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A Note on Two Different Definitions of Reference Surface of Deformed Rubber-Like Shells

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Abstract: The mid surface of undeformed rubber-like shells undergoing finite rotations and finite strains has been considered to be the reference surface of the undeformed configuration. There have been two different definitions for the reference surface of deformed rubber-like shells, undergoing finite rotations and finite strains, used by several researchers. In this study, some comments on the stress-resultants defined relative to the mentioned two reference surfaces of deformed incompressible rubber-like shells under axisymmetrical effects, considering transverse normal and transverse shear deformations, are presented.

Keywords: rubber-like shell, finite strain, finite rotation, finite deformation, reference surface.

1 Introduction

Considering the mid surface of undeformed rubber-like shells undergoing finite rotations and finite strains as the reference surface of the undeformed configuration, there have been two alternative definitions for the reference surface of deformed rubber-like shells, undergoing finite rotations and finite strains [Taber (1987); Simmonds (1986); Libai and Simmonds (1998); Yükseler (1996a); Yükseler (1996b)]:

(i) Theory II : The reference surface of a deformed shell is composed of the particles which have been on the reference surface of the undeformed shell, and therefore

$$\xi_{\mathrm{II}}|_{\xi_0=0} = 0, \tag{1}$$

where ξ_0 denotes the transverse coordinate of a point of the undeformed shell measured from the reference surface S_0 of the undeformed shell space (the perpendicular distance from the reference surface S_0 of the undeformed shell) and ξ_{II} denotes the transverse coordinate of a point of the deformed shell measured from the reference surface S_{II} of the deformed shell space (See Fig. 1 of Yükseler (2003).). The

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subscript 0 is used to denote that the related parameter belongs to the undeformed configuration, and the subscript II is used to denote that the concerning parameter belongs to Theory II. In most of the relevant studies, Theory II has been used, e.g. [Makowski and Stumpf (1989); Taber (1985); Stumpf and Makowski (1986); Brodland and Cohen (1987); BaŞar and Kratzig (1989); Yükseler (2008)].

(ii) Theory I : The material composition of the reference surface S_{I} of the deformed shell is not necessarily same as that of the reference surface S_0 of the undeformed shell and

$$\int_{-t/2}^{t/2} \xi_I d\xi_0 = 0 \tag{2}$$

should be satisfied [Taber (1987); Simmonds (1986); Libai and Simmonds (1998)]. t is the thickness of the undeformed shell. Here and henceforth, the subscript I is used to denote that the concerning parameter belongs to Theory I. If neither the subscript I nor the subscript II is used (except the parameters having the subscript 0), the concerning parameter is considered to depend on the choice of the theory. In some of the relevant references, Theory I has been used, e.g. [Yükseler (1996a); Yükseler (1996b); Taber (1988); Taber (1989); Koçak and Yükseler (1999)].

The numerical results corresponding to these two theories have been compared in various studies [Taber (1987); Yükseler (1996a); Yükseler (1996b); Koçak and Yükseler (1999); Koçak and Yükseler (2000); Koçak and Yükseler (2002)]. The parameters affecting the differences between the solutions corresponding to the two different definitions of the reference surface of deformed rubber-like shells of revolution, undergoing axisymmetric finite strains and finite rotations, have been presented in Yükseler (2003), in which an asymptotic analysis has been used. Accordingly, the differences between the solutions to problems pertaining to incompressible rubber-like shells of revolution, undergoing axisymmetric finite rotations and finite strains including transverse shear and transverse normal strains, corresponding to the two definitions of the reference surface of deformed rubber-like shells, namely Theory I and Theory II, are increased if (i) the thickness of shell is increased, (ii) the bending strains are increased, and (iii) the extensional strains are decreased. An important emphasis has been noted in Yükseler (2003) that there has been no claim that one of the two approaches is better than the other, the corresponding solutions are being relative to the chosen reference surface. The present study can be considered to be a modest extension of the study presented in Yükseler (2003).

2 Preliminary analysis

Assuming that the material fibers normal to the reference surface of the undeformed shell are assumed to remain straight but not necessarily normal to the reference surface of the deformed shell [Taber (1987); Yükseler (1996a); Yükseler (1996b); Makowski and Stumpf (1989)], the natural base vectors of the undeformed and deformed shell spaces and, then, the metric tensors in the undeformed and deformed configurations can be obtained [Yükseler (2003)]. Through an asymptotic analysis, the incompressibility condition [Green and Zerna (1975)] has yielded the difference between the transverse coordinates corresponding to Theory I and Theory II, as a measure of the difference between the locations of the mentioned two reference surfaces of the deformed shell [Yükseler (2003)]:

$$\xi_{\mathrm{I}}^{*} - \xi_{\mathrm{II}}^{*} = \varepsilon[k_{\phi\mathrm{II}}^{*} / \{\lambda_{\theta\mathrm{II}}^{2}(\overline{\lambda}_{\phi\mathrm{II}})^{3}\} + k_{\theta\mathrm{II}}^{*} / \{\lambda_{\theta\mathrm{II}}^{3}(\overline{\lambda}_{\phi\mathrm{II}})^{2}\}]/6 + O(\varepsilon^{3}) \tag{3}$$

where

$$\begin{aligned} k_{\theta}^{*} &= \overline{k}_{\theta}^{*} - \overline{\lambda}_{\phi}\lambda_{\theta}^{2}k_{\theta 0}^{*}, \\ k_{\phi}^{*} &= \overline{k}_{\phi}^{*} - \overline{\lambda}_{\phi}^{2}\lambda_{\theta}k_{\phi 0}^{*} \end{aligned} \tag{4}$$

