

A Fast Product of Conditional Reduction Method for System Failure Probability Sensitivity Evaluation

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> Abstract: System reliability sensitivity analysis becomes difficult due to involving the issues of the correlation between failure modes whether using analytic method or numerical simulation methods. A fast conditional reduction method based on conditional probability theory is proposed to solve the sensitivity analysis based on the approximate analytic method. The relevant concepts are introduced to characterize the correlation between failure modes by the reliability index and correlation coefficient, and conditional normal fractile the for the multi-dimensional conditional failure analysis is proposed based on the two-dimensional normal distribution function. Thus the calculation of system failure probability can be represented as a summation of conditional probability terms, which is convenient to be computed by iterative solving sequentially. Further the system sensitivity solution is transformed into the derivation process of the failure probability correlation coefficient of each failure mode. Numerical examples results show that it is feasible to apply the idea of failure mode relevancy to failure probability sensitivity analysis, and it can avoid multi-dimension integral calculation and reduce complexity and difficulty. Compared with the product of conditional marginal method, a wider value range of correlation coefficient for reliability analysis is confirmed and an acceptable accuracy can be obtained with less computational cost.

> Keywords: Probability of failure; sensitivity; approximate analytical method; correlation coefficient; conditional marginal method

1 Introduction

Reliability sensitivity analysis plays an important role in study the influence level of parameter distribution variability of random variables to system reliability. Sensitivity analysis can provide how uncertainty in the output of a model can be apportioned to different source of uncertainty in the model input factor [1,2]. It is necessary to obtain the information about the importance ranking of each random variable since useful guidance can be obtained for structural reliability design and maintenance to various engineering applications, such as ocean and civil engineering structures [3–6], mechanical systems [7–9], software systems [10], etc.



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To measure the sensitivity of system failure probability, we can compute the partial derivative of the failure probability with respect to the distribution parameter such as mean and standard deviation of random variable. Currently, numerical simulation methods based on Monte Carlo simulation (MCS) and approximate analytical methods are two major methods of reliability sensitivity analysis. Compared to approximate analytical methods, MCS can obtain more accurate results, but one main disadvantage is that the amount of calculation is too large when involved in structural analysis. Alternative approaches have been developed in which surrogate models as an alternative to approximate the limit state functions with explicit expressions to reduce computational effort. A hybrid method based on surrogate model and simulation sampling methods [11,12] proposed to deal with the "curse of computational cost" for the problem with small failure probability for reliability sensitivity analysis. Papaioannou et al. [13] and Pradlwarter et al. [14] researched the sequential importance sampling method and line sampling simulation method for reliability sensitivity analysis. Lacaze et al. [15] presented an approach based on the approximation of the Dirac's Delta to estimate the gradient of the probability of failure with respect to the design variables using Crude Monte Carlo and Subset Simulation. The main drawback of the proposed approach is that the computational effort required grows very fast for increasing number of random variables.

Approximate analytical methods are widely employed [16] due to an acceptable accuracy with less computational cost. Sues et al. [17] presented a hybrid approach to quantify the contribution of each random variable to the system failure probability by using FORM to find the most probable failure point and performing overclosed form approximations for the limit state functions at the most probable failure point. To improve the calculation accuracy of the reliability for a nonlinear limit state function, Dong et al. [18] built a number of hyperplanes near the design point by first-order Tayler series expansion, thus the reliability sensitivities can be estimated more accurately by the derived equations based on the equivalent hyperplane. Zhu et al. [19] proposed a probability model d to evaluate the MPP using cumulative distribution function for improve accuracy and stable results for complex problems. Kang et al. [20] develops a method for evaluating multivariate normal integrals in which the method compounds two components coupled by union or intersection sequentially until the system becomes a single compound event.

System reliability analysis based on the first order approximate analytical methods can be reduced to the solution of multidimensional normal integral. First order multinormal method proposed by Tang et al. [21] is to convert the joint failure probability problem of multiple limit states into the product of the failure probability of multiple single limit states through the conditional probability method. To avoid the problems of linearization and optimization of performance function by using tedious iterative calculation in First order multinormal method, product of Conditional Marginal and improved PCM [22,23] with higher calculation efficiency are proposed to impove the computational accuracy and efficiency of system reliability. In addition, the moment methods [24–26] are presented for reliability sensitivity to avoid solving the derivative function and design point of performance function. However, for highly nonlinear problems, the efficiency of such methods still needs to be improved. Global sensitivity analysis [27–29] is proposed in engineering design and reliability evaluation to reflect the influence of input variable uncertainty on output variable uncertainty from a global perspective, and an overview of available methods by structuring them into local and global methods is presented in Borgonovo et al. [30].

