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Carrying Capacity of Pressure Vessels under Hydrostatic Pressure

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Abstract: To use material effective and keep pressure vessel safety, large deformation analysis for pressure vessel is very important. Until 2007, the elastic-plastic stress analysis method, that is the first time all over the world, is provided in ASME VIII-2 edition 2007 for boiler and pressure vessel standard that Finite Element Method is used with large deformation analysis. But there is no common recognized direct solution for the carrying capacity of pressure vessels yet and this restrict the application of large deformation analysis in pressure vessel design. This paper investigates the carrying capacity of pressure vessels under hydrostatic pressure, based on the elastic-plastic theory. Firstly, to understand the large deformation characteristic of pressure vessel, the expressions of pressure and strain of thin-walled cylindrical and spherical vessels under internal pressure is reviewed. Secondly, to investigate the solution of carrying capacity of pressure vessels, the plastic instability criterion is derived. Further, the method to obtain the carrying capacity of pressure vessels is given for all pressure vessel material and two representative examples for analysis solutions of cylindrical and spherical pressure vessel respectively are given. The proposed research can be used for the elastic-plastic stress analysis method of pressure vessels safely.

Keywords: Pressure vessel, carrying capacity, large deformation, elastic-plastic.

1 Introduction

To avoid accident of pressure vessels, the first pressure vessel standard was established in 1914, it is based on the linear elastic analysis with the small deformation assumption and is usually called Design by Formulae(DBF) or Design-by-Rule(DBR) in the Volume VIII Part 1 of ASME Boiler and Pressure Vessel Code. The Design-by-Analysis(DBA) concept based on stress analysis was first intro-

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duced in 1963 with the publication of the nuclear vessels code and was accepted in 1968 as the Volume ¢ø Part 2 of ASME Boiler and Pressure Vessel Code, but it is still based on the small deformation assumption. The DBF and DBA are broadly used in the world(Ling, 2000). The elastic-plastic stress analysis method as a part of DBA approach, with the large deformation as a precondition, was first introduced in the Volume VIII Part 2 of ASME Boiler and Pressure Vessel Code(2007) that was promulgated in 2007, and it aroused a broad attention in the pressure vessel area. In this method, Finite Element Method is used with considering true stress vs. true strain relationship and large deformation. It is a technical breakthrough as a milestone that considering strain hardening effect and structural deformation. Anyway, large deformation analysis is much more complicated than small deformation analysis, there is not any well-recognized theoretical analysis method available for the carrying capacity of pressure vessels, therefore, only FEM can be used for large deformation analysis, that limits engineering application of large deformation analysis and directly affects its widespread use.

Early on 1964, while summarizing design methods of pressure vessel, Langer(1963) pointed out design methods of pressure vessel can be more rationalized if considering effects of material strain hardening and non-linear structural deformation. In order to reduce safety factor in ASME code to lower cost of pressure vessels, Upitis and MoKhtarian(1998) studied actual safety margin of pressure vessels, and indicated that burst pressure of pressure vessel is related to structural deformation and material strain hardening exponent in 1997. Many researchers focus on study of plastic instability for cylindrical and spherical pressure vessels, it represent the carrying capacity of pressure vessels under hydrostatic pressure. For material with true stress vs. true strain relationship of $\overline{\sigma} = A \cdot \overline{\varepsilon}^{\,n}$, Sachs and Lubahn(1946)¡¢Swift(1952) deduced plastic deformation instability criteria for thin-walled cylindrical and spherical vessels under internal pressure respectively. Cooper(1957) and Hill(1950) deduced plastic deformation instability strain for thin-walled cylindrical and spherical vessels under internal pressure respectively. For material with true stress vs. true strain relationship of $\overline{\sigma} = A \cdot (B + \overline{\varepsilon})^n$, Mellor(1962) deduced plastic deformation instability criteria and plastic instability strain for thin-walled cylindrical and spherical vessels under internal pressure. Other researchers, such as Hillier(1965,1965,1966), Lankford and Saibel(1947), George(1969), Davis(1945), Rawe and Corn(1969) studied instability strain for the similar type of above pressure vessels. In 1958, Svensson(1958) provided plastic instability pressure expressions for cylindrical and spherical pressure vessels for material of $\overline{\sigma} = A \cdot \overline{\varepsilon}^{\,n}$. Recently, Zhu and Leis(2006), Law(2007) worked on similar topics. Truong and Blachut(2009) worked on plastic instability pressure of toroidal shells. However, they are limited with typical material and lack of universality.

