Prestress Force Identification for Eccentrically Prestressed Concrete Beam from Beam Vibration Response

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Abstract: The measurement of residue prestress force is one main issue for condition and performance assessment of prestressed concrete beam bridge. This paper proposes a vibration based parameter estimation technique for this purpose. Under given form external excitation, beam velocity responses at multiple points are collected firstly. The prestress force of the beam is then identified based on the minimization of the least square difference between the measured response and the baseline response. A numerical study on a beam of variant length, subjected to a constant prestress force with variant eccentricity, is conducted to show the effect of prestress force and the effect of bending moment due to eccentricity on fundamental frequency of the beam. The results show that this vibration based method for prestress force identification is both theoretically feasible and practically workable.

Keywords: prestress force, identification, effect of eccentricity, sensitivity

1 Introduction

Prestressed concrete beam bridge is one of the most popular types of bridge in highway system. Interest in the safety and load assessment of this type of bridge is increased in recent years. Since prestress force is one of the most important parameters, an accurate and efficient measurement technique is important. Most of the in-service bridges have not been equipped with sensors on prestressed tendons. Consequently, prestress force cannot be measured from popular detective methods nowadays. Several researchers tried to estimate the residue prestress force by vibration test. Abraham, Park and Stubbs (1995) reported that the effect of prestress force on the mode shape and amplitude is tiny. Saiidi, Douglas and Feng (1994) studied on the variation trend of natural frequencies to the axial force, and a concept of effective rigidity was used to account for this trend. Lu and Law (2006) reported

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a method based on sensitivity for prestress force identification, and the method was used to identify a PC beam under axial prestress force. Li and Sun (2008) made a further study on the identification of eccentricity compression force with the sensitivity method. Kim, Park, Hong, Cho and Na (2009) reported the effectiveness of vibration-impedance-based method on the identification of prestress force loss. Moreover, several researchers studied on the effect of dead load on the vibration characteristic of beams. Takabatake, H. (1990) established the governing equation of a beam under transverse dead load using Hamilton principal. The effect of dead load was obtained by solving the governing equation using Galerkin method.

In this study, a method based on vibration response measurement is proposed to estimate the residue prestress force. The proposed method takes both the effect of prestress force and the effect of eccentricity into account. Firstly, the responses of the beam subjected to an eccentric prestress force under a given form external excitation are collected by sensors. Secondly, the responses of a FEM beam, with zero initial prestress force, are calculated. There is a difference between the two responses above. Thirdly, the sensitivities of dynamic responses to prestress force and the ones to bending moment due to eccentricity are calculated. Then, a difference of prestress force is obtained via the relationship between sensitivities and responses. Consequently, the FEM is updated. Finally, the prestress force is identified when the steps above get a converged result. Numerical study is carried out to verify the effectiveness of this method. Numerical studies of a beam with variant length, subjected to a constant prestress force with variant eccentricity, are conducted to show the effect of prestress force and the effect of bending moment due to eccentricity on fundamental frequency of the beam.

2 Method

For the structure subjected to an arbitrary loading, the differential equation of motion can be written as:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \tag{1}$$

where M, C and K are the mass, damping and stiffness matrix, respectively. F is the excitation vector.

Suppose a beam subjected to an eccentric prestressed force as shown in Fig. 1. The beam can be modeled as n finite elements of beam. So the stiffness matrix and mass matrix can be written as:

$$K = \bar{K} - K_G + K_M = \sum_{i=1}^{n} \left(\bar{k}^i - k_G^i + k_M^i \right) \quad , \quad M = \sum_{i=1}^{n} m^i \tag{2}$$
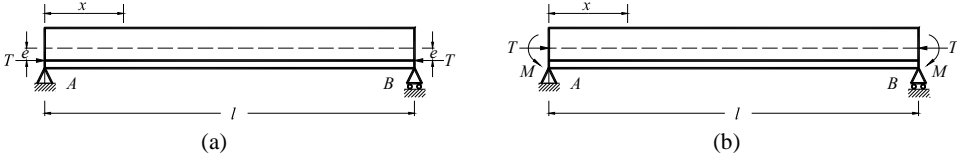
Figure 1: (a) Eccentrically prestressed beam with a straight unbonded tendon; (b) transferring the eccentric prestress force to the center of the cross section of the beam as the superposition of an axial prestress force and a moment couple.