Noted that most of the above methods based on the fact that the random variable is an independent standard normal variable. For correlated non-normal variables, some representative transformation methods, such as Orthogonal transformation, Rosenblatt transformation, and Nataf transformation are often used to convert them into independent standard normal variables. The Nataf transformation is, in fact, a form of copula, a wider framework for expressing the stochastic dependence of random variables [31], and the proposed sensitivities have the potential to be generalized to a wider class of copulas by future research. Nataf transformation is recommended by Wu et al. [32] to be used for reliability analysis involving correlated input variables due to its accuracy and applicability, and the analytical derivatives of the Nataf transformation with respect to distribution parameters are presented by Hesam et al. [33].

In this paper, a fast product of conditional reduction method for reliability sensitivity analysis is presented based on the principle of conditional marginal. In order to overcome the computational complexity and deficiency of PCM caused by repeated iterations of each conditional fractile derivation and correlation coefficient, an improved correlation coefficient based on conditional marginal probability is introduced. As a result the burden solving process of system sensitivity can be carried out by the coefficient derivation of failure probability corresponding to all failure modes. And the system sensitivity analysis is convenient to be realized without involving second-order or higher-order joint probability calculation. The correlation coefficient formula is suggested according to the approximate two-dimensional conditional marginal probability formula and detailed expressions of the sensitivities with respect to the random variables are presented. The accuracy of the method is investigated and some conclusions are drawn.

2 The System Failure Probability Analysis

Series system and parallel system are the two most basic forms modeled in the structural system. Let $M_j = g_j(x_1, ..., x_n)$, j = 1, 2, ..., m, be a set of *m* given failure modes, the expressions of reliability thus the system failure probability for series system and parallel system are as follows:

$$P_{fs} = P\left(\bigcup_{j=1}^{m} \left\{ M_j\left(x\right) \le 0 \right\} \right), \quad P_{fp} = P\left(\bigcap_{j=1}^{m} \left\{ M_j\left(\mathbf{x}\right) \le 0 \right\} \right)$$
(1)

In the standard normal space U, the failure probability P_{fs} can be written as Eq. (2) by using a m-dimensional standard multinormal integral in which the limit state tangent plane of $M_j(\mathbf{u}) \approx -\boldsymbol{\alpha}_j^T \mathbf{u} + \beta_j$ is used to replace the limit state surface of each failure mode at the maximum possible failure point according to FORM.

$$P_{fs} = P\left(\bigcup_{j=1}^{m} \left\{-\boldsymbol{\alpha}_{j}^{\mathrm{T}}\boldsymbol{u} \leq -\beta_{j}\right\}\right) = 1 - \Phi_{m}\left(\boldsymbol{\beta}, \boldsymbol{\rho}\right), \quad P_{fp} = P\left(\bigcap_{j=1}^{m} \left\{-\boldsymbol{\alpha}_{j}^{\mathrm{T}}\boldsymbol{u} \leq -\beta_{j}\right\}\right) = \Phi_{m}\left(-\boldsymbol{\beta}, \boldsymbol{\rho}\right)$$
(2)

In which $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)^T$ is a vector of reliability index, and $\boldsymbol{\alpha}_j$ is a vector of unit direction cosine of the linearization hyperplane. $\boldsymbol{\rho}$ is correlation coefficient matrix and the correlation coefficient between any two failure modes M_i and M_j is given as $\rho_{ij} = \boldsymbol{\alpha}_i^T \cdot \boldsymbol{\alpha}_j$.

It can be seen that the core of system reliability calculation lies in the solution of multidimensional normal integral. Based on the conditional probability theory, Pendey suggested a product of conditional marginal (PCM) and its improved method (I-PCM) to evaluate $\Phi_m(-\beta,\rho)$, in which the expression of $\Phi_m(-\beta,\rho)$ is $\Phi_m(\beta,\rho) = P\left[(X_m \le \beta_m) \mid \bigcap_{k=1}^{m-1} (X_k \le \beta_k)\right] \times$

 $P\left[(X_{m-1} \leq \beta_{m-1}) \mid \bigcap_{k=1}^{m-2} (X_k \leq \beta_k)\right] \times \cdots \times \Phi(\beta_1) = \Phi\left(\beta_{m|(m-1)}\right) \times \Phi\left(\beta_{(m-1)|(m-2)}\right) \cdots \times \Phi(\beta_1) \approx \prod_{i=1}^{m} \Phi\left(\beta_{(m+1-i)|(m-i)}\right), \text{ in which } \Phi \text{ denotes the standard Gaussian cumulative probabilistic distribution function, and } \beta_{(m+1-i)|(m-i)} \text{ represents a conditional normal fractile.}$

The method represents the multi-normal integral as a product of conditional probability terms. According to the principle of PCM, an iteration procedure is available as the way of Tab. 1. The detailed calculation formulas of PCM can refer to the work by Pandey et al. [22,23].