In this paper, the expressions of pressure and strain of thin-walled cylindrical and spherical vessels under internal pressure is reviewed firstly. Secondly the plastic instability criterion for thin-walled vessels under internal pressure is analyzed. Then the load carrying capacity of pressure vessels are analyzed. Finally, two conclusions are given.

2 Expressions of equivalent stress and equivalent strain for structures under plain stress proportional loading

Plain stress proportional loading can be expressed as:

$$\sigma_3 = 0, \quad \sigma_2 = x\sigma_1$$

where σ_1, σ_2, and σ_3 are principal stresses, x is stress coefficient of proportion.

From Levy-Mises equation, strain increment tensor are proportional to stress deviator tensor, i.e.

$$\frac{d\varepsilon_1}{s_1} = \frac{d\varepsilon_2}{s_2} = \frac{d\varepsilon_3}{s_3}$$

where, ε_1, ε_2, ε_3 are principal strain, s_1, s_2, s_3 are principal value of stress deviator tensor, and $s_i = \sigma_i - \sigma_m$ $(i = 1,2,3)$,σ_m is mean normal stress.

Simplifying above, then

$$s_1 = \frac{2-x}{3}\sigma_1, \quad s_2 = \frac{2x-1}{3}\sigma_1, \quad s_3 = -\frac{1+x}{3}\sigma_1$$

$$\frac{d\varepsilon_1}{2-x} = \frac{d\varepsilon_2}{2x-1} = -\frac{d\varepsilon_3}{1+x} \tag{1}$$

Equivalent stress is:

$$\begin{aligned}\overline{\sigma} &= \sqrt{\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]\Big/2} \\ &= \sigma_1\left(1-x+x^2\right)^{1/2}\end{aligned} \tag{2}$$

where $\overline{\sigma}$ is Mises equivalent stress.

From equation (2),

$$d\overline{\sigma} = d\sigma_1 \left(1 - x + x^2\right)^{1/2} \tag{3}$$

Equivalent strain increment is:

$$\begin{aligned} d\overline{\varepsilon} &= \sqrt{\tfrac{2}{9}\left[(d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2\right]} \\ &= d\varepsilon_1 \cdot \tfrac{2}{2-x} \cdot \left(1 - x + x^2\right)^{1/2} \end{aligned} \tag{4}$$

where $\overline{\varepsilon}$ is Mises equivalent strain¡£

From equation (1) and (4), then

$$\frac{\overline{\varepsilon}}{2\left(1 - x + x^2\right)^{\frac{1}{2}}} = \frac{\varepsilon_1}{2 - x} = \frac{\varepsilon_2}{2x - 1} = -\frac{\varepsilon_3}{1 + x} \tag{5}$$

3 Expressions of pressure and strain of thin-walled vessels under internal pressure

3.1 Cylindrical vessels

If there is no shape change for pressure vessel, principal stress expressions are always tenable for thin-walled cylindrical and spherical vessels if substituting instant diameter and instant thickness of the pressure vessel into principal stress expressions.

Stresses for thin-walled cylindrical vessels are:

$$\sigma_3 = 0, \sigma_1 = \frac{pr}{t}, \sigma_2 = \frac{pr}{2t}, x = 1/2$$

where p is internal pressure, r,t represent median radii and wall thickness respectively, and they vary with internal pressure changes.

After integral of equation $d\varepsilon_1 = \frac{dr}{r}$ for cylindrical vessels, then

$$r = r_{in} e^{\varepsilon_1}$$

Similarly, from equation $d\varepsilon_3 = \frac{dt}{t}$, we can get

$$t = t_{in} e^{\varepsilon_3}$$

where r_{in} and t_{in} represent original mediate radii and wall thickness, respectively. Then

$$\sigma_1 = \frac{pr}{t} = p \cdot \frac{r_{in}}{t_{in}} \cdot e^{\varepsilon_1 - \varepsilon_3} \tag{6}$$

Substitute $x = 1/2$ into equation (5), then

$$\frac{\overline{\varepsilon}}{\sqrt{3}} = \frac{\varepsilon_1}{3/2} = \frac{\varepsilon_2}{0} = -\frac{\varepsilon_3}{3/2}$$

