\lambda^y(\bar{f}(t)) \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^n \frac{1}{j!} \left\langle \frac{1}{g(\bar{f}(t))} (\bar{f}(t))^j \middle| (x)_{n,\lambda} \right\rangle_\lambda (y)_{j,\lambda}. \end{aligned} \quad (22)$$

This gives the following lemma.

**Lemma 2.1.** For  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ , we have

$$s_{n,\lambda}(x) = \sum_{j=0}^n \frac{1}{j!} \left\langle \frac{1}{g(\bar{f}(t))} (\bar{f}(t))^j \middle| (x)_{n,\lambda} \right\rangle_\lambda (x)_{j,\lambda}.$$

From (10), we note that

$$B_{n,\lambda}^{(k)}(x) \sim \left( \frac{e_\lambda(t) - 1}{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}, t \right)_\lambda, \quad (n \geq 0). \quad (23)$$

By Lemma 1, we get

$$\begin{aligned} B_{n,\lambda}^{(k)}(x) &= \sum_{j=0}^n \frac{1}{j!} \left\langle \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} t^j \middle| (x)_{n,\lambda} \right\rangle_\lambda (x)_{j,\lambda} \\ &= \sum_{j=0}^n \frac{(n)_j}{j!} \left\langle \sum_{m=0}^{\infty} B_{m,\lambda}^{(k)} \frac{t^m}{m!} \middle| (x)_{n-j,\lambda} \right\rangle_\lambda (x)_{j,\lambda} \\ &= \sum_{j=0}^n \binom{n}{j} B_{n-j,\lambda}^{(k)}(x)_{j,\lambda}. \end{aligned} \quad (24)$$

Therefore, by (24), we obtain the following theorem:

**Theorem 2.1.** For  $n \geq 0$ , we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{j=0}^n \binom{n}{j} B_{n-j,\lambda}^{(k)}(x)_{j,\lambda}.$$

Before proceeding further, we will plot the degenerate poly-Bernoulli polynomial  $B_{n,\lambda}^{(k)}(x)$ , for  $k = 2$  and  $n = 2, 3, 4$ . From (10), we recall that the generating function of the degenerate poly-Bernoulli numbers  $B_{n,\lambda}^{(2)} = B_{n,\lambda}^{(2)}(0)$  is given by

$$\frac{\text{Ei}_{2,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t) - 1} = \frac{\sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} \left(\frac{1}{\lambda} \sum_{l=1}^{\infty} (\lambda)_l \frac{t^l}{l!}\right)^n}{n^{2(n-1)!}}}{\sum_{n=1}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(2)} \frac{t^n}{n!}. \tag{25}$$

Using (25), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda}^{(2)} \frac{t^n}{n!} &= 1 + \frac{3}{4}t(-1 + \lambda) + \frac{5}{72}t^2(5 - 6\lambda + \lambda^2) + \frac{1}{288}t^3(-53 + 74\lambda - 31\lambda^2 + 10\lambda^3) \\ &\quad + \frac{1}{43200}t^4(5477 - 9195\lambda + 5045\lambda^2 - 525\lambda^3 - 802\lambda^4) + \dots \end{aligned} \tag{26}$$

Thus we have

$$\begin{aligned} B_{0,\lambda}^{(2)} &= 1, B_{1,\lambda}^{(2)} = \frac{3}{4}(-1 + \lambda), B_{2,\lambda}^{(2)} = \frac{5}{36}(5 - 6\lambda + \lambda^2), B_{3,\lambda}^{(2)} = \frac{1}{48}(-53 + 74\lambda - 31\lambda^2 + 10\lambda^3), \\ B_{4,\lambda}^{(2)} &= \frac{1}{1800}(5477 - 9195\lambda + 5045\lambda^2 - 525\lambda^3 - 802\lambda^4). \end{aligned} \tag{27}$$

From Theorem 2.1 and (27), we finally have

$$\begin{aligned} B_{2,\lambda}^{(2)}(x) &= \frac{5}{36}(5 - 6\lambda + \lambda^2) + \frac{3}{2}(-1 + \lambda)x + x(x - \lambda), \\ B_{3,\lambda}^{(2)}(x) &= \frac{1}{48}(-53 + 74\lambda - 31\lambda^2 + 10\lambda^3) + \frac{15}{36}(5 - 6\lambda + \lambda^2)x + \frac{9}{4}(-1 + \lambda)x(x - \lambda) + x(x - \lambda)(x - 2\lambda), \\ B_{4,\lambda}^{(2)}(x) &= \frac{1}{1800}(5477 - 9195\lambda + 5045\lambda^2 - 525\lambda^3 - 802\lambda^4) + \frac{1}{12}(-53 + 74\lambda - 31\lambda^2 + 10\lambda^3)x \\ &\quad + \frac{5}{6}(5 - 6\lambda + \lambda^2)x(x - \lambda) + 3(-1 + \lambda)x(x - \lambda)(x - 2\lambda) + x(x - \lambda)(x - 2\lambda)(x - 3\lambda). \end{aligned} \tag{28}$$

