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Determination of Physical Properties of Euler Bernoulli Beam via the Method of Inverse Vibration Problem

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Abstract: In this study, some physical properties of an Euler–Bernoulli beam was tried to be estimated by using the method of inverse vibration problem. The Euler –Bernoulli beam was first modeled and simulated on the ANSYS program for different boundary condition to obtain data to be used as experimental input to the optimization program aroused during the solution of inverse problem. An numerical model of the Euler-Bernoulli beam with unknown parameters was also developed using a two-dimensional finite element model. Then, these two models were embedded into the optimization program to form the objective function to be minimized using genetic algorithms. After minimizing the squared difference of the natural frequencies from these two models, the unknown parameters of the beam was found. The estimated values were finally compared with the expected values and a very good correspondence was observed.

Keywords: Inverse problem, inverse vibration, FEM, Euler-Bernoulli beam, Genetic algorithm, free vibration, ANSYS, MATLAB.

1 Introduction

Beams find an important area of applications in many mechanical and civil engineering structures. As a result, studies on their statics and dynamics stability analysis have gained important place among mechanics researches, and hence, a vast amount of study has been carried out on this area lately. However, designing beams representing pre-specified behavior or suitable to any working conditions is a hard task because of large number of unknown parameters appearing in their designs. Consequently, to be able to overcome this difficulty and to estimate beam parameters, the method of inverse vibration problem has found its place in the design of beams.

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The method of inverse vibration problem can be basically used to estimate unknown parameters by using data obtained from experiments or computer simulations. There are some literatures using the inverse vibration for parameter estimations. For instance, Huang et al. [1] attempted inverse vibration method to solve a forced vibration problem raised in cutting tools which were modeled as Euler-Bernoulli beam. In the numerical solutions conjugate gradient method was utilized and the simulation results for the beam displacements were used for estimating the external forces on the cutting tool. Huang [2] tackled an inverse nonlinear forced vibration problem and solved it by using conjugate gradient method. In the solution, experimental results were used to estimate external forces on a damped multi degree of freedom system. Chiwiacowsky et al. [3] used dynamics inverse problem to assess damages in buildings through the use of experimental vibration measurements. Marinov et al. [4] utilized the variational imbedding method to solve the inverse problem arisen during the estimation of unknown coefficients of Euler-Bernoulli equation. Gladwell [5-7] developed finite element model for an in-line two-degree-of-freedom systems and solved it as an inverse vibration problem. Mass and stiffness matrices were written in a closed form procedure in such a way to minimize the mass.

Dynamic stability of Euler-Bernoulli beams were also investigated in some studies. For instance, Ozturk [8] solved the free vibration problem of a pre-stressed curved beam whose model is obtained through the finite element model of a largely deflected cantilever beam. The same problem was solved with ANSYS and the results are compared with the previous ones. Karaagac et al. [9] developed a finite element model for the lateral buckling of a cantilever beam with an edge crack using Euler-Bernoulli beam approach. Results from this model were compared with the experimental findings. Rossit and Laura [10] investigated a cantilever beam hanged by a mass-spring system at the free end by treating the beam as an Euler-Bernoulli beam.

In this paper, inverse vibration problem was utilized to find physical properties of an Euler-Bernoulli beam from its measured vibration frequencies. In the method proposed, the difference between the measured frequencies and the ones from numerical model with unknown parameters is minimized to be able to choose the best solution among infinitely many possible solutions that can arise in an inverse method. Simulation results from an ANSYS model were used to imitate the experimental data. Genetic algorithms are used in the optimizations to ensure not to be trapped by local minima.

2 Mathematical Model

According to Euler-Bernoulli beam theory, the deformations caused by the transverse shear stresses are accepted as zero [11]. The bending behavior of an Euler-Bernoulli beam was shown on the Figure 1.

Axes and cross section of the beam under consideration have been shown in Figure 2. The letter L represents the beam length, I the area moment of inertia, A the cross sectional area, b the width of the beam, and h the height of the beam.

Stress-strain relationship for an Euler-Bernoulli beam is given by the Hooke's law

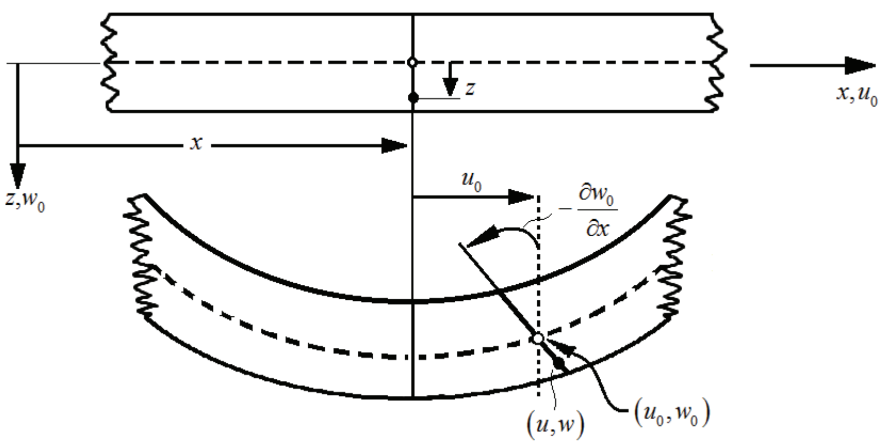

Figure 1: Euler-Bernoulli beam theory [11]