where $\bar{K}$, K_G and K_M are the global elastic stiffness matrix, geometric stiffness matrix caused by axial force and stiffness matrix caused by the moment due to eccentricity, respectively. $\bar{k}^i$, k_G^i, k_M^i and m^i are the elemental elastic stiffness matrix, geometric stiffness matrix, and mass matrix, respectively.

The following elemental elastic stiffness matrix, geometric stiffness matrix, stiffness matrix caused by the moment due to eccentricity and mass matrix are adopted in this study, as below.

$$
\bar{k}^i = \frac{2EI}{L^3}
\begin{pmatrix}
6 & 3L & -6 & 3L \\
3L & 2L^2 & -3L & L^2 \\
-6 & -3L & 6 & -3L \\
3L & L^2 & -3L & 2L^2
\end{pmatrix}, \quad
k_G^i = \frac{T}{30L}
\begin{pmatrix}
36 & 3L & -36 & 3L \\
3L & 4L^2 & -3L & -L^2 \\
-36 & -3L & 36 & -3L \\
3L & -L^2 & -3L & 4L^2
\end{pmatrix},
$$

$$
k_M^i = \frac{EA}{60L} \cdot \left(\frac{d_2 - d_1}{L}\right)^2 \cdot
\begin{pmatrix}
36 & 3L & -36 & 3L \\
3L & 4L^2 & -3L & -L^2 \\
-36 & -3L & 36 & -3L \\
3L & -L^2 & -3L & 4L^2
\end{pmatrix},
$$

$$
m^i = \frac{\bar{m}L}{420}
\begin{pmatrix}
156 & 22L & 54 & -13L \\
22L & 4L^2 & 13L & -3L^2 \\
54 & 13L & 156 & -22L \\
-13L & -3L^2 & -22L & 4L^2
\end{pmatrix}
\tag{3}
$$

where L, $\bar{m}$ and T are the length, the mass per meter and the prestress force of element, respectively. EI and EA are the flexural rigidity and compressive rigidity of the cross section of the beam. d_2 and d_1 are the deflection at the left end and the right end of the element, respectively. Since the prestress force T is set to be positive, the item K_G is minus in Eq. (2) to indicate the softening effect of prestress force on stiffness.

Rayleigh damping is used, and the damping matrix C is written as

$$
C = a_0 M + a_1 K
\tag{4}
$$

where a_0 and a_1 are the two Rayleigh damping coefficients.

Take the first differential of the dynamic response with respect to the prestress force T, Eq. (1) becomes

$$[M]\left\{\frac{\partial \ddot{x}}{\partial T}\right\} + [C]\left\{\frac{\partial \dot{x}}{\partial T}\right\} + [K]\left\{\frac{\partial x}{\partial T}\right\} + \left[\frac{\partial C}{\partial T}\right]\{\dot{x}\} + \left[\frac{\partial K}{\partial T}\right]\{x\} = 0 \tag{5}$$

It is noted that $[M]$ is not dependent on T, and thus the partial derivative $\frac{\partial M}{\partial T}$ in Eq. (5) disappears.

According to Eq. (2) and Eq. (4),

$$\begin{aligned}
\frac{\partial K}{\partial T} &= \frac{\partial (\bar{K} - K_G + K_M)}{\partial T} = -\frac{\partial K_G}{\partial T} + \frac{\partial K_M}{\partial T} \\
\frac{\partial C}{\partial T} &= \frac{\partial (a_0 M + a_1 K)}{\partial T} = a_1 \frac{\partial K}{\partial T} = a_1 \left(-\frac{\partial K_G}{\partial T} + \frac{\partial K_M}{\partial T}\right)
\end{aligned} \tag{6}$$

Since the deflections of the beam under bending moment caused by eccentric prestress force T can be written as

$$d_x = \frac{Te}{2EI}x(l-x) \tag{7}$$

where x is the distance from the left end of the beam to the point of deflection, e is the eccentricity of the prestress force. Obviously, d_1 and d_2 are dependent on T, and the partial derivative $\frac{\partial K_M}{\partial T}$ appears in Eq. (7).

