Computer Modeling in
Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2021.015310

ARTICLE

A Pseudo-Spectral Scheme for Systems of Two-Point Boundary Value Problems with Left and Right Sided Fractional Derivatives and Related Integral Equations

I. G. Ameen[1], N. A. Elkot[2], M. A. Zaky[3,*], A. S. Hendy[4,5] and E. H. Doha[2]

[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt

[2]Department of Mathematics, Faculty of Science, Cairo University, Giza, 12613, Egypt

[3]Department of Applied Mathematics, Physics Division, National Research Centre, Dokki, Cairo, 12622, Egypt

[4]Department of Computational Mathematics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, Yekaterinburg, 620002, Russia

[5]Department of Mathematics, Faculty of Science, Benha University, Benha, 13511, Egypt

[*]Corresponding Author: M. A. Zaky. Email: ma.zaky@yahoo.com; zaky.nrc@gmail.com

Received: 08 December 2020 Accepted: 04 March 2021

ABSTRACT

We target here to solve numerically a class of nonlinear fractional two-point boundary value problems involving left- and right-sided fractional derivatives. The main ingredient of the proposed method is to recast the problem into an equivalent system of weakly singular integral equations. Then, a Legendre-based spectral collocation method is developed for solving the transformed system. Therefore, we can make good use of the advantages of the Gauss quadrature rule. We present the construction and analysis of the collocation method. These results can be indirectly applied to solve fractional optimal control problems by considering the corresponding Euler–Lagrange equations. Two numerical examples are given to confirm the convergence analysis and robustness of the scheme.

KEYWORDS

Spectral collocation method; weakly singular integral equations; two-point boundary value problems; convergence analysis

1 Introduction

Fractional-order differential operators have recently risen to prominence in the modelling of several processes. The mathematical models involving these operators have also attracted much attention, a survey of recent activity is given in [1–3]. The issue we address in this paper is to construct and analyse a spectral collocation method to solve the following nonlinear system of Caputo fractional two-point boundary value problems:

$$\begin{cases} {}_{-1}^{C}D_z^{\mu} u(z) = f_L\left(z, u(z), v(z)\right), \\ {}_{z}^{C}D_1^{\mu} v(z) = f_R\left(z, u(z), v(z)\right), \\ u(-1) = u_0, \quad u'(-1) = u_1, \quad v(1) = v_0, \quad v'(1) = v_1, \quad \mu \in (1, 2), \end{cases} \tag{1}$$

where f_R and f_L: $[-1, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and satisfy the Lipschitz condition (64), and ${}_{-1}^{C}D_z^{\mu}$ and ${}_{z}^{C}D_1^{\mu}$ are the left- and right-sided Caputo fractional derivatives, respectively (see definition 2). In case of $\mu = 2$, then ${}_{-1}^{C}D_z^{\mu}$ and ${}_{z}^{C}D_1^{\mu}$ coincide with the usual second order derivative $u''(z)$ and $v''(z)$, and the system (1) recovers the integer-order system of two point boundary value problems.

Because the fractional-order differential operators are nonlocal with weakly singular kernels, the numerical discretization of the fractional models is more change than the classical schemes. There are several analytical schemes to solve fractional differential equations, such as the Green's function method, the Mellin transform method, the Laplace transform method, the Fourier transform method, and so on [4–7]. However, analytical methods are rare for most of fractional differential equations, e.g., with non linearities or linear equations with time-dependent coefficients. Hence, constructing efficient numerical approaches is of great importance in practical applications.

Many numerical schemes have been developed to solve the fractional differential equations, mostly with the finite element methods (e.g., [8–11] and the references therein) and the finite difference methods (e.g., [12–18] and the references therein). Since spectral methods are capable of providing high-order accurate numerical approximations with less degrees of freedoms [19–23], they have been widely used for numerical approximations of fractional differential equations [24–29] or its related integral equations [30–37]. In particular, well designed spectral methods appear to be particularly attractive to tackle the difficulties associated with the weakly singular kernels of the fractional differential operators and the integral equations [38,39].

The system of fractional two-point boundary value problems (1) can be converted to an equivalent weakly singular nonlinear system of Volterra integral equations (15). The key idea of the presented approach is to solve (15) using the Legendre spectral collocation scheme. The aim of that convert to (15) is to approximate the related integral terms by the Gauss quadrature formula. The presented method has spectral convergence. This theoretical estimate is confirmed by two numerical test examples. Specifically, our strategies and contributions are highlighted as follows:

(i) The system of fractional two-point boundary value problems is recast into an equivalent weakly singular nonlinear system of Volterra integral equations.
(ii) The Legendre spectral collocation method is applied to the transformed equation.
(iii) The convergence analysis of the Legendre collocation method under the L^2-norms is derived.

The structure of the paper is as follows. In Section 2, we introduce some necessary definitions, notations and lemmas. The Legendre spectral collocation scheme is presented in Section 3. The convergence analysis is provided in Section 4. In Section 5, numerical examples are performed to confirm the efficiency of the numerical method. A brief conclusion is highlighted in Section 6.

2 Mathematical Preliminaries

In this section, we provide some notations, definitions, and some useful lemmas about the fractional differential and integral operators [4] and the Jacobi polynomials.

Definition 1. Let $t \in [-1, 1]$, for $\alpha > 0$, the left and right Riemann-Liouville fractional integrals of order μ are defined, respectively, as:

$$\begin{aligned} {}_{-1}I_z^{\mu} u(z) &= \frac{1}{\Gamma(\mu)} \int_{-1}^{z} (z-\tau)^{\mu-1}\, u(\tau)\, d\tau, \\ {}_{z}I_1^{\mu} \upsilon(z) &= \frac{1}{\Gamma(\mu)} \int_{z}^{1} (\tau-z)^{\mu-1}\, \upsilon(\tau)\, d\tau, \end{aligned} \tag{2}$$

where $\Gamma(.)$ is the usual Gamma function.

Definition 2. The left- and right-sided Caputo fractional derivatives are defined as:

$$\begin{aligned} ({}_{-1}^{C}D_z^{\mu} u)(z) &= \frac{1}{\Gamma(m-\mu)} \int_{-1}^{z} (z-\tau)^{m-\mu-1}\, u^{(m)}(\tau)\, d\tau, \\ ({}_{z}^{C}D_1^{\mu} u)(z) &= \frac{(-1)^m}{\Gamma(m-\mu)} \int_{z}^{1} (\tau-z)^{m-\mu-1}\, u^{(m)}(\tau)\, d\tau, \end{aligned} \tag{3}$$

where $m-1 < \mu < m,\ m \in \mathbb{N}$.

Definition 3. The left- and right-sided Riemann-Liouville fractional derivatives are defined as:

$$\begin{aligned} ({}_{-1}^{RL}D_z^{\mu} u)(z) &= \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dz^m} \int_{-1}^{z} (z-\tau)^{m-\mu-1}\, u(\tau)\, d\tau, \\ ({}_{z}^{RL}D_1^{\mu} u)(z) &= \frac{(-1)^m}{\Gamma(m-\mu)} \frac{d^m}{dz^m} \int_{z}^{1} (\tau-z)^{m-\mu-1}\, u(\tau)\, d\tau. \end{aligned} \tag{4}$$

It is worthy to mention here that the left- and right-sided Caputo fractional derivatives satisfy the following fundamental properties

Theorem 1. There hold [4]

$${}_{a}I_z^{\alpha}({}_{a}^{C}D_z^{\mu} u(z)) = {}_{a}^{C}D_z^{\mu-\alpha} u(z) - \sum_{j=\lceil \mu-\alpha \rceil}^{\lceil \mu \rceil -1} \frac{u^{(j)}(a)}{\Gamma(j+\alpha-\mu+1)} (z-a)^{j+\alpha-\mu}, \tag{5}$$

$${}_{z}I_T^{\alpha}({}_{z}^{C}D_T^{\mu} u(z)) = {}_{z}^{C}D_T^{\mu-\alpha} u(z) - \sum_{j=\lceil \mu-\alpha \rceil}^{\lceil \mu \rceil -1} \frac{u^{(j)}(T)}{\Gamma(j+\alpha-\mu+1)} (T-z)^{j+\alpha-\mu}, \tag{6}$$

where $\mu \geq \alpha$.