Now, using (28) and with the help of mathematica we plot  $B_{2,\lambda}^{(2)}(x)$ ,  $B_{3,\lambda}^{(2)}(x)$ , and  $B_{4,\lambda}^{(2)}(x)$  in the following Figs. 1–3.

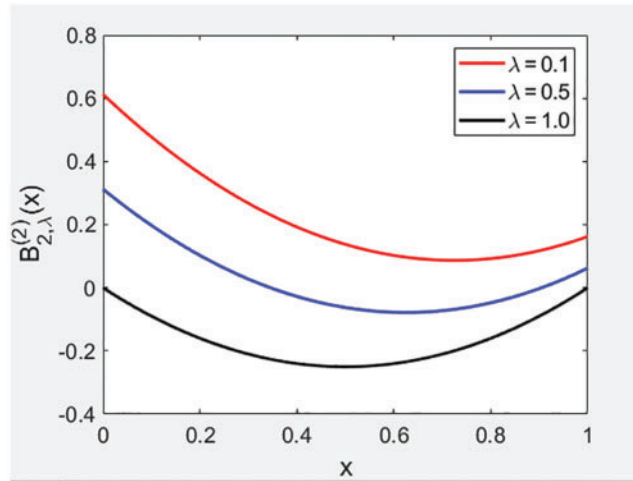


Figure 1:  $B_{2,\lambda}^{(2)}(x)$

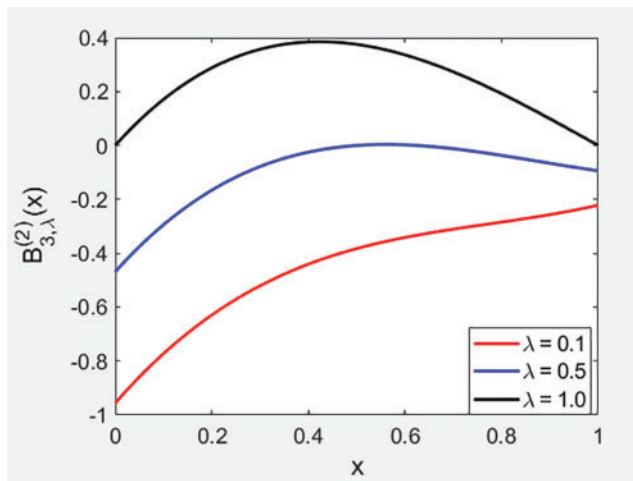


Figure 2:  $B_{3,\lambda}^{(2)}(x)$

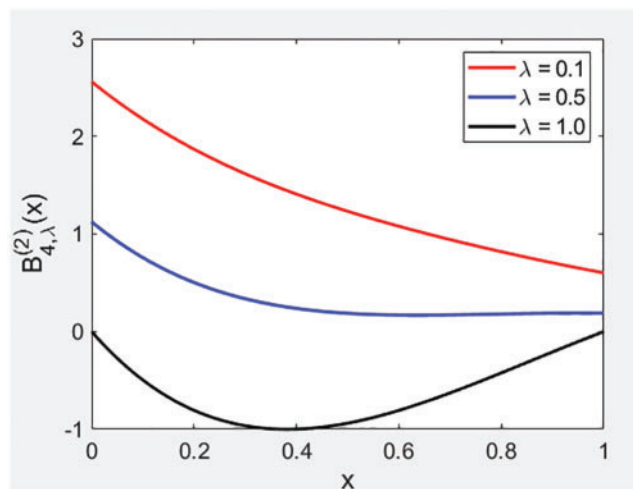


Figure 3:  $B_{4,\lambda}^{(2)}(x)$



By (17) and (23), we get

$$\left( \frac{e_\lambda(t) - 1}{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))} \right)_\lambda B_{n,\lambda}^{(k)}(x) = (x)_{n,\lambda}, \quad (n \geq 0). \tag{29}$$

