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Multiple Cracked Beam Modeling and Damage Detection using Frequency Response Function

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Abstract: An efficient approach to identify multiple cracks in a slender beam using the frequency response function data is presented. It is formulated in a general form from the dynamic stiffness equation of motion for a structure and then applied to a slender beam. The cracks are modeled by rotational springs and the frequency response function is computed based on a spectral element model by the spectral finite element. The procedure gives a linear relationship explicitly between the changes of the measured frequency response function and crack parameters. The inverse problem is solved iteratively for the depths and locations of the cracks through the sensitivity-based model updating. Some numerically simulated tests on beam examples are provided for validating the feasibility of the method to identify the cracks. The results are generally agreement with the target values. Finally, the effect of noise on the damage identification is discussed in the numerical examples.

Keywords: Cracks, Spectral element, Inverse problems, FRF, Vibration measurement.

1 Introduction

Due to deterioration of materials, overloading, severe environment condition and insufficient maintenance, the performance of civil infrastructural systems will degrade during their service life. Cracks or crack-like damage are observed in the components of the degraded structure. The dynamic characteristic of the cracked components should be monitored more carefully. This has generated many researches on the vibration monitoring of components with one or more cracks (O.S. Salawu(1997), A.C. Chasalevris and C.A. Papadopoulos(2006), A.S.Sekhar(2008)).

The majority of studies in the crack identification of a beam concern single crack problem (H.P.Lin(2004)). Generally, crack is not singly distributed in a structural component. The case of multiple cracks is received the same degree of attention

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(X.Q.Zhang *et al* (2010), D.P.Patil and S.K.Maiti (2003, 2005)), recently. Lee (2009a) applied the Newton-Raphson method to identify k cracks using $2k$ natural frequencies of the cracked beam, and he (2009b) had also used this method to detect multiple cracks in a beam using the vibration amplitudes instead of the natural frequencies.

The FRF-data (U.Lee and J.Shin(2002a, 2002b), S.J.Araujo *et al* (2005)) had also been used for the past two decades because of some advantages over using the natural frequencies or modal shape. The major advantage of using FRFs is more damage information can be obtained in a desired high frequency range than modal shape. For structural numerical modeling and dynamic characteristic analysis in high frequency range, the spectral element method is more attractive than the classical FEM method. The spectral element method uses the frequency dependant shape functions in contrast to the constant polynomial shape functions employed by finite element analysis to construct the element mass and stiffness matrices. Thus it can accurately model structural dynamic characteristic in high frequency range.

The objective of this study is to present a simple method for modeling transverse vibration of beams with multiple cracks based on massless rotational spring model and spectral element method. The formulation helps to calculate response data and estimate cracks parameters through sensitivity-based model updating technique.

2 Crack modeling

The problem considered here is a beam element with a transverse crack (as shown in Fig.1). The element has two nodes and two degrees of freedoms per node which are transverse displacement and rotation. The length of element is$L$, and its area of cross-section is$A$. The normalized depth and location of a crack are $\beta_1 = L_1/L$ and$\alpha_1 = h_{d1}/h$, respectively. The cracks can be represented by massless rotational springs. The equivalent spring stiffness for open crack (T.G.Chondros *et al* (1998)) is given by

$$K = \frac{EI}{6\pi h(1-v^2)f(\alpha)} \tag{1}$$

$$\begin{aligned} f(\alpha) =& 0.6272\alpha^2 - 1.04533\alpha^3 + 4.5948\alpha^4 - 9.973\alpha^5 + 20.2948\alpha^6 \\ & - 33.0351\alpha^7 + 47.1063\alpha^8 - 40.7556\alpha^9 + 19.6\alpha^{10} \end{aligned} \tag{2}$$

where h is height and b is width of the beam. E, I and v are Young's modulus, moment of inertia and Poisson ration, respectively.