The following formulas introduce the relationship between the Riemann-Liouville and the Caputo fractional derivatives [5].

$${}_{a}D_z^{\mu} u(z) = {}_{a}^{C}D_z^{\mu} u(z) + \sum_{j=0}^{m-1} \frac{u^{(j)}(a)}{\Gamma(j-\mu+1)} (z-a)^{j-\mu}, \tag{7}$$

$$ {}_zD_T^{\mu}u(z) = {}_z^C D_T^{\mu}u(z) + \sum_{j=0}^{m-1} \frac{u^{(j)}(T)}{\Gamma(j-\mu+1)} (T-z)^{j-\mu}. \tag{8} $$

Let $\theta > 0$ and $\vartheta > -1$, then

$$ {}_aI_z^{\theta}(z-a)^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+\theta+1)}(z-a)^{\vartheta+\theta}, $$

and for $\theta \in (m-1, m)$ with $m \in \mathbb{N}$ and $\vartheta \neq 0$,

$$ {}_a^C D_z^{\theta}(z-a)^{\vartheta} = \begin{cases} 0, & \vartheta \le m-1, \ \vartheta \in \mathbb{N}_0, \\ \dfrac{\Gamma(\vartheta+1)}{\Gamma(\vartheta-\theta+1)}(z-a)^{\vartheta-\theta}, & \vartheta > m-1, \ \vartheta \in \mathbb{R}. \end{cases} $$

Now, we give some basic properties of the Jacobi polynomials and related Jacobi–Gauss interpolation. For $\nu,\ \upsilon > -1$, the Jacobi polynomials $J_i^{\nu,\upsilon}(\zeta)$, $\zeta \in \Lambda = [-1, 1]$ of degree i form a complete $L^2_{\omega^{\nu,\upsilon}}(\Lambda)$ orthogonal system with the weight function $\omega^{\nu,\upsilon} = (1-\zeta)^{\nu}(1+\zeta)^{\upsilon}$, i.e.,

$$ \int_{\Lambda} J_i^{\nu,\upsilon}(\zeta)\, J_j^{\nu,\upsilon}(\zeta)\, \omega^{\nu,\upsilon}(\zeta) d\zeta = \gamma_i^{\nu,\upsilon}\, \delta_{i,j} \tag{9} $$

where, $\delta_{i,j}$ is the Kronecker function, and

$$ \gamma_i^{\nu,\upsilon} = \frac{2^{\nu+\upsilon+1}\,\Gamma(i+\nu+1)\,\Gamma(i+\upsilon+1)}{(2\,i+\nu+\upsilon+1)\, i!\ \Gamma(i+\nu+\upsilon+1)}. \tag{10} $$

Denote $P_N(\Lambda)$ the space of all polynomials of degree less than or equal to N and $\{\bar{\omega}_i^{\nu,\upsilon},\ x_i^{\nu,\upsilon}\}_{i=0}^N$ are the set of weights and nodes of the Gauss-Jacobi interpolation. The associated Gauss-Jacobi integration formula can be written as:

$$ \int_{\Lambda} Q(x)\, \omega^{\nu,\upsilon}(x) dx \approx \sum_{i=0}^{N} Q(x_i)\, \bar{\omega}_i^{\nu,\upsilon}. \tag{11} $$

The formula (11) is exact for any $Q(x) \in P_{2N+1}(\Lambda)$. Accordingly,

$$ \sum_{k=0}^{N} J_i^{\nu,\upsilon}(x_k^{\nu,\upsilon})\, J_j^{\nu,\upsilon}(x_k^{\nu,\upsilon})\, \bar{\omega}_k^{\nu,\upsilon} = \gamma_i^{\nu,\upsilon}\, \delta_{i,j}, \quad \forall\, 0 \le i+j \le 2N+1. \tag{12} $$

For any $u \in C[-1, 1]$, the Gauss-Jacobi interpolating operator $\mathcal{J}_{x,N}^{\nu,\upsilon}\colon C[-1, 1] \to P_N(\Lambda)$ is determined uniquely by

$$ \mathcal{J}_{x,N}^{\nu,\upsilon} u(x_j^{\nu,\upsilon}) = u(x_j^{\nu,\upsilon}), \quad 0 \le j \le N. \tag{13} $$

The above condition indicates that $\mathcal{J}_{x,N}^{\nu,\upsilon} u = u$, $\forall\, u \in P_N$. Consequently, since $\mathcal{J}_{x,N}^{\nu,\upsilon} u \in P_N$, then we can write

$$\mathcal{J}_{x,N}^{\nu,\upsilon} u(x) = \sum_{i=0}^{N} \hat{\upsilon}_i^{\nu,\upsilon} J_i^{\nu,\upsilon}(x), \quad \textit{where } \hat{\upsilon}_i^{\nu,\upsilon} = \frac{1}{\gamma_i^{\nu,\upsilon}} \sum_{j=0}^{N} u(x_j)\, J_j^{\nu,\upsilon}(x_j)\, \bar{\omega}_j^{\nu,\upsilon}. \tag{14}$$

The Legendre polynomials can be obtained directly from the properties of the Jacobi polynomials by setting $\nu = \upsilon = 0$ as $L_i(x) = J_i^{0,0}(x)$. In the following sections, we drop the parameters ν and υ whenever $\nu = \upsilon = 0$.

3 The Pseudo-Spectral Method

We consider the system of two-point fractional boundary value problems (1) with homogeneous boundary conditions. There is no loss of generality since this can always be accomplished by a simple change of variables.

$$\begin{cases} {}_{-1}^{C}D_z^{\mu} u(z) = f_L\left(z, u(z), v(z)\right), \\ {}_{z}^{C}D_1^{\mu} v(z) = f_R\left(z, u(z), v(z)\right), \\ u(-1) = u'(-1) = v(1) = v'(1) = 0, \quad \mu \in (1, 2). \end{cases} \tag{15}$$

The above system is equivalent to the following system of weakly singular integral equations:

$$\begin{cases} u(x) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{-1}^{x} (x-\sigma)^{\mu-1} f_L(\sigma, u(\sigma), \upsilon(\sigma))\, d\sigma, \\ \upsilon(x) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{x}^{1} (\sigma - x)^{\mu-1} f_R(\sigma, u(\sigma), \upsilon(\sigma))\, d\sigma. \end{cases} \tag{16}$$

The variable transformations $\sigma_1(x, \zeta) = \frac{x+1}{2}\zeta - \frac{1-x}{2}$ and $\sigma_2(x, \zeta) = \frac{x+1}{2} - \frac{x-1}{2}\zeta$ are used in the first and second equations of the system (16) to convert the intervals $(-1, x)$ and $(x, 1)$ to the unit interval Λ as follows

$$u(x) = \frac{1}{\Gamma(\mu)} \left(\frac{1+x}{2}\right)^{\mu} \int_{-1}^{1} (1-\zeta)^{\mu-1} f_L(\sigma_1(x,\zeta), u(\sigma_1(x,\zeta)), v(\sigma_1(x,\zeta))\, d\zeta, \tag{17}$$

$$v(x) = \frac{1}{\Gamma(\mu)} \left(\frac{1-x}{2}\right)^{\mu} \int_{-1}^{1} (1+\zeta)^{\mu-1} f_R(\sigma_2(x,\zeta), u(\sigma_2(x,\zeta)), v(\sigma_2(x,\zeta)))\, d\zeta. \tag{18}$$