Now, we observe

$$\begin{aligned} \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{l^{k-1}} \frac{1}{l!} (\log_\lambda(1+t))^l \frac{1}{e_\lambda(t) - 1} \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{l^{k-1}} \sum_{j=l}^{\infty} S_{1,\lambda}(j, l) \frac{t^j}{j!} \frac{1}{e_\lambda(t) - 1} \\ &= \sum_{j=1}^{\infty} \sum_{l=1}^j \frac{(1)_{l,\lambda}}{l^{k-1}} S_{1,\lambda}(j, l) \frac{t^j}{j!} \frac{1}{e_\lambda(t) - 1} \\ &= \sum_{j=0}^{\infty} \sum_{l=1}^{j+1} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \frac{t^j}{j!} \frac{t}{e_\lambda(t) - 1} \\ &= \sum_{j=0}^{\infty} \sum_{l=1}^{j+1} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \frac{t^j}{j!} \sum_{m=0}^{\infty} \beta_{m,\lambda} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \sum_{l=1}^{j+1} \binom{n}{j} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \beta_{n-j,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{30}$$

From (29) and (30), we note that

$$\begin{aligned} B_{n,\lambda}^{(k)}(x) &= \left( \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} \right)_\lambda (x)_{n,\lambda} \\ &= \left( \sum_{m=0}^{\infty} \sum_{j=0}^m \sum_{l=1}^{j+1} \binom{m}{j} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \beta_{m-j,\lambda} \frac{t^m}{m!} \right)_\lambda (x)_{n,\lambda} \\ &= \sum_{m=0}^n \sum_{j=0}^m \sum_{l=1}^{j+1} \binom{m}{j} \binom{n}{m} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \beta_{m-j,\lambda} (x)_{n-m,\lambda}. \end{aligned} \tag{31}$$

Therefore, by (31), we obtain the following theorem:

**Theorem 2.2.** For  $n \geq 0$ , we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{m=0}^n \sum_{j=0}^m \sum_{l=1}^{j+1} \binom{m}{j} \binom{n}{m} \frac{(1)_{l,\lambda} S_{1,\lambda}(j+1, l)}{l^{k-1}(j+1)} \beta_{m-j,\lambda}(x)_{n-m,\lambda}.$$

By (10) and (11), we get

$$\begin{aligned} B_{n,\lambda}^{(k)}(y) &= \left\langle \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} e_\lambda^y(t) \middle| (x)_{n,\lambda} \right\rangle & (32) \\ &= \left\langle \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t} \middle| \left( \frac{t}{e_\lambda(t) - 1} e_\lambda^y(t) \right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}(y) \left\langle \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t} \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\ &= \sum_{l=0}^n \frac{\binom{n}{l} \beta_{l,\lambda}(y)}{(n-l+1)} \langle \text{Ei}_k(\log_\lambda(1+t)) \middle| (x)_{n-l+1,\lambda} \rangle_\lambda \\ &= \sum_{l=0}^n \frac{\binom{n}{l} \beta_{l,\lambda}(y)}{(n-l+1)} \left\langle \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}m!} (\log_\lambda(1+t))^m \middle| (x)_{n-l+1,\lambda} \right\rangle_\lambda \\ &= \sum_{l=0}^n \frac{\binom{n}{l} \beta_{l,\lambda}(y)}{(n-l+1)} \left\langle \sum_{j=1}^{\infty} \left( \sum_{m=1}^j \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(j, m) \right) \frac{t^j}{j!} \middle| (x)_{n-l+1,\lambda} \right\rangle_\lambda \\ &= \sum_{l=0}^n \frac{\binom{n}{l} \beta_{l,\lambda}(y)}{(n-l+1)} \sum_{m=1}^{n-l+1} \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(n-l+1, m) \\ &= \sum_{l=0}^n \sum_{m=1}^{n-l+1} \frac{\binom{n}{l} \beta_{l,\lambda}(y)}{(n-l+1)} \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(n-l+1, m). \end{aligned}$$

Therefore, by (32), we obtain the following theorem:

**Theorem 2.3.** For  $n \geq 0$ , we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n \sum_{m=1}^{n-l+1} \binom{n}{l} \frac{(1)_{m,\lambda} S_{1,\lambda}(n-l+1, m)}{m^{k-1}(n-l+1)} \beta_{l,\lambda}(x).$$

The fully degenerate Bell polynomials are defined as

$$e_\lambda^x (e_\lambda(t) - 1) = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{33}$$

Note that

$$\sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} = e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}. \tag{34}$$

By comparing the coefficients on both sides of (34), we have

$$\lim_{\lambda \rightarrow 0} \text{Bel}_{n,\lambda}(x) = \text{Bel}_n(x), \quad (n \geq 0),$$

where  $\text{Bel}_n(x)$  are the ordinary Bell polynomials.