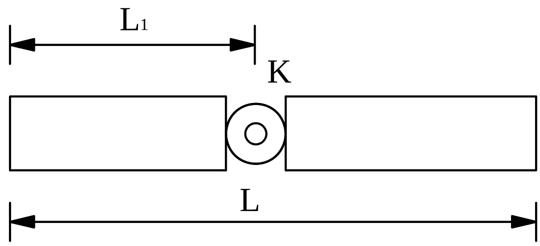

Figure 1: Beam element with a crack simulated by rotational spring

For spectral element modeling, the elemental displacements have the following form for the left and right parts of the beam

$$\begin{aligned} u_1 &= a_1 \sin kx + a_2 \cos kx + a_3 \sinh kx + a_4 \cosh kx \\ u_2 &= a_5 \sin kx + a_6 \cos kx + a_7 \sinh kx + a_8 \cosh kx \end{aligned} \tag{3}$$

where $k = \sqrt[4]{\omega^2 m/EI}$ is the wavenumber, $a_i, (i = 1, 2, \cdots 8)$ is the coefficients of the element given by boundary conditions.
When $x = 0$, at the left end of the element

$$u_1(x) = q_1, \quad \frac{\partial u_1(x)}{\partial x} = q_2 \tag{4}$$

When $x = L_1$, at the crack location of the element

$$\begin{aligned} & u_1(x) = u_2(x) \\ & K\left(\frac{\partial u_2(x)}{\partial x} - \frac{\partial u_1(x)}{\partial x}\right) = EI\frac{\partial^2 u_1(x)}{\partial x^2} \\ & \frac{\partial^2 u_1(x)}{\partial x^2} = \frac{\partial^2 u_2(x)}{\partial x^2} \\ & \frac{\partial^3 u_1(x)}{\partial x^3} = \frac{\partial^3 u_2(x)}{\partial x^3} \end{aligned} \tag{5}$$

When $x = L$, at the right end of the element

$$u_2(x) = q_3, \frac{\partial u_2(x)}{\partial x} = q_4 \tag{6}$$

From Eqs. (3)-(6), the boundary conditions can be written in a matrix form as

$$\begin{Bmatrix} q_1 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ k & 0 & k & 0 & 0 & 0 & 0 & 0 \\ \sin kL_1 & \cos kL_1 & \sinh kL_1 & \cosh kL_1 & -\sin kL_1 & -\cos kL_1 & -\sinh kL_1 & -\cosh kL_1 \\ K_T k\cos kL_1 - k^2 \sin kL_1 & -K_T k\sin kL_1 - k^2\cos kL_1 & K_T k\cosh kL_1 + k^2\sinh kL_1 & K_T k\sinh kL_1 + k^2\cosh kL_1 & -K_T k\cos kL_1 & K_T k\sin kL_1 & -K_T k\cosh kL_1 & -K_T k\sinh kL_1 \\ -k^2\sin kL_1 & -k^2\cos kL_1 & k^2\sinh kL_1 & k^2\cosh kL_1 & k^2\sin kL_1 & k^2\cos kL_1 & -k^2\sinh kL_1 & -k^2\cosh kL_1 \\ -k^3\cos kL_1 & k^3\sin kL_1 & k^3\cosh kL_1 & k^3\sinh kL_1 & k^3\cos kL_1 & -k^3\sin kL_1 & -k^3\cosh kL_1 & -k^3\sinh kL_1 \\ 0 & 0 & 0 & 0 & \sin kL & \cos kL & \sinh kL & \cosh kL \\ 0 & 0 & 0 & 0 & k\cos kL & -k\sin kL & k\cosh kL & k\sinh kL \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{7}$$

Eq. (7) can be simplified to be

$$\mathbf{q} = \mathbf{P} \cdot \mathbf{A} \tag{8}$$

The nodal forces at the two sides of the element is expressed in the matrix form as follows by Eq. (3)

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} -k^3 & 0 & k^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & k^2 & 0 & -k^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k^3 \cos kL & -k^3 \sin kL & -k^3 \cosh kL & -k^3 \sinh kL \\ 0 & 0 & 0 & 0 & -k^2 \sin kL & -k^2 \cos kL & k^2 \sinh kL & k^2 \cosh kL \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{9}$$

Eq. (9) can be simplified to be

$$\mathbf{F} = \mathbf{Q} \cdot \mathbf{A} \tag{10}$$

Eliminating the coefficients vector **A** from Eq. (8) and Eq.(10), the nodal forces to nodal displacements relation can be obtained

$$\mathbf{F} = \mathbf{Q} \cdot \mathbf{P}^{-1} \cdot \mathbf{q} = \mathbf{z} \cdot \mathbf{q} \tag{11}$$

where **z** is the dynamic stiffness matrix of the spectral element with a crack.