The Legendre spectral collocation scheme for (17) and (18) is to seek $u_N(x)$ and $v_N(x) \in \mathbb{P}_N(\Lambda)$ with $N \geq 1$, such that

$$u_N(x) = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \left[\left(\frac{x+1}{2}\right)^{\mu} \int_{-1}^{1} (1-\zeta)^{\mu-1}\, \mathcal{J}_{\zeta,N}^{\mu-1,0} f_L(\sigma_1(x,\zeta), u_N(\sigma_1(x,\zeta)), v_N(\sigma_1(x,\zeta))\, d\zeta \right] \tag{19}$$

$$v_N(x) = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \left[\left(\frac{1-x}{2} \right)^{\mu} \int_{-1}^{1} (1+\zeta)^{\mu-1} \mathcal{J}_{\zeta,N}^{0,\mu-1} f_R(\sigma_2(x,\zeta), u_N(\sigma_2(x,\zeta)), v_N(\sigma_2(x,\zeta))) d\zeta \right]. \tag{20}$$

We now describe the implementation procedure of (19), (20) in detail. We consider the following Legendre approximations

$$\begin{aligned} u_N(x) &= \sum_{r=0}^{N} \hat{u}_r L_r(x), \\ \mathcal{J}_{x,N} \mathcal{J}_{\zeta,N}^{\mu-1,0} &\left(\left(\frac{1+x}{2} \right)^{\mu} f_L(\sigma_1(x,\zeta), u_N(\sigma_1(x,\zeta)), v_N(\sigma_1(x,\zeta)) \right) \\ &= \sum_{q=0}^{N} \sum_{q'=0}^{N} \hat{\rho}_{q,q'} L_q(x) J_{q'}^{\mu-1,0}(\zeta), \end{aligned} \tag{21}$$

and

$$\begin{aligned} v_N(x) &= \sum_{s=0}^{N} \hat{v}_s L_s(x), \\ \mathcal{J}_{x,N} \mathcal{J}_{s,N}^{0,\mu-1} &\left(\left(\frac{1-x}{2} \right)^{\mu} f_R(\sigma_2(x,\zeta), u_N(\sigma_2(x,\zeta)), v_N(\sigma_2(x,\zeta))) \right) \\ &= \sum_{r=0}^{N} \sum_{r'=0}^{N} \hat{\xi}_{r,r'} L_r(x) J_{r'}^{0,\mu-1}(\zeta). \end{aligned} \tag{22}$$

Then, by (21) and (9) direct computations lead to

$$\begin{aligned} &\frac{1}{\Gamma(\mu)} \int_{-1}^{1} (1-\zeta)^{\mu-1} \mathcal{J}_{x,N} \mathcal{J}_{\zeta,N}^{\mu-1,0} \left(\left(\frac{1+x}{2} \right)^{\mu} f_L(\sigma_1(x,\zeta), u_N(\sigma_1(x,\zeta)), v_N(\sigma_1(x,\zeta))) \right) d\zeta \\ &= \frac{1}{\Gamma(\mu)} \sum_{q=0}^{N} \sum_{q'=0}^{N} \hat{\rho}_{q,q'} L_q(x) \int_{-1}^{1} (1-\zeta)^{\mu-1} J_{q'}^{\mu-1,0}(\zeta) d\zeta \\ &= \frac{2^{\mu}}{\Gamma(\mu+1)} \sum_{q=0}^{N} \hat{\rho}_{q,0} L_q(x). \end{aligned} \tag{23}$$

Applying (12) to (21), one can verify readily that

$$\hat{\rho}_{q,0} = \frac{\mu\,(2q+1)}{2^{\mu+1}} \sum_{i=0}^{N}\sum_{j=0}^{N} \left(\frac{x_i+1}{2}\right)^{\mu}$$
$$\times f_L\left(\sigma_1(x_i,\,\zeta_j^{\mu-1,0}),\,u_N(\sigma_1(x_i,\,\zeta_j^{\mu-1,0})),\,v_N(\sigma_1(x_i,\,\zeta_j^{\mu-1,0}))\right) L_q(x_i)\bar{\omega}_i\,\bar{\omega}_j^{\mu-1,0}. \tag{24}$$

Similarly

$$\frac{1}{\Gamma(\mu)}\int_{-1}^{1}(1+s)^{\mu-1}\,\mathcal{J}_{x,N}\mathcal{J}_{\zeta,N}^{0,\mu-1}\left(\left(\frac{1-x}{2}\right)^{\mu} f_R(\sigma_2(x,\,\zeta),\,u_N(\sigma_2(x,\,\zeta)),\,v_N(\sigma_2(x,\,\zeta))\right)d\zeta$$
$$= \frac{1}{\Gamma(\mu)}\sum_{r=0}^{N}\sum_{r'=0}^{N}\hat{\xi}_{r,r'}L_r(x)\int_{-1}^{1}(1+\zeta)^{\mu-1}\,J_{r'}^{0,\mu-1}(s)d\zeta \tag{25}$$
$$= \frac{2^{\mu}}{\Gamma(\mu+1)}\sum_{r=0}^{N}\hat{\xi}_{r,0}L_r(x).$$

Using (12)–(22) yields

$$\hat{\xi}_{r,0} = \frac{\mu\,(2r+1)}{2^{\mu+1}}\sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{1-x_i}{2}\right)^{\mu}$$
$$\times f_R(\sigma_2(x_i,\,\zeta_j^{0,\mu-1}),\,u_N(\sigma_2(x_i,\,\zeta_j^{0,\mu-1})),\,v_N(\sigma_2(x_i,\,\zeta_j^{0,\mu-1}))L_r(x_i)\bar{\omega}_i\,\bar{\omega}_j^{0,\mu-1}. \tag{26}$$

Hence, by using (17)–(26) we deduce that

$$\sum_{i=0}^{N}\hat{u}_iL_i(x) = \frac{2^{\mu}}{\Gamma(\mu+1)}\sum_{i=0}^{N}\hat{\rho}_{i,0}L_i(x), \tag{27}$$

$$\sum_{i=0}^{N}\hat{v}_iL_i(x) = \frac{2^{\mu}}{\Gamma(\mu+1)}\sum_{i=0}^{N}\hat{\xi}_{i,0}L_i(x). \tag{28}$$

Finally, using (9) we obtain

$$\hat{u}_i = \frac{2^{\mu}}{\Gamma(\mu+1)}\hat{\rho}_{i,0}, \quad 0 \le i \le N, \tag{29}$$

$$\hat{v}_i = \frac{2^{\mu}}{\Gamma(\mu+1)}\hat{\xi}_{i,0}, \quad 0 \le i \le N. \tag{30}$$

This system of equations can be solved for $\hat{u}_i$ and $\hat{v}_i$. Then by using (21) and (22), we obtain an approximate solution for the problem (1).