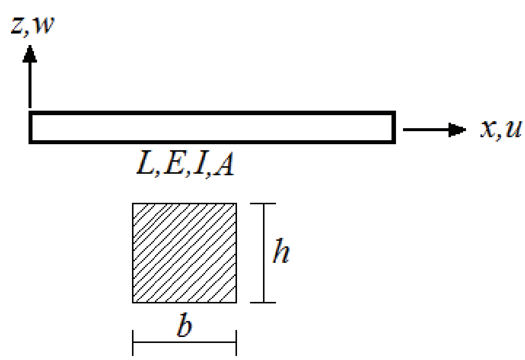

Figure 2: Coordinates and Geometry of Euler-Bernoulli beam

as [11]

$$\sigma = E\varepsilon \tag{1}$$

where E is the modulus of elasticity while σ representing the stress. Material was assumed to be linear isotropic.

The relation between the strain ε and the axial displacement u is given by [12]

$$\varepsilon = \frac{du}{dx} \tag{2}$$

The relationship between the axial displacement u and the rotation of the cross-section θ is obtained as [12].

$$du = -d\theta z \tag{3}$$

The relationship between the deflection of beam w and the rotation of the cross-section θ can be written as

$$\theta \cong \frac{dw}{dx} \tag{4}$$

from the Euler-Bernoulli beam theory [11]. If the equations (4) and (3) are substituted into equation (2),

$$\varepsilon = -\frac{d^2w}{dx^2}z \tag{5}$$

can be obtained [11]. The curvature of the beam from equation (5) can be written as [12]

$$\kappa = \frac{d^2w}{dx^2} \tag{6}$$

2.1 Finite Element Model

A planar beam bending element with two nodes, each having two-degree-of-freedom, was chosen obeying Euler-Bernoulli beam theory. The beam element having the degree of freedom as the beam deflection "w" and the rotation of the cross-section "θ" was depicted in Figure 3.

The potential energy for a beam element in bending vibration is given as [12]

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{7}$$

The kinetic energy of a beam element in bending vibration is

$$T = \frac{1}{2}\int_{0}^{l} \rho A \left(\frac{dw}{dt}\right)^{2} dx \tag{8}$$

If the potential and kinetic energy expressions are substituted in the Hamilton principle [12]

$$L = \int_{t_0}^{t_1} \delta(T-U)dt + \int_{t_0}^{t_1} \delta W dt \tag{9}$$

is obtained, where W expresses the work done by external forces. If the equation (9) is minimized for a system under undamped free vibration, the equation of motion of a beam undergoing bending vibrations can be obtained as [13]

$$[m^e]\{\ddot{q}^e\} + [k^e]\{q\} = 0 \tag{10}$$

where $[m^e]$ and $[k^e]$ are the mass and stiffness matrices, respectively. If the curvature of the Euler-Bernoulli beam is re-written

$$\kappa = \frac{d^2w}{dx^2} = \frac{d^2}{dx^2}\{w\} \rightarrow \kappa = Dw \tag{11}$$

where $\boldsymbol{D}$ is the linear differential operator and $\boldsymbol{w}$ displacement. Beam displacement equation is then [14]

$$w = Nd \tag{12}$$

where $\boldsymbol{N}$ represents shape function and $\boldsymbol{d}$ nodal displacement operator. The nodal displacement operator is given as

$$d = \begin{Bmatrix} w_i \\ \frac{dw_i}{dx} \end{Bmatrix} = \begin{Bmatrix} w_i \\ \theta_i \end{Bmatrix} \tag{13}$$

If the equation (13) is substituted in equation (12) the expression

$$\kappa = Dw = DNd \rightarrow \kappa = Bd \tag{14}$$

is obtained [14], where the expression $\boldsymbol{B}$ is the strain-displacement matrix. If the stiffness matrix for the bending beam element is given as [12, 14]

$$[k^e] = EI \int_{A} B^T B dA \tag{15}$$

The mass matrix for the bending beam element is also given as [12, 14]

$$[m^e] = \rho A \int_A N^{\mathrm{T}} N dx \tag{16}$$

If the mass and stiffness matrices developed for the bending beam element are combined together so as to represent an Euler-Bernoulli beam

$$\begin{aligned} [M] &= \sum_{e=1}^{n} m^e \\ [K] &= \sum_{e=1}^{n} k^e \end{aligned} \tag{17}$$

are obtained, where the matrices [M] and [K] are respectively the global mass and stiffness matrices, and n the number of finite elements used in the model.