The elemental stiffness matrix is obtained and then assembled into the global stiffness matrix. The compatibility of the elemental stiffness matrix is satisfied by constraining each element displacement to match the displacement at the nodes. Hence,

$$\mathbf{F} = \mathbf{Z} \cdot \mathbf{q} \tag{12}$$

where **Z** is the global dynamic stiffness matrix, which is the reciprocal of the transfer function. This matrix includes the information of the crack, and is frequency dependent.

3 Sensitivity based crack detection

For the detection of k cracks, the vector of updating parameters is

$\theta = [\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k]^T.$

The measurement vector is denoted to be $\mathbf{q}^0 = [q_{11}^0, q_{12}^0, \dots, q_{mn}^0]^T$, where m is the number of excitation frequency points, n is the number of vibration amplitude points selected. The corresponding vibration amplitude computed from the numerical model is$\mathbf{q} = [q_{11}, q_{12}, \dots, q_{mn}]^T$.

The procedure for crack detection is applied as follows:

Assume initial parameters of $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots \alpha_k, \beta_k$;

Compute the vibration amplitude using Eq. (12) and the residual error is $\delta\mathbf{q} = \mathbf{q}^0 - \mathbf{q}$.

The sensitivity matrix, $\mathbf{S}$, is the first derivative of $\mathbf{q}$ with respect to the updating parameters $\boldsymbol{\theta}$. The sensitivity matrix $\mathbf{S}$

$$S = \begin{bmatrix} \frac{\partial q_{11}}{\partial \alpha_1} & \frac{\partial q_{11}}{\partial \beta_1} & \frac{\partial q_{11}}{\partial \alpha_2} & \frac{\partial q_{11}}{\partial \beta_2} & \cdots & \frac{\partial q_{11}}{\partial \alpha_k} & \frac{\partial q_{11}}{\partial \beta_k} \\ \frac{\partial q_{12}}{\partial \alpha_1} & \frac{\partial q_{12}}{\partial \beta_1} & \frac{\partial q_{12}}{\partial \alpha_2} & \frac{\partial q_{12}}{\partial \beta_2} & \cdots & \frac{\partial q_{12}}{\partial \alpha_k} & \frac{\partial q_{12}}{\partial \beta_k} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial q_{mn}}{\partial \alpha_1} & \frac{\partial q_{mn}}{\partial \beta_1} & \frac{\partial q_{mn}}{\partial \alpha_2} & \frac{\partial q_{mn}}{\partial \beta_2} & \cdots & \frac{\partial q_{mn}}{\partial \alpha_k} & \frac{\partial q_{mn}}{\partial \beta_k} \end{bmatrix} \tag{13}$$

and the derivatives for $\partial q_{11}/\partial \alpha_1$ is readily computed as

$$\frac{\partial q_{11}}{\partial \alpha_1} = \frac{q_{11}(\alpha_1 + \delta, \beta_1, \alpha_2, \cdots, \beta_k) - q_{11}(\alpha_1, \beta_1, \alpha_2, \cdots, \beta_k)}{\delta} (|\delta| \ll 1) \tag{14}$$

Solve the equation

$$\mathbf{S} \cdot \delta\boldsymbol{\theta} = \delta\mathbf{q} \tag{15}$$

The perturbation in the parameter vector is given as

$$\delta\boldsymbol{\theta} = [\mathbf{S}^\mathbf{T}\mathbf{S}]^{-1}\mathbf{S}^\mathbf{T}\delta\mathbf{q} \tag{16}$$

Update the crack parameters

$$\boldsymbol{\theta}_{new} = \boldsymbol{\theta}_{old} + \delta\boldsymbol{\theta} \tag{17}$$

To avoid overshoots in the early stage an underrelaxation is achieved during the foremost iterations ..#.

$$\boldsymbol{\theta}_{new} = \boldsymbol{\theta}_{old} + 0.25\delta\theta \tag{18}$$

Solving Eq. (16) is equivalent to minimizing the function

$$J(\delta\boldsymbol{\theta}) = ||\mathbf{S}\delta\boldsymbol{\theta} - \delta\mathbf{q}||^2 \tag{19}$$

Iterate the procedures (b)-(e) until the following criterion is fulfilled

$$\frac{||\boldsymbol{\theta}_{k+1} - \boldsymbol{\theta}_k||}{||\boldsymbol{\theta}_k||} \le tolerance \tag{20}$$

where k denotes the kth iteration. The tolerance is taken as 1×10^{-9} in this study.