4 Convergence Analysis

4.1 Auxiliary Lemmas

In this section, we introduce some functional spaces. We denote $\partial_x^m g(x)$ to be the mth derivative of g i.e., $\partial_x^m g(x) := \frac{d^m g}{dx^m}(x)$. We also denote the $L^2_{\omega^{\nu,\mu}}(\Lambda)$ inner product and norm by

$$(g, h)_{\omega^{\nu,\mu}} := \int_\Lambda g(x)\, h(x)\, \omega^{\nu,\mu} dx, \tag{31}$$

$$\|g\|_{\omega^{\nu,\mu}} := \left(\int_\Lambda |g(x)|^2\, \omega^{\nu,\mu} dx\right)^{\frac{1}{2}}. \tag{32}$$

Definition 4. Let $s \geq 1$ be an integer. The Sobolev space $H^s_{\omega^{\nu,\mu}}$ is defined as

$$H^s_{\omega^{\nu,\mu}}(\Lambda) := \{g \in L^2_{\omega^{\nu,\mu}}(\Lambda) : \partial_x^m g \in L^2_{\omega^{\nu,\mu}}(\Lambda),\ 0 \leq m \leq s\}, \tag{33}$$

with the inner product and norm

$$(g, v)_{H^s_{\omega^{\nu,\mu}}} = \sum_{m=0}^{s} (\partial_x^m g, \partial_x^m v)_{\omega^{\nu,\mu}}, \tag{34}$$

$$\|g\|_{H^s_{\omega^{\nu,\mu}}} = (g, v)^{\frac{1}{2}}_{H^s_{\omega^{\nu,\mu}}}. \tag{35}$$

Definition 5. For a non-negative integer s. The weighted Jacobi non-uniformly Sobolev space $B^s_{\omega^{\nu,\mu}}$ is defined as

$$B^s_{\omega^{\nu,\mu}}(\Lambda) := \{g : \partial_x^m g \in L^2_{\omega^{\nu+m,\mu+m}}(\Lambda),\ 0 \leq m \leq s\}. \tag{36}$$

with the inner product, norm, and semi-norm

$$(g, v)_{B^s_{\omega^{\nu,\mu}}} = \sum_{m=0}^{s} (\partial_x^m g, \partial_x^m v)_{\omega^{\nu+m,\mu+m}}, \tag{37}$$

$$\|g\|_{B^s_{\omega^{\nu,\mu}}} = (g, g)^{\frac{1}{2}}_{B^s_{\omega^{\nu+m,\mu+m}}}. \tag{38}$$

In particular, $L^2(\Lambda) = B^0_{\omega^{0,0}}(\Lambda)$ and $\|.\| = \|.\|_{\omega^{0,0}}$. It is obvious that $H^s_{\omega^{\nu,\mu}}(\Lambda)$ is a subspace of $B^s_{\omega^{\nu,\mu}}(\Lambda)$, that is

$\| u \|_{B^s_{\omega^{\nu,\mu}}} \leq c \| u \|_{H^s_{\omega^{\nu,\mu}}}$.

The space $L^\infty(\Lambda)$ is defined with the norm

$$\| u \|_\infty = ess \sup_{x \in \Lambda} |u(x)|. \tag{39}$$

Lemma 1. (cf. [40]) Let $\nu, \mu > -1$, then for any $u \in B^s_{\omega^{\nu,\mu}}(\Lambda)$ with $s \geq 1$ and $0 \leq k \leq s \leq N+1$,

$$\|\partial_x^k (u - \mathcal{J}^{\nu,\mu}_{x,N} u)\|_{\omega^{\nu+k,\mu+k}} \leq c N^{k-s} \|\partial_x^s u\|_{\omega^{\nu+k,\mu+k}}, \tag{40}$$

where $\mathcal{J}_{x,N}^{\nu,\mu}$ is the Gauss-Jacobi interpolation operator.

Lemma 2. (cf. [40]) For any $u \in H^s(\Lambda)$ with $0 \le s \le N+1$,

$$\|u - \mathcal{J}_{x,N} u\|_\infty \le c N^{\frac{3}{4}-s} \|\partial_x^s u\|_\infty. \tag{41}$$

Lemma 3. (cf. [41]) Let $\{F_i(x)\}_{i=0}^N$ be the Nth interpolation Lagrange polynomials related to the $N+1$ Gauss points of the Jacobi polynomial. Then,

$$\|\mathcal{J}_N^{\nu,\mu}\|_\infty := \max_{x\in\Lambda} \sum_{i=0}^N |F_i(x)| = \begin{cases} O(\log N), & -1 < \nu,\ \mu \le -\dfrac{1}{2}, \\ O(N^{\gamma+\frac{1}{2}}), & \gamma = \max(\nu, \mu), \quad \text{otherwise}. \end{cases} \tag{42}$$

Let $\zeta_i^{\mu-1,0}$ be the Gauss-Jacobi nodes in Λ and $\sigma_{1,i}^{\mu-1,0} = \sigma_1(x, \zeta_i^{\mu-1,0})$. The mapped Gauss-Jacobi interpolating operator ${}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0}$: $C(-1, x) \to P_N(-1, x)$ is given as

$${}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} u(\sigma_i^{\mu-1,0}) = u(\sigma_i^{\mu-1,0}), \quad 0 \le i \le N. \tag{43}$$

Hence

$${}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} u(\sigma_i^{\mu-1,0}) = u(\sigma_i^{\mu-1,0}) = u(\sigma_1(x, \zeta_i^{\mu-1,0})) = {}_x\mathcal{J}_{\sigma,N}^{\mu-1,0} u(\sigma_1(x, \zeta_i^{\mu-1,0})), \tag{44}$$

and

$${}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} u(\sigma) = \mathcal{J}_{\zeta,N}^{\mu-1,0} u(\sigma_1(x, \zeta))\Big|_{\zeta = \frac{2\sigma_1}{1+x} + \frac{1-x}{1+x}}. \tag{45}$$

It is not difficult to obtain the following results

$$\begin{aligned} &\int_{-1}^x (x-\sigma)^{\mu-1}\, {}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} f_L(\sigma, u(\sigma), v(\sigma)) d\sigma \\ &= \left(\frac{1+x}{2}\right)^\mu \int_{-1}^1 (1-\zeta)^{\mu-1} \mathcal{J}_{\zeta,N}^{\mu-1,0} f_L(\sigma_1(x,\zeta), u(\sigma_1(x,\zeta)), v(\sigma_1(x,\zeta))) d\zeta \\ &= \left(\frac{1+x}{2}\right)^\mu \sum_{j=0}^N f_L\left(\sigma_1(x, \zeta_j^{\mu-1,0}), u(\sigma_1(x, \zeta_j^{\mu-1,0})), v(\sigma_1(x, \zeta_j^{\mu-1,0}))\right) \bar{\omega}_j^{\mu-1,0} \\ &= \left(\frac{1+x}{2}\right)^\mu \sum_{j=0}^N f_L\left(\sigma_{1,j}^{\mu-1,0}, u(\sigma_{1,j}^{\mu-1,0}), v(\sigma_{1,j}^{\mu-1,0})\right) \bar{\omega}_j^{\mu-1,0}. \end{aligned} \tag{46}$$

Similarly

$$\begin{aligned} &\int_{-1}^x (x-\sigma)^{\mu-1} \left({}_x\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} f_L(\sigma, u(\sigma), v(\sigma))\right)^2 d\sigma \\ &= \left(\frac{1+x}{2}\right)^\mu \sum_{j=0}^N f_L^2\left(\sigma_{1,j}^{\mu-1,0}, u(\sigma_{1,j}^{\mu-1,0}), v(\sigma_{1,j}^{\mu-1,0})\right) \bar{\omega}_j^{\mu-1,0}. \end{aligned} \tag{47}$$