Substituting Eq. (6) into Eq. (5), the following ODE is obtained

$$[M]\left\{\frac{\partial \ddot{x}}{\partial T}\right\} + [C]\left\{\frac{\partial \dot{x}}{\partial T}\right\} + [K]\left\{\frac{\partial x}{\partial T}\right\} = \left(\frac{\partial K_G}{\partial T} - \frac{\partial K_M}{\partial T}\right)(a_1\{\dot{x}\} + \{x\}) \tag{8}$$

Define $S = \frac{\partial x}{\partial T}$ as the sensitivity of dynamic response to prestress force and $\{P\} = \left(\frac{\partial K_G}{\partial T} - \frac{\partial K_M}{\partial T}\right)(a_1\{\dot{x}\} + \{x\})$. Then Eq. (8) becomes

$$[M]\{\ddot{S}\} + [C]\{\dot{S}\} + [K]\{S\} = \{P\} \tag{9}$$

S can be obtained by solving Eq. (8). According to the definition of sensitivity, the relationship between dynamic response and prestress force can be found as

$$\{S\} \cdot \delta T = \{\delta x\} \tag{10}$$

where $\{\delta x\}$ is the difference between the response of the beam subjected to eccentric prestress force T and the response of the beam subjected to $T + \delta T$, under the same excitation.

Then δT can be obtained by solving Eq. (10) using least square method. Since $\delta T - \{\delta x\}$ relationship is nonlinear, numerical iterations are generally required to get converged results. The convergence principle to stop the iteration is set to be

$$\frac{|T_k - T_{k-1}|}{|T_k|} = \frac{|\delta T_k|}{|T_k|} \leq r \tag{11}$$

where r, set to be 1×10^{-5}, is a threshold value to stop the iteration.

3 Effect of prestress force and eccentricity on fundamental frequency

A simply-supported beam is studied. The width and depth of the beam are both 20cm. The mass per meter of the beam and Young's modulus of concrete are 100kg/m and 32.5Gpa, respectively. The prestress force is set to be 100kN. The length of the beam varies from 1.8m to 12m. The eccentricity (denoted by e) varies from 0 to 0.1m with 0.02m increment. The variation of fundamental frequency is shown in Fig. 2 and Fig. 3. We denote the fundamental frequency of the beam without prestress force as the reference frequency ω_0. Put an axial prestress force T onto the beam, the fundamental frequency changes to ω_1. When the eccentricity is varying, the fundamental frequency changes to ω_2. Denote $\alpha = \frac{\omega_1 - \omega_0}{\omega_0} \times 100\%$ as the softening effect of prestress force. Denote $\beta = \frac{\omega_2 - \omega_1}{\omega_0} \times 100\%$ as the stiffening effect of eccentricity. Denote $\varepsilon = e/h$, where h is the depth of the beam.

From Fig. 2 and Fig. 3 we can find that, as h/l decreases, the absolute values of α and β both have a notable increase. That means as the beam getting more slender, the effect of prestress force and the effect of eccentricity on the fundamental frequency both increase significantly, thus should not be ignored. Moreover, as the eccentricity increases, the effect of eccentricity on the fundamental frequency has an increase.