Thus

$$\varepsilon_1 - \varepsilon_3 = \sqrt{3}\overline{\varepsilon} \tag{7}$$

Substitute $x = 1/2$ into equation (2), then

$$\overline{\sigma} = \frac{\sqrt{3}}{2}\sigma_1 \tag{8}$$

Substitute equation (7) and equation (8) into equation (6), then

$$\frac{2}{\sqrt{3}}\overline{\sigma} = p \cdot \frac{r_{in}}{t_{in}} \cdot e^{\sqrt{3}\overline{\varepsilon}}$$

i.e.

$$p = \frac{2}{\sqrt{3}} \cdot \frac{t_{in}}{r_{in}} \cdot \overline{\sigma} \cdot e^{-\sqrt{3}\overline{\varepsilon}} \tag{9}$$

3.2 Spherical vessels

Stress expressions for thin-walled spherical vessels under internal pressure are:

$$\sigma_3 = 0, \quad \sigma_1 = \frac{pr}{2t}, \quad \sigma_2 = \frac{pr}{2t}, \quad x = 1$$

From $r = r_{in}e^{\varepsilon_1}$,$t = t_{in}e^{\varepsilon_3}$, we can obtain

$$\sigma_1 = \frac{pr}{2t} = p \cdot \frac{r_{in}}{2t_{in}} \cdot e^{\varepsilon_1 - \varepsilon_3} \tag{10}$$

Substitute $x = 1$ into equation (2), then

$$\overline{\sigma} = \sigma_1 \tag{11}$$

Substitute $x = 1$ into equation(5), then

$$\frac{\overline{\varepsilon}}{2} = \frac{\varepsilon_1}{1} = \frac{\varepsilon_2}{1} = -\frac{\varepsilon_3}{2}$$

Thus

$$\varepsilon_1 - \varepsilon_3 = \frac{3}{2}\overline{\varepsilon} \tag{12}$$

Substitute equation (11) and (12) into equation (10), then

$$\overline{\sigma} = p \cdot \frac{r_{in}}{2t_{in}} \cdot e^{3/2\overline{\varepsilon}}$$

i.e.,

$$p = 2 \cdot \frac{t_{in}}{r_{in}} \cdot \overline{\sigma} \cdot e^{-3/2\overline{\varepsilon}} \tag{13}$$

Equation(9) and Equation (13) represent the relationship between internal pressure p with Mises equivalent stress$\overline{\sigma}$ and equivalent strain $\overline{\varepsilon}$ for thin-walled cylinder and

spherical vessels respectively under internal pressure. Since $\overline{\sigma}$ and $\overline{\varepsilon}$ are satisfied true stress-true strain relationship of material under monotone loading, they are not independent variables. In other words, $\overline{\sigma}$ can be obtained by function $\overline{\sigma}(\overline{\varepsilon})$ if $\overline{\varepsilon}$ was defined. r_{in} and t_{in} are constants that represents original mediate radii and thickness respectively. Thus equation (9) and (13) contain 2 variables, p and $\overline{\varepsilon}$, i.e. pressure and strain, they are expressions of pressure-strain relationship for thin-walled cylinder and spherical vessels respectively under internal pressure and are firstly deduced by Deng and Chen(2010).

When the deformation of pressure vessel is defined, the corresponding pressure can be obtained if associated with material true stress-true strain curve. Thus, the practical value of Equation(9) and Equation (13) is equivalent to principal stress equations of thin-walled cylindrical and spherical vessels with considering non-linear structural deformation effect.

4 Plastic instability criterion for thin-walled vessels under internal pressure

4.1 Cylindrical vessel

Differentiate equation (9), then

$$\frac{dp}{d\overline{\varepsilon}} = \frac{2}{\sqrt{3}} \cdot \frac{t_{in}}{r_{in}} \cdot \left(\frac{d\overline{\sigma}}{d\overline{\varepsilon}} \cdot e^{-\sqrt{3}\overline{\varepsilon}} + \overline{\sigma} \cdot e^{-\sqrt{3}\overline{\varepsilon}} \left(-\sqrt{3} \right) \right)$$

Plastic instability criterion for thin-walled cylindrical vessels under internal pressure is

$$\frac{dp}{d\overline{\varepsilon}} = 0$$

Thus

$$\frac{d\overline{\sigma}}{d\overline{\varepsilon}} = \frac{1}{1/\sqrt{3}} \overline{\sigma} \tag{14}$$