4 Numerical Tests

To validate the efficiency of the method, some numerically simulated damage identification tests are used to evaluate the feasibility of the presented method. A program is written in Matlab to compute the displacement q_{ij} when the beam is excited at frequency f_i just as presented in Eq. (12). The length, height, width, density, Young's modulus and the Poisson ration of a cantilever beam are L=0.8m, h=0.02m, b=0.02m, ρ=7850kg/m3, E=181GPa and v=0.29, respectively.

Fig.2 shows the depths and locations of the double cracks placed on the cantilever beam. Analytically predicted FRFs are used for the numerically simulated cracks estimation tests. The vibration amplitude at $x = 0.24, 0.4, 0.64$m are computed by applying a harmonic point force at$x = 0.8$m. It is worthwhile to note that how to choose the excitation frequency points is very important for the successful damage identification. Lee and Shin (2002b) recommended to choose the frequency points near the resonance peaks in the low-frequency range. Therefore, the excitation frequencies are set to be $f_1 = 80$Hz, $f_2 = 160$Hz, $f_3 = 240$Hz and $f_4 = 320$Hz.

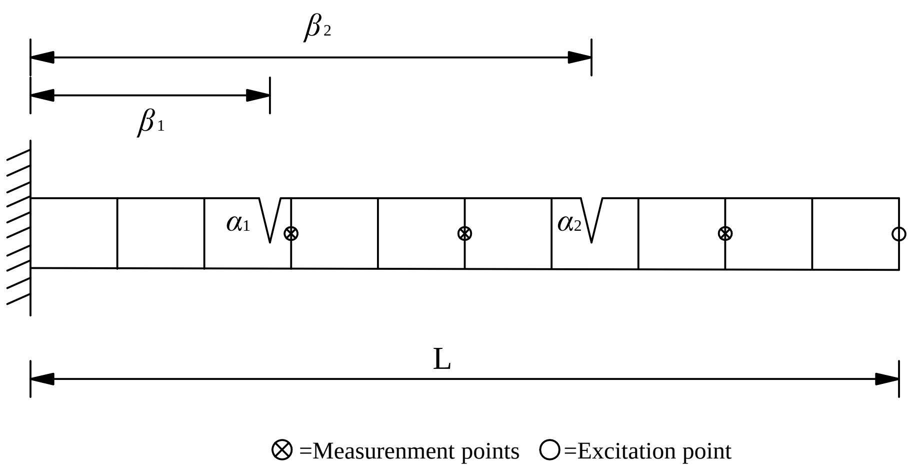

Figure 2: Cantilever beam with double cracks considered for numerically simulated tests

Table 1 presents a typical set of multiple damaged cases and the predicted values. The depth and location of cracks are generated as random numbers. The accuracy of predictions shown in Table 1 indicates that the method can accurately identify the depth and location of multiple cracks.