4 Case study on prestress force identification

A simply-supported beam is studied. The width, depth and length of the beam are 20cm, 20cm and 6m respectively. The mass per meter of the beam and Young's modulus of concrete are 100kg/m and 32.5Gpa, respectively. The first three natural frequencies are 9.083, 36.361 and 82.069 Hz, obtained by solving the frequency equation. Figure 4 illustrates the procedure of the identification of the prestress force. The beam is subjected to four different levels of eccentric prestress forces of 30, 60, 90, 120 kN, with constant eccentricity of 0.1m. The excitation force is shown in Fig. 4(a). The response of the beam with or without prestress force is obtained under the excitation above, as shown in Fig. 4(b). Then the differences of responses and sensitivities $\{S\}$ were calculated, as shown in Fig. 4(c) and Fig.
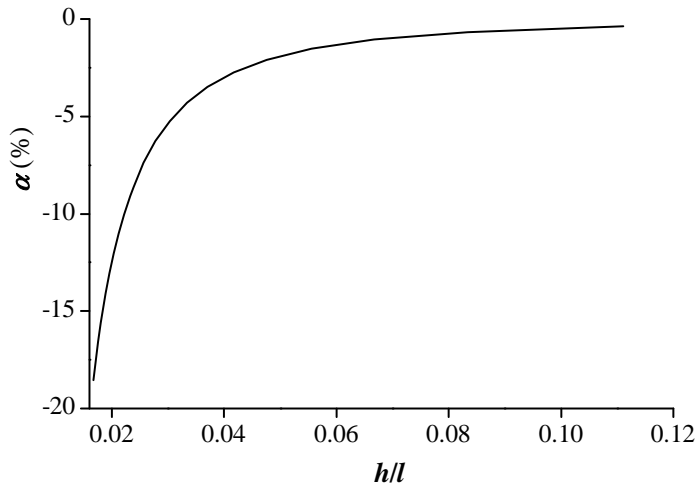
Figure 2: Effect of axial prestress force on fundamental frequency when *h/l* varies.
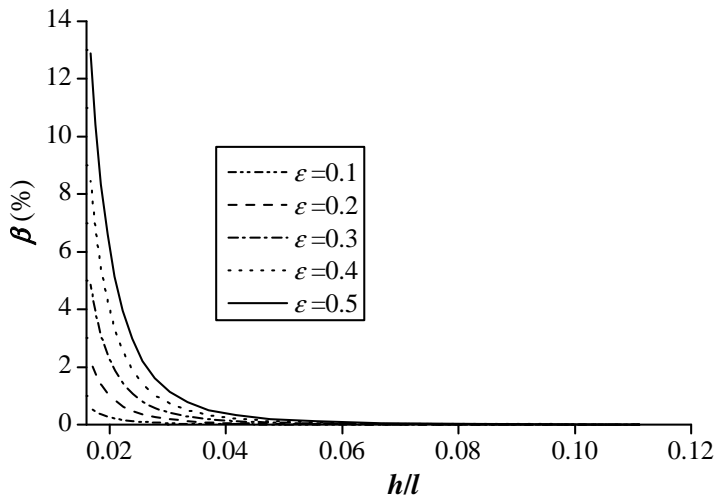

Figure 3: Effect of eccentricity on fundamental frequency when *h/l* varies.