4.2 Spherical vessel

Differentiate equation (13), then

$$\frac{dp}{d\overline{\varepsilon}} = 2 \cdot \frac{t_{in}}{r_{in}} \cdot \left(\frac{d\overline{\sigma}}{d\overline{\varepsilon}} \cdot e^{-3/2\overline{\varepsilon}} + \overline{\sigma} \cdot e^{-3/2\overline{\varepsilon}} (-3/2) \right)$$

Plastic instability criterion for thin-walled spherical vessels under internal pressure is

$$\frac{dp}{d\overline{\varepsilon}} = 0$$

Thus

$$\frac{d\overline{\sigma}}{d\overline{\varepsilon}} = \frac{1}{2/3}\overline{\sigma} \tag{15}$$

Cooper(1957) and Hill(1950) and other researchers had derived plastic instability criterion equation (14) and (15) for thin-walled cylindrical and spherical vessels under internal pressure before. But the deriving process in this paper is the simplest one, and it is based on equation (9) and (13), it is independent of material type.

5 Plastic instability pressures for thin-walled pressure vessels under internal pressure

The pressure-strain curve can be obtained by calculations of equation (9) and (13). The maximum pressure value from the curve is plastic instability pressure for thin-walled pressure vessels under internal pressure. This is identical to the result that is gotten from Finite Element analysis results. The method to calculate the pressure-strain curve of pressure vessels in this paper is more efficient and clear than FE analysis, and easy to be used in engineering applications.

By now, for all sorts of pressure vessel materials, plastic instability pressures of thin-walled cylindrical and spherical vessel under internal pressure can be completely calculated individually. While true stress vs. true strain function of material is known, equivalent strain and equivalent stress for thin-walled cylindrical and spherical vessels under internal pressure in plastic instability can be calculated by simultaneous equation (14) and (15). Substituting equation (9) and (13), the plastic instability pressures for thin-walled pressure vessels under internal pressure can be obtained directly.

For example, material which true stress vs. true strain relationship is $\overline{\sigma} = A \cdot \overline{\varepsilon}^{\,n}$, the ultimate tensile strength of the material can be expressed as

$$\sigma_b = An^n e^{-n}$$

where σ_b is the ultimate tensile strength of material.

When the thin-walled cylindrical vessels under internal pressure at plastic instability, substituting $\overline{\sigma} = A \cdot \overline{\varepsilon}^{\,n}$ to equation (14) for thin-walled cylindrical, the equivalent strain and equivalent stress are:

$$\overline{\sigma} = A \cdot \left(\frac{n}{\sqrt{3}} \right)^{n}$$

$$\overline{\varepsilon} = \frac{n}{\sqrt{3}}$$

Substituting above three equations to equation (9), the plastic instability pressure function for thin-walled cylindrical vessels under internal pressure can be expressed as

$$p_{plin} = \frac{2}{\left(\sqrt{3}\right)^{n+1}} \cdot \sigma_b \cdot \frac{t_{in}}{r_{in}} \tag{16}$$

Similarly, combining with equation (15) and equation (10), the plastic instability pressure function for thin-walled spherical vessels under internal pressure can be expressed as

$$p_{plin} = \left(\frac{2}{3} \right)^{n} \cdot 2 \cdot \sigma_b \cdot \frac{t_{in}}{r_{in}} \tag{17}$$

Svensson(1958) had derived equation (16) and equation (17) too, the expressions of plastic instability pressure for thin-walled cylindrical and spherical vessels respectively under internal pressure. In this paper, all the derivations for plastic instability criterions and instability pressures are based on equation (9) and (13), and the process is simple and clear. It would be quite easy to derive theoretical expressions of plastic instability pressure for thin-walled cylindrical and spherical vessels under internal pressure with any other type pressure vessel materials by this paper analysis method.

6 Conclusion

(1) The expressions of pressure and strain of thin-walled cylindrical and spherical vessels under internal pressure are $p = \frac{2}{\sqrt{3}} \cdot \frac{t_{in}}{r_{in}} \cdot \overline{\sigma} \cdot e^{-\sqrt{3}\overline{\varepsilon}}$ and $p = 2 \cdot \frac{t_{in}}{r_{in}} \cdot \overline{\sigma} \cdot e^{-3/2\overline{\varepsilon}}$

respectively. And their practical value is equivalent to principal stress equations of thin-walled cylindrical and spherical vessels with considering non-linear structural deformation effect.

(2) For any sort of pressure vessel material, the instability pressures of thin-walled cylindrical and spherical vessels under internal pressure can be obtained from the pressure-strain curves according to the expressions of pressure and strain of thin-walled cylindrical and spherical vessels, that is quite similar to Finite Element analysis method, but it is more efficient and easy to be used in engineering applications.