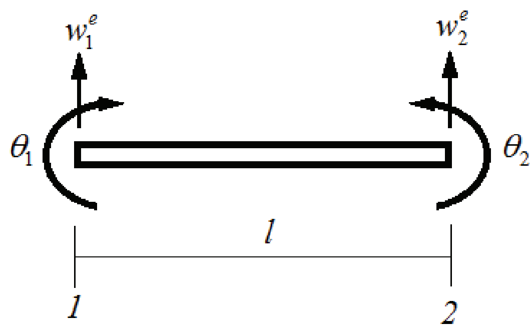

Figure 3: Plane beam bending element

2.2 *Development of Shape Function*

Plain beam element shown in Fig. 3 has two nodes with two degrees of freedom, and hence it has four degrees of freedom in total. To form shape function, cubic polynomial with four terms for each degree of freedom has been chosen as the displacement shape function.

$$w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{18}$$

Beam displacement can be written in the following form [15]:

$$w = \{P\}^{\mathrm{T}} \{a\} \tag{19}$$

In Eqn. (19), a is the coefficient vector and P is the interpolation polynomial term vector. Coefficient and the interpolation polynomial term vector are given in the following format [15]

$$\begin{aligned} \{a\} &= \left\{a_1 \quad a_2 \quad a_3 \quad a_4\right\}^{\mathrm{T}} \\ \{P\} &= \left\{1 \quad x \quad x^2 \quad x^3\right\}^{\mathrm{T}} \end{aligned} \tag{20}$$

If the displacement polynomial given in Eqn. (18) is substituted in Eqn. (12) in a matrix form and expanded for each node [15],

$$\{d\} = [X]\{a\} \tag{21}$$

is obtained. X in Eqn. (21) represents the expanded displacement matrix of dimension 4x4. Then, Eqn. (21) is solved for coefficient vector

$$\{a\} = [X]^{-1}\{d\} \tag{22}$$

If Eqn. (22) is substituted in Eqn. (19), one may obtain

$$w = \{P\}^{\mathrm{T}}[X]^{-1}\{d\} \tag{23}$$

If Eqn. (23) is substituted in Eqn. (12) and reorganized, the shape function is developed in the form below

$$N = \{P\}^{\mathrm{T}}[X]^{-1} \tag{24}$$

2.3 Dynamics Analysis

The equation of motion for the beam undergoing an undamped free vibration was given in Eqn. (10). The equation of motion for the global system is

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \tag{25}$$

for which a harmonic solution can be proposed in the following form:

$$\{q\} = \{\psi\}\, sin(\omega t) \tag{26}$$

If Eqn. (26) is substituted in Eqn. (25)

$$-[M]\{\psi\}\,\omega^2 \sin(\omega t) + [K]\{\psi\}\sin(\omega t) = 0 \tag{27}$$

can be obtained. If Eqn. (27) is further reorganized, it takes the following eigenvalue problem form

$$\left([K] - \omega^2 [M]\right)\{\psi\} = 0 \tag{28}$$

where $\lambda = \omega^2$ are eigenvalues representing vibration frequencies while ψ are eigenvectors representing vibration modes.

3 Forming Objective Function for Genetic Algorithm use

There are infinitely many possible solutions in the solution of inverse problems, and hence, some form of optimization is necessary to choose the best solution amongst them. Furthermore, there is a possibility of getting trapped in local minima in such optimizations of multimodal problems. Therefore, genetic algorithms are utilized to make sure that global minima are searched for the solutions.

Objective functions are needed in the optimizations and the sum of squared difference between the frequencies obtained from simulations and numerical model was accepted as the objective function for optimizations. Natural frequencies are chosen as optimization parameters as they provide more information about systems with fewer data, and this further leads to less computation time. The objective function is used in the objective function evaluations is

$$FF(t) = \min \sum_{i=1}^{n} \left(\{\omega_{ANSYS}(t)\} - \{\omega_{Model}(t)\} \right)^2 \tag{29}$$

where ω_{ANSYS} represents the natural frequencies obtained from ANSYS, which imitates experimental data, while ω_{Model} represents the frequencies obtained from numerical model, which includes unknown system parameters.

In the optimizations with Genetic Algorithms, settings are of great importance because small changes result in large difference in solutions. Settings were decided after a long period of trial and error as shown in Table 1 [13].