We denote $\mathcal{J}$ to be the identity operator. Then, for any $1 \leq s \leq N+1$ we have

$$\int_{-1}^{x} (x-\sigma)^{\mu-1} \left| \mathcal{J} - {}_{x}\tilde{\mathcal{J}}_{\sigma,N}^{\mu-1,0} u(\sigma) \right|^2 d\sigma$$

$$= \left(\frac{1+x}{2}\right)^{\mu} \int_{-1}^{1} (1-\zeta)^{\mu-1} \left| \left(\mathcal{J} - \mathcal{J}_{\zeta,N}^{\mu-1,0}\right) u(\sigma_1(x,\zeta)) \right|^2 d\zeta$$

$$\leq c\, N^{-2s} \left(\frac{1+x}{2}\right)^{\mu} \int_{-1}^{1} (1-\zeta)^{\mu+s-1} (1+\zeta)^s \left| \partial_\zeta^s u(\sigma_1(x,\zeta)) \right|^2 d\zeta$$

$$= c\, N^{-2s} \int_{-1}^{x} (x-\sigma)^{\mu+s-1} (1+\sigma)^s \left| \partial_\zeta^s u(\sigma) \right|^2 d\sigma \tag{48}$$

Let $\zeta_i^{0,\mu-1}$ be the Jacobi-Gauss nodes in Λ and $\zeta_{2,i}^{0,\mu-1} = \sigma_2(x, \zeta_i^{0,\mu-1})$. The mapped Jacobi-Gauss interpolation operator ${}_{x}\tilde{\mathcal{J}}_{\zeta,N}^{0,\mu-1}: C(x,1) \to P_N(x,1)$ is defined by

$${}_{x}\tilde{\mathcal{J}}_{\sigma,N}^{0,\mu-1} v(\zeta_i^{0,\mu-1}) = v(\zeta_i^{0,\mu-1}), \quad 0 \leq i \leq N. \tag{49}$$

Hence,

$${}_{x}\tilde{\mathcal{J}}_{\sigma,N}^{0,\mu-1} v(\zeta_i^{0,\mu-1}) = v(\zeta_i^{0,\mu-1}) = v(\sigma_2(x, \zeta_i^{0,\mu-1})) = {}_{x}\mathcal{J}_{\zeta,N}^{0,\mu-1} v(\sigma_2(x, \zeta_i^{0,\mu-1})) \tag{50}$$

and

$${}_{x}\tilde{\mathcal{J}}_{\zeta,N}^{0,\mu-1} v(\zeta) = \mathcal{J}_{\zeta,N}^{0,\mu-1} v(\sigma_2(x,\zeta)) \Big|_{\zeta = \frac{2\sigma_2}{1-x} - \frac{1+x}{1-x}} \tag{51}$$

We can also derive the following results

$$\int_{x}^{1} (\sigma - x)^{\mu-1} {}_{x}\tilde{\mathcal{J}}_{\sigma,N}^{0,\mu-1} f_R(\sigma, u(\sigma), v(\sigma)) d\sigma$$

$$= \left(\frac{1-x}{2}\right)^{\mu} \int_{-1}^{1} (1+\zeta)^{\mu-1} \mathcal{J}_{\zeta,N}^{0,\mu-1} f_R(\sigma_2(x,\zeta), u(\sigma_2(x,\zeta)), v(\sigma_2(x,\zeta))) d\zeta$$

$$= \left(\frac{1-x}{2}\right)^{\mu} \sum_{j=0}^{N} f_R\left(\sigma_2(x, \zeta_j^{0,\mu-1}), u(\sigma_2(x, \zeta_j^{0,\mu-1}), v(\sigma_2(x, \zeta_j^{0,\mu-1}))\right) \bar{\varpi}_j^{0,\mu-1}$$

$$= \left(\frac{1-x}{2}\right)^{\mu} \sum_{j=0}^{N} f_R\left(\sigma_{2,j}^{0,\mu-1}, u(\sigma_{2,j}^{0,\mu-1}), v(\sigma_{2,j}^{0,\mu-1})\right) \bar{\varpi}_j^{0,\mu-1}. \tag{52}$$

Similarly

$$\int_{x}^{1} (\sigma - x)^{\mu-1} \left({}_{x}\tilde{\mathcal{J}}_{\sigma,N}^{0,\mu-1} f_R(\sigma, u(\sigma), v(\sigma))\right)^2 d\zeta$$

$$= \left(\frac{1-x}{2}\right)^{\mu} \sum_{j=0}^{N} f_R^2 \left(\sigma_{2,j}^{0,\mu-1}, u(\sigma_{2,j}^{0,\mu-1}), v(\sigma_{2,j}^{0,\mu-1})\right) \bar{\omega}_j^{0,\mu-1}, \tag{53}$$

and

$$\int_x^1 (\sigma-x)^{\mu-1} \left|\mathcal{J} - {}_x\tilde{\mathcal{J}}_{\sigma,N}^{0,\mu-1} v(\sigma)\right|^2 d\sigma$$

$$= \left(\frac{1-x}{2}\right)^{\mu} \int_{-1}^{1} (1+\zeta)^{\mu-1} \left|\left(\mathcal{J} - \mathcal{J}_{\zeta,N}^{0,\mu-1}\right) v(\sigma_2(x,\zeta))\right|^2 d\zeta$$

$$\leq c N^{-2s} \left(\frac{1-x}{2}\right)^{\mu} \int_{-1}^{1} (1+\zeta)^{\mu+s-1} (1-\zeta)^s \left|\partial_\zeta^s v(\sigma_2(x,\zeta))\right|^2 d\zeta$$

$$= c N^{-2s} \int_x^1 (\sigma-x)^{\mu+s-1} (1-\sigma)^s \left|\partial_\zeta^s v(\sigma)\right|^2 d\sigma. \tag{54}$$

4.2 Error Analysis in L^2-Norm

In this section, we analyze the numerical errors of the systems (19) and (20). Let $E = |E_u| + |E_v|$, where $E_u = u - u_N$, $E_v = v - v_N$ and denote by $\mathcal{J}$ to be the identity operator.

Lemma 4. The following inequality holds

$$\|E\| \leq \|E_u\| + \|E_v\| \leq \sum_{i=1}^{6} \|E_i\| \tag{55}$$

where

$$E_1 = u(x) - \mathcal{J}_{x,N} u(x),$$

$$E_2 = v(x) - \mathcal{J}_{x,N} v(x),$$

$$E_3 = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{-1}^{x} (x-\sigma)^{\mu-1} (\mathcal{J} - {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}) f_L(\sigma, u(\sigma), v(\sigma)) d\sigma,$$

$$E_4 = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{x}^{1} (\sigma-x)^{\mu-1} (\mathcal{J} - {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}) f_R(\sigma, u(\sigma), v(\sigma)) d\sigma,$$

$$E_5 = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{-1}^{x} (x-\sigma)^{\mu-1} \, {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0} \left(f_L(\sigma, u(\sigma), v(\sigma)) - f_L(\sigma, u_N(\sigma), v_N(\sigma))\right) d\sigma,$$

$$E_6 = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{x}^{1} (\sigma-x)^{\mu-1} \, {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1} \left(f_R(\sigma, u(\sigma), v(\sigma)) - f_R(\sigma, u_N(\sigma), v_N(\sigma))\right) d\sigma.$$

Proof. It follows from (16) and (46) that

$$\mathcal{J}_{x,N} u(x) = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{-1}^{x} (x-\sigma)^{\mu-1} f_L(\sigma, u(\sigma), v(\sigma)) d\sigma, \tag{56}$$

$$u_N(x) = \frac{1}{\Gamma(\mu)} \mathcal{J}_{x,N} \int_{-1}^{x} (x-\sigma)^{\mu-1} \, {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0} f_L(\sigma, u_N(\sigma), v_N(\sigma)) d\sigma. \tag{57}$$