From (33), we note that

$$\text{Bel}_{n,\lambda}(x) \sim (1, \log_\lambda(1+t))_\lambda. \tag{35}$$

Assume that

$$\text{Bel}_{n,\lambda}^{(k)}(x) = \sum_{m=0}^n c_{n,m} \text{Bel}_{m,\lambda}(x), \quad (n \geq 0). \tag{36}$$

Then, by (20), we get

$$\begin{aligned} c_{n,m} &= \frac{1}{m!} \left\langle \frac{\text{Ei}_{k,\lambda}(\log(1+t))}{e_\lambda(t) - 1} (\log_\lambda(1+t))^m \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{l=m}^{\infty} \frac{S_{1,\lambda}(l,m)}{l!} \left\langle \frac{\text{Ei}_{k,\lambda}(\log(1+t))}{e_\lambda(t) - 1} t^l \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{l=m}^n \binom{n}{l} S_{1,\lambda}(l,m) \left\langle \frac{\text{Ei}_{k,\lambda}(\log(1+t))}{e_\lambda(t) - 1} \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\ &= \sum_{l=m}^n \binom{n}{l} S_{1,\lambda}(l,m) B_{n-l,\lambda}^{(k)}. \end{aligned} \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem:

**Theorem 2.4.** For  $n \geq 0$ , we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{m=0}^n \left( \sum_{l=m}^n \binom{n}{l} S_{1,\lambda}(l,m) B_{n-l,\lambda}^{(k)} \right) \text{Bel}_{m,\lambda}(x).$$

Let  $\mathbb{P}_n = \{p(x) \in \mathbb{P} \mid \deg p(x) \leq n\}$ . Then  $\mathbb{P}_n$  is an  $(n + 1)$ -dimensional vector space over  $\mathbb{C}$ . For  $r = 1$  in (8), we have

$$\beta_{n,\lambda}(x) \sim \left( \frac{e_\lambda(t) - 1}{t}, t \right)_\lambda, \quad (n \geq 0). \tag{38}$$

For  $p(x) \in \mathbb{P}_n$ , we let

$$p(x) = \sum_{l=0}^n A_l \beta_{l,\lambda}(x) \tag{39}$$

Then, by (15), we get

$$\begin{aligned} \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right) t^m \middle| p(x) \right\rangle_\lambda &= \sum_{l=0}^n A_l \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right) t^m \middle| \beta_{l,\lambda}(x) \right\rangle_\lambda \\ &= \sum_{l=0}^n A_l m! \delta_{m,l} = A_m m!, \end{aligned} \tag{40}$$

where  $0 \leq m \leq n$ .

Therefore, by (38) and (39), we obtain the following theorem:

**Theorem 2.5.** For  $p(x) \in \mathbb{P}_n$ , we have

$$p(x) = \sum_{l=0}^n A_l \beta_{l,\lambda}(x),$$

where

$$A_l = \frac{1}{l!} \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right) t^l \middle| p(x) \right\rangle_\lambda.$$

Let  $p(x) = B_{n,\lambda}^{(k)}(x) \in \mathbb{P}_n$ . Then, by Theorem 6, we get

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n A_l \beta_{l,\lambda}(x), \tag{41}$$

where

$$\begin{aligned} A_l &= \frac{1}{l!} \left\langle \frac{e_\lambda(t) - 1}{t} t^l \middle| B_{n,\lambda}^{(k)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} \left\langle \frac{e_\lambda(t) - 1}{t} \middle| B_{n-l,\lambda}^{(k)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} \left\langle \frac{e_\lambda(t) - 1}{t} \middle| \frac{1}{n-l+1} (t)_\lambda B_{n-l+1,\lambda}^{(k)}(x) \right\rangle_\lambda \end{aligned} \tag{42}$$

$$\begin{aligned}
 &= \frac{\binom{n}{l}}{n-l+1} \langle e_\lambda(t) - 1 | B_{n-l+1,\lambda}^{(k)}(x) \rangle_\lambda \\
 &= \frac{\binom{n}{l}}{n-l+1} (B_{n-l+1}^{(k)}(1) - B_{n-l+1}^{(k)}) \\
 &= \frac{\binom{n}{l}}{n-l+1} \sum_{m=0}^{n-l} \binom{n-l+1}{m} B_{m,\lambda}^{(k)}(1)_{n-l+1-m,\lambda}.
 \end{aligned}$$

Therefore, by (41) and (42), we obtain the following theorem:

**Theorem 2.6.** For  $n \geq 0$ , we have

$$\begin{aligned}
 B_{n,\lambda}^{(k)}(x) &= \sum_{l=0}^n \binom{n}{l} \left( \frac{B_{n-l+1}^{(k)}(1) - B_{n-l+1}^{(k)}}{n-l+1} \right) \beta_{l,\lambda}(x) \\
 &= \sum_{l=0}^n \left( \frac{\binom{n}{l}}{n-l+1} \sum_{m=0}^{n-l} B_{m,\lambda}^{(k)}(1)_{n-l+1-m,\lambda} \binom{n-l+1}{m} \right) \beta_{l,\lambda}(x).
 \end{aligned}$$

From (8), we note that

$$\beta_{n,\lambda}^{(r)}(x) \sim \left( \left( \frac{e_\lambda(t) - 1}{t} \right)^r, t \right)_\lambda. \tag{43}$$

For  $p(x) \in \mathbb{P}_n$ , we let

$$p(x) = \sum_{l=0}^n C_l \beta_{l,\lambda}^{(r)}(x). \tag{44}$$

Then we have

$$\begin{aligned}
 \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right)^r t^m \middle| p(x) \right\rangle_\lambda &= \sum_{l=0}^n C_l \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right)^r t^m \middle| \beta_{l,\lambda}^{(r)}(x) \right\rangle_\lambda \\
 &= \sum_{l=0}^n C_l m! \delta_{m,l} = m! C_m.
 \end{aligned} \tag{45}$$

Therefore, by (44) and (45), we obtain the following theorem:

**Theorem 2.7.** For  $p(x) \in \mathbb{P}_n$ , we have

$$p(x) = \sum_{l=0}^n C_l \beta_{l,\lambda}^{(r)}(x),$$

where

$$C_l = \frac{1}{l!} \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right)^r t^l \middle| p(x) \right\rangle_\lambda.$$

Let  $p(x) = B_{n,\lambda}^{(k)}(x) \in \mathbb{P}_n$ . Then, by Theorem 8, we get

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n C_l \beta_{l,\lambda}^{(r)}(x), \quad (46)$$

where

$$\begin{aligned} C_l &= \frac{1}{l!} \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right)^r t^l \middle| B_{n,\lambda}^{(k)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} \left\langle \left( \frac{e_\lambda(t) - 1}{t} \right)^r \middle| B_{n-l,\lambda}^{(k)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} r! \left\langle \frac{1}{t^r} \frac{1}{r!} (e_\lambda(t) - 1)^r \middle| B_{n-l,\lambda}^{(r)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} \sum_{m=0}^{n-l} \frac{r! m!}{(m+r)!} S_{2,\lambda}(m+r, r) \frac{1}{m!} \left\langle t^m \middle| B_{n-l,\lambda}^{(k)}(x) \right\rangle_\lambda \\ &= \binom{n}{l} \sum_{m=0}^{n-l} \frac{\binom{n-l}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r, r) \langle 1 \middle| B_{n-l-m,\lambda}^{(k)}(x) \rangle_\lambda \\ &= \binom{n}{l} \sum_{m=0}^{n-l} \frac{\binom{n-l}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r, r) B_{n-l-m,\lambda}^{(k)}. \end{aligned} \quad (47)$$

Therefore, by (46) and (47), we obtain the following theorem:

**Theorem 2.8.** For  $n \geq 0$ , we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} \left( \sum_{m=0}^{n-l} \frac{\binom{n-l}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r, r) B_{n-l-m,\lambda}^{(k)} \right) \beta_{l,\lambda}^{(r)}(x).$$

### 3 Conclusion

The study of degenerate versions of some special polynomials was initiated by Carlitz and has spurred increased interests by some mathematicians in recent times. This study unveiled many interesting results, not only from arithmetical and combinatorial perspectives but also in their applications to differential equations, identities of symmetry and probability theory.

Recently, the  $\lambda$ -umbral calculus was developed starting from the question, what if the usual exponential function is replaced by the degenerate exponential functions in the generating function of a Sheffer sequence. This question led us to the introduction of the concepts like  $\lambda$ -linear functionals,  $\lambda$ -differential operators and  $\lambda$ -Sheffer sequences.

In this paper, the degenerate poly-Bernoulli polynomials were investigated using three different tools, namely a formula about representing a  $\lambda$ -Sheffer sequence by another, a formula coming from the generating functions of  $\lambda$ -Sheffer sequences and a formula arising from the definitions for  $\lambda$ -Sheffer sequences. Then, among other things, we represented the degenerate poly-Bernoulli polynomials by Carlitz Bernoulli polynomials and degenerate Stirling numbers of the first kind, by fully degenerate Bell polynomials and degenerate Stirling numbers of the first kind, and by higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

As one of our future projects, we want to continue to investigate the degenerate special numbers and polynomials by using the recently developed  $\lambda$ -umbral calculus.

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