Table 1: Genetic Algorithm Settings

Population size	30
Selection	Stochastic uniform
Mutation	Adaptive feasible
Mutation rate	0.01
Crossover	Arithmetic

4 Simulation data

In the study performed in ANSYS, Euler-Bernoulli beam is modeled with clamped-free (CF), clamped-clamped (CC), and simple-simple supported (SS) boundary conditions. In the model, BEAM3 element from ANSYS element library was used for the Euler-Bernoulli beam. The BEAM3 element is a single axis elastic beam element with two nodes and three degrees of freedom on each node, which can be used to model both Euler-Bernoulli and Timoshenko beams.

Geometric and physical properties of Euler-Bernoulli beam to be used in ANSYS are provided in Table 2 below.

Table 2: Geometric and physical properties of Euler-Bernoulli beam

$E = 210000(10^6)\,\mathrm{N/m^2}$		$\rho = 7850\,\mathrm{Kg/m^3}$
$L = 1\mathrm{m}$	$b = 0.01\mathrm{m}$	$h = 0.005\mathrm{m}$

Table 3: Natural frequencies of Euler-Bernoulli beam from simulations (Hz)

Mode no	Clamped-Clamped (CC)	Clamped-Free (CF)	Simply supported (SS)
1	26.58	4.178	11.73
2	73.27	26.18	46.91
3	143.64	73.30	105.53
4	237.42	143.63	187.60
5	354.63	237.41	293.09
6	495.26	354.61	422.00
7	659.28	495.21	574.32
8	846.67	659.21	750.01
9	1057.40	846.58	949.07
10	1291.50	1057.30	1171.5

5 Parameter estimation

Numerical model for the Euler-Bernoulli beam was constructed in MATLAB, and the optimizations are realized in Genetic Algorithm Toolbox in MATLAB.

The finite element model for the Euler-Bernoulli beam was constructed with a mesh size of 500x500, and its elasticity module E and density ρ were estimated for different boundary conditions. The estimates are tabulated together with the objective function evaluations and percent errors.

6 Results and discussion

In this study, physical properties of an Euler-Bernoulli beam were estimated based on its known/measured natural frequencies. The simulation results obtained from ANSYS were assumed to be experimental data. On the other hand, the numerical

Table 4: Natural frequency (Hz) estimates for Euler-Bernoulli beam

Mode no	Clamped-Clamped (CC)	Clamped-Free (CF)	Simply supported (SS)
1	26.56	4.17	11.72
2	73.22	26.16	46.87
3	143.53	73.25	105.45
4	237.27	143.54	187.47
5	354.43	237.28	292.92
6	495.04	354.46	421.80
7	659.08	495.07	574.13
8	846.55	659.11	749.89
9	1057.46	846.60	949.11
10	1291.81	1057.50	1171.78

Table 5: Elasticity module (E) and density (ρ) estimates for Euler-Bernoulli beam

	Clamped-Clamped (CC)		Clamped-Free (CF)		Simply supported (SS)	
	E (N/m^2)	ρ (kg/m^3)	E (N/m^2)	ρ (kg/m^3)	E (N/m^2)	ρ (kg/m^3)
Desired	2.1x10^{11}	7850	2.1x10^{11}	7850	2.1x10^{11}	7850
Estimated	2.026x10^{11}	7584,86	2.07x10^{11}	7755.30	2.111x10^{11}	7904.566
Error (%)	3.52	3.38	1.428	1.2	-0.52	-0.695
Objective function evaluations ($FF(t)$)	0.003788		0.1278		0.2276	

model with unknown parameters was established in MATLAB. Finally, the natural frequencies from these two sources were combined in the objective function to estimate the unknown physical parameters.

The inverse vibration problem was tackled as an optimization problem and solved using Genetic Algorithm Toolbox of MATLAB. Elasticity module (E) and density (ρ) of the beam were estimated simultaneously for different boundary conditions and compared with the real values.

The proposed method gives base not only for estimation of physical properties of materials used in a system but also for estimation of initial conditions and/or boundary conditions, etc.

When the results obtained from the optimizations are compared with the real values, very good correspondence is observed within a maximum error of 3.5%. Although the result would be satisfactory for most of the applications, further improvements might be sought for by improving the following items:

- Timoshenko beam theory which takes shear deformations into account would be used.
- Damping effects would be added to the model.
- Nonlinear analysis would be performed.
- More suitable finite element would be chosen to increase the effectiveness of finite element method, and further a shape function suitable to this element would be chosen.
- Mesh density of the model would be increased for improved computations.
- More data and some constraints would be added to the objective function to improve the Genetic Algorithm solution.
- Genetic Algorithm itself would be improved with more intelligence such as Fuzzy and/or Neural Networks, or more advanced mutation and crossover methods would be embodied for getting improved results.

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