Subtracting (57) from (56) yields

$$\mathcal{J}_{x,N}u(x) - u_N(x) = \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1} \times \left(f_L(\sigma, u(\sigma), v(\sigma)) - {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0} f_L(\sigma, u_N(\sigma), v_N(\sigma))d\sigma\right) \tag{58}$$

Similarly, from (16) and (52) we deduce that

$$\mathcal{J}_{x,N}v(x) = \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1} f_R(\sigma, u(\sigma), v(\sigma))d\sigma, \tag{59}$$

$$v_N(x) = \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1}\,{}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1} f_R(\sigma, u_N(\sigma), v_N(\sigma))d\sigma. \tag{60}$$

Subtracting (60) from (59) yields

$$\mathcal{J}_{x,N}v(x) - v_N(x) = \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1} \times \left(f_R(\sigma, u(\sigma), v(\sigma)) - {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1} f_R(\sigma, u_N(\sigma), v_N(\sigma))d\sigma\right), \tag{61}$$

and adding (58) and (61) yields

$$\begin{aligned}
&\mathcal{J}_{x,N}u(x) - u_N(x) + \mathcal{J}_{x,N}v(x) - v_N(x) \\
&= \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}\left(f_L(\sigma, u(\sigma), v(\sigma)) - {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0} f_L(\sigma, u_N(\sigma), v_N(\sigma))d\sigma\right) \\
&\quad + \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1}\left(f_R(\sigma, u(\sigma), v(\sigma)) - {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1} f_R((\sigma, u_N(\sigma), v_N(\sigma))d\sigma\right).
\end{aligned} \tag{62}$$

The above equation can be rewritten as

$$\begin{aligned}
&\mathcal{J}_{x,N}u(x) - u_N(x) + \mathcal{J}_{x,N}v(x) - v_N(x) \\
&= \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}(\mathcal{J} - {}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}) f_L(\sigma, u(\sigma), v(\sigma))d\sigma \\
&\quad + \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}\,{}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}\left(f_L(\sigma, u(\sigma), v(\sigma)) - f_L(\sigma, u_N(\sigma), v_N(\sigma))\right) d\sigma \\
&\quad + \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1}(\mathcal{J} - {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}) f_R(\sigma, u(\sigma), v(\sigma))d\sigma \\
&\quad + \frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{x}^{1}(\sigma-x)^{\mu-1}\,{}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}\left(f_R(\sigma, u(\sigma), v(\sigma)) - f_R(\sigma, u_N(\sigma), v_N(\sigma))\right) d\sigma.
\end{aligned} \tag{63}$$

The desired result follows immediately from the above.

Throughout this section, we suppose that f_L and f_R fulfil the Lipschitz conditions

$$|f_L(z, u_1(z), v_1(z)) - f_L(z, u_2(z), v_2(z))| \le L_{11}|u_1 - u_2| + L_{11}|v_1 - v_2|,$$
$$|f_r(z, u_1(z), v_1(z)) - f_r(z, u_2(z), v_2(z))| \le L_{22}|u_1 - u_2| + L_{22}|v_1 - v_2|, \tag{64}$$

with the Lipschitz positive constants L_{11} and L_{22}, are chosen such that

$$L_{11} + L_{22} \le \frac{\Gamma(\lambda+1)}{2^{\lambda}}. \tag{65}$$

Theorem 2. Let $u_N(x)$ and $v_N(x)$ be the solutions of the systems of Eqs. (17) and (18), respectively, $u \in B^s_{\omega^{s,s}}(\Lambda)$ and $v \in B^s_{\omega^{s,s}}(\Lambda)$, with integer $1 \le s \le N+1$ and $N \ge 1$. Then we have the following estimate

$$\|E_u\| + \|E_v\| \le cN^{-s}\left(\left\|\partial_x^s u\right\|_{\omega^{s,s}} + \left\|\partial_x^s v\right\|_{\omega^{s,s}} + \left\|\partial_x^s f_L(\cdot, u)\cdot), v(\cdot))\right\|_{\omega^{\mu+s-1,s}} + \left\|\partial_x^s f_R(\cdot, u)\cdot), v(\cdot))\right\|_{\omega^{s,\mu+s-1}}\right). \tag{66}$$

Proof. Using Lemma (4), we get

$$\|E_1\| \le cN^{-s}\|\partial_x^s u(x)\|, \tag{67}$$

and

$$\|E_2\| \le cN^{-s}\|\partial_x^s v(x)\|. \tag{68}$$

We now estimate $\|E_3\|$. Using the Gauss-Legendre integration formula (11), we have

$$\begin{aligned}\|E_3\| &= \left\|\frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}(\mathcal{J} - {}_x\bar{\mathcal{J}}^{\mu-1,0}_{\sigma,N})f_L(\sigma, u(\sigma), v(\sigma))d\sigma\right\| \\ &= \left[\frac{1}{\Gamma(\mu)}\int_{-1}^{1}\left(\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}(\mathcal{J} - {}_x\bar{\mathcal{J}}^{\mu-1,0}_{\sigma,N})f_L(\sigma, u(\sigma), v(\sigma))d\sigma\right)^2 dx\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{\Gamma(\mu)}\sum_{j=0}^{N}\bar{\omega}_j\left(\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}(\mathcal{J} - {}_{x_j}\bar{\mathcal{J}}^{\mu-1,0}_{\sigma,N})f_L(\sigma, u(\sigma), v(\sigma))d\sigma\right)^2\right]^{\frac{1}{2}},\end{aligned} \tag{69}$$

using the Cauchy-Schwarz inequality, we further get

$$\begin{aligned}\|E_3\| &\le \left[\frac{1}{\Gamma(\mu)}\sum_{j=0}^{N}\bar{\omega}_j\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}d\sigma\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|(\mathcal{J} - {}_{x_j}\bar{\mathcal{J}}^{\mu-1,0}_{\sigma,N})f_L(\sigma, u(\sigma), v(\sigma))\right|^2 d\sigma\right]^{\frac{1}{2}} \\ &\le \left[\frac{1}{\Gamma(\mu+1)}\sum_{j=0}^{N}\bar{\omega}_j(x_j+1)^{\mu}\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|(\mathcal{J} - {}_{x_j}\bar{\mathcal{J}}^{\mu-1,0}_{\sigma,N})f_L(\sigma, u(\sigma), v(\sigma))\right|^2 d\sigma\right]^{\frac{1}{2}},\end{aligned} \tag{70}$$

making use of (48) leads to

$$\|E_3\| \le c\,N^{-s} \max_{0\le j\le N} \left[\int_{-1}^{x_j} (x_j-\sigma)^{\mu+s-1}(1+\sigma)^s \left|\partial_\sigma^s f_L(\sigma,\, u(\sigma),\, v(\sigma))\right|^2 d\sigma\right]^{\frac{1}{2}}$$
$$\le c\,N^{-s} \left\|\partial_\sigma^s f_L(\sigma,\, u(\sigma),\, v(\sigma))\right\|_{\omega^{\mu+s-1,s}}. \tag{71}$$

Using similar arguments leads to

$$\|E_4\| = \frac{1}{\Gamma(\mu)} \left\|\mathcal{J}_{x,N} \int_x^1 (\sigma-x)^{\mu-1}(\mathcal{J} - {}_x\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}) f_R(\sigma,\, u(\sigma),\, v(\sigma))d\sigma\right\|$$
$$\le c\,N^{-s} \left\|\partial_\sigma^s f_R(\sigma,\, u(\sigma),\, v(\sigma))\right\|^2_{\omega^{s,\mu+s-1}}. \tag{72}$$