(3) The plastic instability criterions and instability pressures of thin-walled cylindrical and spherical vessels are derived, and they are consistent with previous studies of other researchers, but the deriving process is the simplest one, and the analysis method can be easily applied to all type pressure vessel materials when the true stress vs. true strain relationship of pressure vessel material is known.

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References

ASME Boiler and Pressure vessel Committee (2007): 2007 ASME Boiler & pressure vessel code,?-Division 2,alternative rules, rules for construction of pressure vessels.

Cooper, W. E. (1957): The significance of the tensile test to pressure vessel design. *Welding Research Council, WRC Supplement, Pressure Vessels.*

Davis, E. A. (1945): Yielding and fracture of medium-carbon steel under combined stress. *Journal of Applied Mechanics.* Vol.12, no1, pp.A13-A24.

Deng, Y. C.; Chen, G. (2010): Expressions of load and structural deformation relationship for cylindrical and spherical vessels under internal pressure. *Proceedings of the ASME 2010 Pressure Vessels and Piping Division/K-PVP Conference. July 18-22,2010,Bellevue,Washington, USA.* pp. PVP2010-25648:1-7.

George, S. (1969): Yielding and plastic instability under biaxial stress in design of metal pressure vessels. *Journal of Materials.* Vol.4, no.2, pp.377-392.

Hill, R. A. (1950): Theory of plastic bulging of a metal diaphragm by lateral pressure. *Philos. Mag.,* Vol.41, no.7, pp.1133-1142.

Hillier, M. J. (1965): Tensile plastic instability of thin tubes–I. *International Journal of Mechanical Sciences.* Vol.7, no.8, pp.531-538.

Hillier, M. J. (1965): Tensile plastic instability of thin tubes–II. *International Jour-*

nal of Mechanical Sciences. Vol.7, no.8, pp. 539-549.

Hillier, M. J. (1966): The inertia effect in the tensile plastic instability of a thin spherical shell. *International Journal of Mechanical Sciences.* Vol.8, pp.61-62.

Langer, B. F. (1964): PVRC interpretive report of pressure vessel research, section 1-design considerations. *Welding Research Council Bulletin.* Vol. 95, pp.1-53.

Lankford, W. T.; Saibel E. (1947): Some problems in unstable plastic flow under biaxial tension. Metals Technology, METY-A, *American Institute of Mining and Metallurgical Engineers.* No.2238, pp.1-12.

Law, M. (2007): Prediction of failure strain and burst pressure in high yield to tensile strength ratio linepipe. *International Journal of Pressure Vessels and Piping.* Vol.84, no.8, pp.487-492.

Ling, J. (2000):The evolution of the ASME boiler and pressure vessel code. *Journal of Pressure Vessel Technology.* Vol. 122, pp. 242-246.

Mellor, P. B. (1962): Tensile instability in thin-walled tubes. *The Journal of Mechanical Engineering Science.* Vol. 4, no.3, pp. 251-256.

Rawe, R. A.; Corn, D. L. (1969): A comparison of the experimental biaxial strength of structural alloys with theoretical predications. *Journal of Materials.* Vol.4, no1, pp.3-18.

Sachs, G; Lubahn, J. D. (1946): Failure of ductile metals in tension. *Trans. ASME.* Vol.68,pp. 277-279.

Svensson, N. L. (1958): The bursting pressure of cylindrical and spherical vessels. *Journal of Applied Mechanics.* Vol.80, pp.89-96.

Swift, H. W. (1952): Plastic instability under plane stress. *Journal of the Mechanics and Physics of Solids.* Vol. 11, no.1, pp.1-18.

Truong, V. V.; Blachut, J. (2009): Plastic instability pressure of Toroidal shells. *Journal of Pressure Vessel Technology.* Vol.131, no.5, pp.051203-1-10.

Upitis, E.; MoKhtarian K. (1998): Evaluation of design margins for section VIII, division 1 and division 2 of the ASME boiler and pressure vessel code. *Welding Research Council Bulletin.* Vol. 435, pp.1-85.

Zhu, X. K.; Leis, B. N. (2006): Theoretical and numerical predictions of burst pressure of pipeline. *Proceedings of PVP2006-ICPVT-11 ,2006 ASME Pressure Vessels and Piping Division Conference, July 23-27,2006,Vancoucer,BC,Canada.* pp.1-10.