are nondimensional curvature change measures [Yükseler (1996b); Yükseler (2003)]; $\overline{\lambda}_{\phi}$ and λ_{θ} are the extensional strain measures; $k_{\phi 0}^{*}$ and $k_{\theta 0}^{*}$ are the nondimensional curvature measures of the reference surface of the undeformed shell; $\overline{k}_{\phi}^{*}$, $\overline{k}_{\theta}^{*}$ are the nondimensional curvature measures of the reference surface of the deformed shell considering the transverse shear strains, respectively. ε is a thickness parameter,

$$\varepsilon = t/(2L) \tag{5}$$

where L is deformation wavelength. ξ^{*} is the nondimensional form of ξ,

$$\xi^{*} = (2\,/\,\mathrm{t})\xi. \tag{6}$$

3 Stress resultants

The nondimensional pseudo-stress resultants, the pseudo-shear stresses and the pseudo-moment resultants per unit length of the reference surface of the undeformed shell space have been defined as[1]

$$\mathrm{N}^*_\alpha = \int_{-1}^{1} \mu \tau^*_{\underline{\alpha\alpha}} \mathrm{d}\xi^*_0,$$
$$Q^*_\alpha = \int_{-1}^{1} \mu \tau^*_{\underline{\alpha}3} \mathrm{d}\xi^*_0, \tag{7}$$
$$\mathrm{M}^*_\alpha = \int_{-1}^{1} \mu \tau^*_{\underline{\alpha\alpha}} \xi^* \mathrm{d}\xi^*_0$$

where $\tau_{\underline{\alpha\alpha}}$ and $\tau_{\underline{\alpha}3}$ are the components of the Piola- Kichhoff stress tensor of the second kind [BaŞar (1987)] and

$$\mu = 1 - 2\varepsilon \mathrm{H}^* \xi^*_0 + \varepsilon^2 \mathrm{K}^* \xi^{*2}_0. \tag{8}$$

H^* and K^* are the nondimensional mean curvature and Gaussian curvature, respectively [BaŞar and Kratzig (1985); Chroscielewski, Pietraszkiewicz and Witkowski (2010)]. The dashes at the bottom of the indices denote that the summation convention is not applied to these indices.

It is clear from the first and second equations of Eqs. (7) that the nondimensional pseudo-stress resultants N^*_α and the pseudo-shear stress Q^* can easily be noted to be invariant of the choice of the mentioned reference surface of the deformed shell. But; as it can be seen from the last of Eqs. (7), the pseudo-moment resultants M^*_α do depend on the choice of the mentioned reference surface (since ξ^* may have the index I or II). Via the last of Eqs. (7), the difference between the pseudo-moment resultants M^*_α relative to Theory I and Theory II can be expressed as

$$\mathrm{M}^*_{\alpha_\mathrm{I}} - \mathrm{M}^*_{\alpha_\mathrm{II}} = \int_{-1}^{1} \mu \tau^*_{\underline{\alpha\alpha}} (\xi^*_\mathrm{I} - \xi^*_\mathrm{II}) d\xi^*_0 \quad , \tag{9}$$

or, using Eq. (3) and the first of Eqs. (7)

$$\mathrm{M}^*_{\alpha_\mathrm{I}} - \mathrm{M}^*_{\alpha_\mathrm{II}} = e N^*_\alpha \tag{10}$$

where

$$e = \xi^*_\mathrm{I} - \xi^*_\mathrm{II}, \tag{11}$$

[1] Considering only the axisymmetrical external effects, one of the two pseudo-shear stresses is equal to zero.

or

$$e = \varepsilon[k^*_{\phi\mathrm{II}} \,/\, \{\lambda^2_{\theta\mathrm{II}}(\overline{\lambda}_{\phi\mathrm{II}})^3\} + k^*_{\theta\mathrm{II}} \,/\, \{\lambda^3_{\theta\mathrm{II}}(\overline{\lambda}_{\phi\mathrm{II}})^2\}]/6 \tag{12}$$

which does not depend on ξ^*_0 .

Equation (10) is nothing but an implication of a very well known expression from the engineering mechanics; resolution of a given force into a force acting at a given point and a couple, the moment of the couple being equal to the moment of the concerning force [Beer and Johnston (1976)]. Therefore; when the pseudo-moment resultants relative to one of the two reference surfaces of the deformed shell are known, it is a very simple algebra to obtain the pseudo-moment resultants relative to the other reference surface of the deformed shell.

4 Discussion

It is shown that

the pseudo-stress resultants and pseudo-shear stresses do no depend on the choice of the reference surface of the deformed rubber-like shell,

the difference between the pseudo-moment resultants defined relative to the mentioned two reference surfaces of the deformed shell is governed by Eqs. (10, 12) and a function of the curvature change measures, strain measures, thickness parameter, and the pseudo-stress resultants.

The arguments made in this study are restricted with the definitions of the stress-resultants defined in Eqs. (7) and similar ones (e.g. those defined in [Taber (1985); Taber (1988); Biricikoglu and Kalnins (1971); Makowski and Stumpf (1986)]). Different definitions of the stress-resultants may not yield the same remarks presented in this study. For example; in case of using ξ^*_0 instead of ξ^* in the last of the Eqs. (7) [Chroscielewski, Pietraszkiewicz and Witkowski (2010); BaŞar and Ding (1990); Schieck, Smolenski and Stumpf (1999)], there would not be any difference between the stress resultants corresponding to the mentioned two theories. Generalizing the last statement; if the transverse coordinate ξ^*, belonging to the deformed configuration, is not used in the expression of a stress resultant, including the higher order stress-resultants corresponding to a higher order shell theory than the first order theory [Reddy and Arciniega (2004)], there would not be any difference between the stress resultants corresponding to the mentioned two theories.

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