We now estimate $\|E_5\|$. Making use of the Gauss-Legendre integration (11) and (46), we have

$$\|E_5\| \le \left\|\frac{1}{\Gamma(\mu)}\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}\,{}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}\Big(f_L(\sigma,\,u(\sigma),\,v(\sigma)) - f_L(\sigma,\,u_N(\sigma),\,v_N(\sigma))\Big)d\sigma\right\|$$

$$= \frac{1}{\Gamma(\mu)}\left[\int_{-1}^{1}\left(\mathcal{J}_{x,N}\int_{-1}^{x}(x-\sigma)^{\mu-1}\,{}_x\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}\Big(f_L(\sigma,\,u(\sigma),\,v(\sigma)) - f_L(\sigma,\,u_N(\sigma),\,v_N(\sigma))\Big)d\sigma\right)^2 dx\right]^{\frac{1}{2}}$$

$$\le \frac{1}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\left(\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\,{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}\Big(f_L(\sigma,\,u(\sigma),\,v(\sigma)) - f_L(\sigma,\,u_N(\sigma),\,v_N(\sigma))\Big)d\sigma\right)^2\right]^{\frac{1}{2}}$$

$$\le \frac{1}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}d\sigma\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}(f_L(\sigma,\,u(\sigma),\,v(\sigma))\right.\right.$$
$$\left.\left. - f_L(\sigma, u_N(\sigma), v_N(\sigma)))\right|^2 d\sigma\right]^{\frac{1}{2}}$$

$$\le \frac{1}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\frac{(x_j+1)^{\mu}}{\mu}\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{\mu-1,0}(f_L(\sigma,\,u(\sigma),\,v(\sigma)) - f_L(\sigma,\,u_N(\sigma),\,v_N(\sigma)))\right|^2 d\sigma\right]^{\frac{1}{2}}.$$

Using (47), we obtain

$$\|E_5\| \le \frac{1}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\frac{(x_j+1)^{2\mu}}{2^{\mu}\,\mu}\sum_{k=0}^{N}\left|(f_L(\sigma_{1,k}^{\mu-1,0},\,u(\sigma_{1,k}^{\mu-1,0}),\,v(\sigma_{1,k}^{\mu-1,0}))\right.\right.$$
$$\left.\left. - f_L(\sigma_k^{\mu-1,0},\,u_N(\sigma_{1,k}^{\mu-1,0}),\,v_N(\sigma_{1,k}^{\mu-1,0})))\right|^2\bar{\omega}_k^{\mu-1,0}\right]^{\frac{1}{2}}.$$

Using (46) and the Lipschitz condition, we get

$$\|E_5\| \le \frac{1}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\frac{(x_j+1)^{2\mu}}{2^\mu\mu}\sum_{k=0}^{N}\bar{\omega}_k^{\mu-1,0}\left|L_{11}\left|u(\sigma_{1,k}^{\mu-1,0})-u_N(\sigma_{1,k}^{\mu-1,0})\right|+L_{12}\left|v(\sigma_{1,k}^{\mu-1,0})-v_N(\sigma_{1,k}^{\mu-1,0})\right|\right|^2\right]^{\frac{1}{2}},$$

For $x_j\in(-1,1)$, one can show that

$$\sum_{j=0}^{N}\omega_j(1-x_j)^\mu\le\frac{8}{3},\quad\forall\mu\in(1,2),$$

hence,

$$\begin{aligned}\|E_5\| &\le \frac{L_{11}}{\Gamma(\mu)}\left[\sum_{j=0}^{N}\bar{\omega}_j\frac{(x_j+1)^\mu}{\mu}\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}\left(\left|u(\sigma)-u_N(\sigma)\right|+\left|v(\sigma)-v_N(\sigma)\right|\right)\right|^2 d\sigma\right]^{\frac{1}{2}}\\
&\le \frac{\sqrt{8\mu/3}L_{11}}{\Gamma(\mu+1)}\max_{0\le j\le N}\left[\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}\left(\left|u(\sigma)-u_N(\sigma)\right|+\left|v(\sigma)-v_N(\sigma)\right|\right)\right|^2 d\sigma\right]^{\frac{1}{2}}\\
&\le \frac{\sqrt{8\mu/3}L_{11}}{\Gamma(\mu+1)}\max_{0\le j\le N}\left[\left(\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left(\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}u(\sigma)-u(\sigma)\right|+\left|{}_{x_j}\bar{\mathcal{J}}_{\sigma,N}^{0,\mu-1}v(\sigma)-v(\sigma)\right|\right)^2 d\sigma\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(\int_{-1}^{x_j}(x_j-\sigma)^{\mu-1}\left(\left|u(\sigma)-u_N(\sigma)\right|+\left|v(\sigma)-v_N(\sigma)\right|\right)^2 d\sigma\right)^{\frac{1}{2}}\right]\\
&\le cN^{-s}\left\|\partial_x^s u\right\|_{\omega^{s,s}}+cN^{-s}\left\|\partial_x^s v\right\|_{\omega^{s,s}}+\frac{\sqrt{8\mu/3}L_{11}}{\Gamma(\mu+1)}\left(\|E_u\|+\|E_v\|\right).\end{aligned}$$

Similarly, we can deduce that

$$\|E_6\|\le cN^{-s}\left\|\partial_x^s u\right\|_{\omega^{s,s}}+cN^{-s}\left\|\partial_x^s v\right\|_{\omega^{s,s}}+\frac{\sqrt{8\mu/3}L_{22}}{\Gamma(\mu+1)}\left(\|E_u\|+\|E_v\|\right).$$

Using the condition (65) and knowing that $\sqrt{\frac{4\mu}{3\times 2^\mu}}\le 1$, we obtain the desired result.

5 Numerical Results

In this section, two numerical examples are provided to illustrate the efficiency and applicability of the proposed method.

Example 1. We consider the following system of fractional differential equations:

$$\begin{cases} {}_{-1}^{C}D_z^{\mu} u(z) = f_L\left(z, u(z), v(z)\right), \\ {}_{z}^{C}D_1^{\mu} v(z) = f_R\left(z, u(z), v(z)\right), \quad z \in (0,1) \quad \mu \in (1,2), \end{cases} \tag{73}$$

The boundary conditions are chosen such that the exact solution is given by $u(z) = (1+z)^{3.5}$ and $v(z) = (1-z)^{2.5}$. In Figs. 1 and 2 we list the L^2-error of the approximate solutions u_N and v_N, respectively, in log scale against various N and μ. In Tab. 1, we list the maximum absolute errors for different values of N and μ.

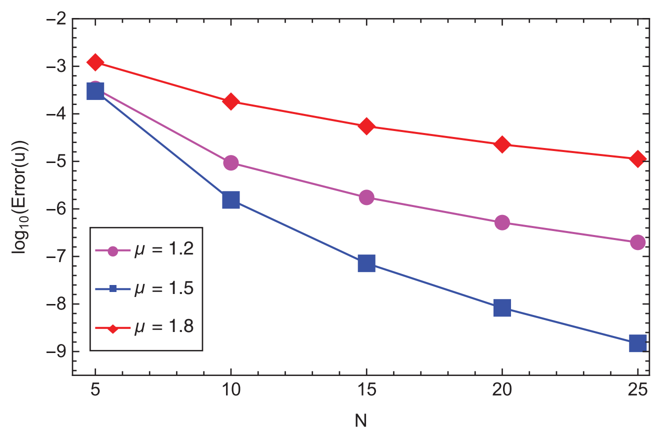

Figure 1: The L^2-error of the approximate solution u_N in log scale *vs.* various N and μ