4(d). After that, δT was calculated by solving Eq. (11) using least square method. Finally the prestress force T was obtained when iteration of the identification converged controlled by Eq. (12). Fig. 4(e) shows the comparison of the responses when the iteration of the identification stopped. The cases considering or ignoring the effect of eccentricity are both conducted. The results of the prestress force identified are shown in Tab. 1.
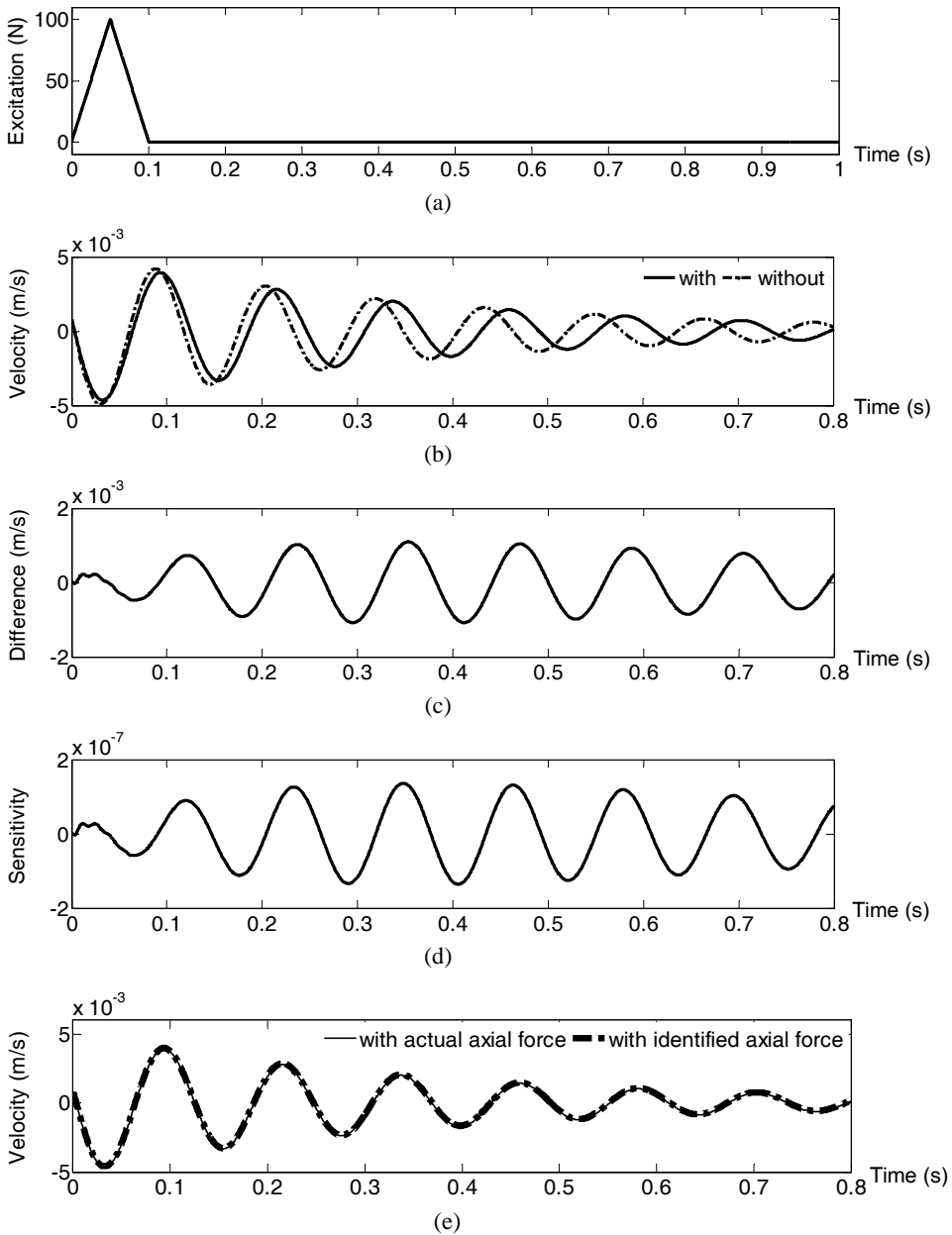
Figure 4: Excitation, velocity response and sensitivity: (a) excitation; (b) comparison of response with and without prestress force; (c) the difference of the responses in (b); (d) sensitivity of velocity response to prestress force; (e) comparison of responses when identification finished.

Results from the simulation above indicate that the prestress force can be identi?ed accurately from the measured dynamic responses. Not only the effect of prestress force, but also the effect of eccentricity should be taken into account to get accurate results of the identification of prestress force.

Table 1: The results of identification of prestress force (T)

Case	Actual T (kN)	Ignoring eccentricity effect		Considering eccentricity effect (kN)
		Identified T (kN)	Error (%)	
1	30	28.413	-5.28922	30
2	60	53.654	-10.5764	60
3	90	75.726	-15.8594	90
4	120	94.636	-21.1364	120

5   Conclusions

A method based on velocity-response-sensitivity to prestress force is proposed for the identification of the prestress force of PC beam. Numerical study is conducted to verify the effectiveness of the proposed method. Results from numerical study show that the effect of prestress force and the effect of eccentricity are both important to get accurate results of identification of prestress force.

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