Table 1: Multiple damage detection cases and results without noise

Case no.	Actual data				Initial guess				Estimated data			
	α_1	α_2	β_1	β_2	α_1	α_2	β_1	β_2	α_1	α_2	β_1	β_2
1	0.18	0.3	0.46	0.62	0.1	0.1	0.4	0.6	0.18	0.3	0.46	0.62
2	0.39	0.41	0.33	0.65	0.1	0.1	0.4	0.6	0.39	0.41	0.33	0.65
3	0.21	0.16	0.31	0.78	0.1	0.1	0.4	0.6	0.21	0.16	0.31	0.78
4	0.21	0.18	0.48	0.6	0.1	0.1	0.4	0.6	0.21	0.18	0.48	0.6
5	0.17	0.35	0.32	0.63	0.1	0.1	0.4	0.6	0.17	0.35	0.32	0.63
6	0.21	0.18	0.21	0.22	0.1	0.1	0.4	0.6	0.21	0.18	0.21	0.22
7	0.22	0.32	0.26	0.70	0.1	0.1	0.4	0.6	0.22	0.32	0.26	0.70
8	0.28	0.30	0.35	0.46	0.1	0.1	0.4	0.6	0.28	0.30	0.35	0.46
9	0.35	0.34	0.27	0.35	0.1	0.1	0.4	0.6	0.35	0.34	0.27	0.35
10	0.20	0.17	0.29	0.45	0.1	0.1	0.4	0.6	0.20	0.17	0.29	0.45
11	0.47	0.23	0.25	0.45	0.1	0.1	0.4	0.6	0.47	0.23	0.25	0.45
12	0.27	0.21	0.31	0.58	0.1	0.1	0.4	0.6	0.27	0.21	0.31	0.58
13	0.22	0.23	0.44	0.51	0.1	0.1	0.4	0.6	0.22	0.23	0.44	0.51
14	0.31	0.20	0.40	0.50	0.1	0.1	0.4	0.6	0.31	0.20	0.40	0.50
15	0.14	0.21	0.37	0.65	0.1	0.1	0.4	0.6	0.14	0.21	0.37	0.65
16	0.26	0.13	0.44	0.53	0.1	0.1	0.4	0.6	0.26	0.13	0.44	0.53
17	0.37	0.36	0.34	0.45	0.1	0.1	0.4	0.6	0.37	0.36	0.34	0.45
18	0.18	0.25	0.47	0.89	0.1	0.1	0.4	0.6	0.18	0.25	0.47	0.89
19	0.45	0.17	0.41	0.62	0.1	0.1	0.4	0.6	0.45	0.17	0.41	0.62
20	0.34	0.20	0.41	0.57	0.1	0.1	0.4	0.6	0.34	0.20	0.41	0.57

To investigate the robustness of the presented algorithm, random experimental noise is simulated in the numerical studies. An $e\%$ random noise is added to the simulated structural FRFs. Structural FRF contaminated with $e\%$ random noise can be described as

$$\bar{\mathbf{q}} = \mathbf{q}\left(1 + \frac{e}{100} \times randn\right) \tag{21}$$

where $\bar{\mathbf{q}}$ and $\mathbf{q}$ are structural FRF with noise and without noise respectively; and *randn* is the random number with a mean equal to zero and a variance equal to one which can be generated by random number generator function in Matlab.

The approach is also applied to estimate the depth and location of double cracks using the contaminated data. The level of noise is controlled and four level of e are set to be 0.1, 0.2, 0.5 and 1, respectively. The parameters α_1, α_2, β_1 and β_2 are 0.2, 0.3, 0.3175 and 0.6813, respectively. The crack parameters are estimated twenty times for each noise level, and the mean values and standard deviations for each noise level are given in Table 2. It shows that the accuracy of damage identification decreases gradually as the level of random noises in FRF increases. Thus, to eliminate the effect of noise and successfully estimate the depth and location of cracks, it is certainly required to acquire accurate experimentally measured FRFs and average the identified results.

Table 2: Mean values and standard deviations of the crack parameters with noise

e	$\bar{\alpha}_1$	σ_{α_1}	$\bar{\alpha}_2$	σ_{α_2}	$\bar{\beta}_1$	σ_{β_1}	$\bar{\beta}_2$	σ_{β_2}
0.1	0.19902	0.00699	0.30048	0.00306	0.31777	0.01364	0.68202	0.00429
0.2	0.21003	0.01625	0.29874	0.00786	0.31216	0.02856	0.68186	0.00963
0.5	0.21915	0.03518	0.29299	0.01474	0.32260	0.02774	0.68314	0.01843
1	0.25090	0.03528	0.27862	0.02272	0.32763	0.03890	0.68602	0.02396

5 Conclusions

A efficient method for estimation both the depth and location of cracks in beams has been presented. The open edge crack is modeled as a rotational spring. A new spectral element modeling of a beam with a transverse open crack has been successfully elaborated. The changes in the frequency response function of the beam due to the crack are used, which are computed using the spectral element method based on Euler-Bernoulli beam theory. The sensitivity-based model updating method has been used for the estimate of the depth and location of cracks. The numerical results indicate the current approach is capable of detecting the size and location of double cracks. Moreover, the cracks can be accurately estimated even if the measurements are polluted with random noise. The accuracy of damage identification decreases gradually as the level of random noises in FRF increases.

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