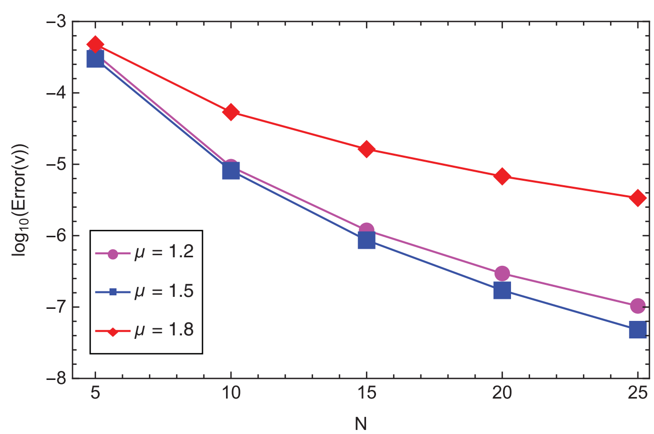

Figure 2: The L^2-error of the approximate solution v_N in log scale *vs.* various N and μ

Table 1: The maximum absolute errors of u and v for different values of N and μ for Example 1

N	μ	Error (u)	Error (v)	μ	Error (u)	Error (v)	μ	Error (u)	Error (v)
5	1.2	1.24×10^{-3}	1.76×10^{-3}	1.5	1.21×10^{-3}	1.75×10^{-3}	1.5	3.82×10^{-3}	1.76×10^{-3}
10		4.51×10^{-5}	7.92×10^{-5}		1.24×10^{-5}	2.08×10^{-5}		5.50×10^{-5}	5.45×10^{-5}
15		8.58×10^{-6}	1.23×10^{-5}		2.66×10^{-7}	1.23×10^{-5}		1.66×10^{-4}	1.58×10^{-5}
20		2.57×10^{-7}	7.39×10^{-7}		1.30×10^{-7}	3.20×10^{-6}		6.88×10^{-5}	6.62×10^{-6}
25		9.82×10^{-7}	1.11×10^{-6}		2.92×10^{-8}	1.11×10^{-6}		3.41×10^{-5}	3.29×10^{-6}

Example 2. We consider the following system of fractional differential equations: [28]:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = \sin x(t) - \dfrac{1}{2}\lambda(t) + \sin(1+t^2) - \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}, \\ {}_t^C D_1^{\alpha} \lambda(t) = -2e^{1+t^2+x(t)} + 2e^{2(1+t^2+x(t))} + \lambda(t)\cos x(t), \\ x(0) = -1, \quad x'(0) = 0, \quad \lambda(1) = \lambda'(1) = 0, \end{cases} \tag{74}$$

In Tab. 2, we compare our results with those reported in [42]. These results indicate that the proposed spectral method is more accurate than the Ritz method [42]. In Tab. 3, we list the maximum absolute errors for different values of N and α. We see in these tables that the results are accurate for even small choices of N. Moreover, to demonstrate the convergence of the proposed method, Figs. 3 and 4 present the logarithmic graphs of L^{∞}-errors of the approximate solutions x_N and λ_N, respectively, for various values of N and α. Clearly, the numerical errors decay as N increases.

Table 2: Comparison between the absolute errors of $x(t)$ obtained by the spectral method with $N = 35$ and method presented in [42] with $N = 6$ for Example 2

t	Method in [42]	Our method
0.1	2.72×10^{-5}	1.681×10^{-8}
0.2	4.50×10^{-5}	1.141×10^{-7}
0.3	4.10×10^{-5}	3.067×10^{-7}
0.4	3.84×10^{-5}	6.020×10^{-7}
0.5	4.86×10^{-5}	1.005×10^{-6}
0.6	6.34×10^{-5}	1.520×10^{-6}
0.7	6.93×10^{-5}	2.145×10^{-6}
0.8	6.52×10^{-5}	2.878×10^{-6}
0.9	6.61×10^{-5}	3.709×10^{-6}

Table 3: The maximum absolute errors of x and λ for different values of N and α for Example 2

N	α	Error (x)	Error (λ)	α	Error (x)	Error (λ)	α	Error (x)	Error (λ)
5	1.2	1.11×10^{-4}	8.09×10^{-5}	1.5	7.63×10^{-4}	5.19×10^{-4}	1.8	1.80×10^{-3}	1.06×10^{-3}
10		1.43×10^{-5}	1.05×10^{-5}		1.42×10^{-4}	9.71×10^{-5}		4.81×10^{-4}	2.82×10^{-4}
15		3.91×10^{-6}	2.87×10^{-6}		4.89×10^{-5}	3.34×10^{-5}		2.07×10^{-4}	1.21×10^{-4}
20		1.51×10^{-6}	1.11×10^{-6}		2.232×10^{-5}	1.52×10^{-5}		1.11×10^{-4}	6.51×10^{-5}
25		7.15×10^{-7}	5.24×10^{-7}		1.199×10^{-5}	8.81×10^{-6}		1.11×10^{-4}	6.51×10^{-5}

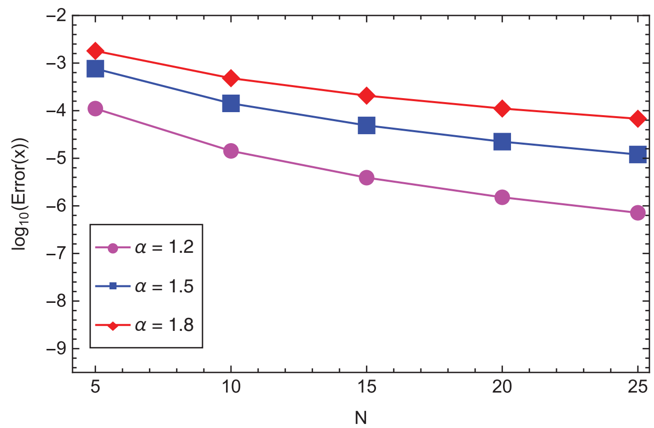

Figure 3: The L^{∞}-error of the approximate solution x_N in log scale *vs.* various N and α

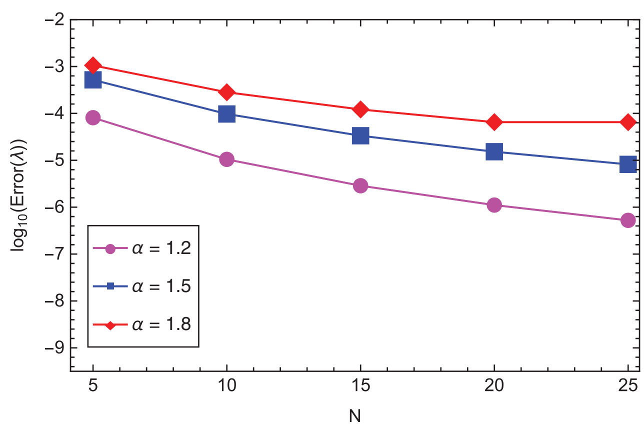

Figure 4: The L^{∞}-error of the approximate solution λ_N in log scale *vs.* various N and α

6 Conclusion

Numerically solving a class of nonlinear fractional two-point boundary value problems involving left- and right-sided fractional derivatives was fulfilled as a target of that work. The way to achieve that was done indirectly by recasting the considered system into a weakly singular integral system for the sake of the possibility of applying the Gauss quadrature rule on the transformed integral system. The construction and analysis of the used collocation method is proposed. The obtained results can be indirectly applied to solve fractional optimal control problems by considering the corresponding Euler–Lagrange equations [43,44]. A numerical example was given to confirm the convergence analysis and robustness of the scheme. Our future work is related to spectral methods for systems of nonlinear fractional differential equations and system of integral equations with non-smooth solutions.

Funding Statement: The Russian Foundation for Basic Research (RFBR) Grant No. 19-01-00019.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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