	The 1th iteration	The 2th iteration		The $(m-1)$ th iteration
β_1				
β_2	$\beta_{2 1}$			
β_3	$\beta_{3 1}$	$\beta_{3 2}$		
•••	•••	•••	•••	
β_m	$\beta_{m 1}$	$\beta_{m 2}$		$\beta_{m (m-1)}$
$[\rho_{ij}]_{m \times m}$	$[\rho_{ij}]_{(m-1)\times(m-1)}$	$[\rho_{ij}]_{(m-2)\times(m-2)}$		$[\rho_{ij}]_{1 \times 1}$

Table 1: Calculation procedure of PCM

It can be drawn from iteration procedure that all elements in lower triangle matrix are need to participation in calculation of the new conditional fractile and correlation coefficient. However it is time-consuming and complex task for further sensitivity analysis due to repeated iterations of the derivation of each conditional fractile and correlation coefficient required, especially when the number of failure mode m is large.

3 The Proposed Method

3.1 The Basic Principle

First, all failure modes in Eq. (1) are rearranged in a descending order according to the failure probability, assuming that

$$P_{f1} \ge P_{f2} \ge, \dots, \ge P_{fm} \tag{3}$$

where $P_{fj} = P(M_j)$ is the failure probability of the *jth* failure mode, and $M_j = g_j(x_1, ..., x_n)$, j = 1, 2, ..., m, be a set of *m* given failure modes.

For simplicity, the method starts with subsystem composed of two failure modes. According to addition rule of probability, the series system failure probability can be represented as flowing form

$$P_{f12} = P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2) = P_{f_1} + P_{f_2} - P_{12}$$
(4)

According to addition rule of probability, the failure probability of subsystem containing three failure modes will be represented as the sum of seven items as $P_{f123} = P_{f_1} + P_{f_2} + P_{f_3} - P_{12} - P_{13} - P_{23} + P_{123}$. It is obviously that the reliability calculation and its sensitivity analysis based on this formula become cumbersome and complicated when the number of failure modes increase due to the correlation between them. The same problem exists with PCM method by analysis of the iterative process shown in Tab. 1. As stated above, the main difficulty associated with Eq. (1) is the correlation treatment between different failure modes. To avoid multidimensional CMES, 2020, vol.125, no.3

integrals involved in expressions, a key efficiency consideration is to develop the method such as reduce dimension.

In this paper, a fast conditional marginal method is introduced by definition of relevance coefficient. In Eq. (4), P_{12} is the joint failure probability that two events occur simultaneously. According to conditional probability formula, P_{12} can be computed [30,34] by $P_{12} = P(E_1 | E_2) P(E_2)$, given in which $P(E_1 | E_2)$ is denoted by correlation coefficient r_{12} , thus Eq. (4) can be written as

$$P_{f12} = P_{f1} + P_{f2} - r_{12}P_{f2} = P_{f1} + (1 - r_{12})P_{f2}$$
(5)

With the use of definition of relevance coefficient as like in Eq. (5), the failure probability with three failure modes would be approximated by $P_{f123} = P(M_1 \cup M_2 \cup M_3) = P(M_{12} \cup M_3) = P_{f12} + P_{f3} - r_{123}P_{f3} = P_{f12} + (1 - r_{123})P_{f3}$. It is shown that it is convenient to calculate failure probability and its sensitivity due to simple expression compared with above methods. According to the idea, the difficulty of presented method lies in how to define the correlation coefficient to ensure the accuracy of the calculation.

Research shows that in case of a bivariate normal distribution, the failure probability condition on $X_2 \leq -\beta_2 | X_1 \leq -\beta_1$ can be well approximated by the standard normal distribution of the same expectation and variance. Combined this approximation with the definition of relevancy, the correlation coefficient r_{12} in this paper is defined as

$$r_{12} \approx \Phi\left(\frac{-\beta_1 - \mu_{12}}{\sigma_{12}}\right) = \Phi\left(\frac{-\beta_1 + \rho_{12}\frac{\varphi\left(-\beta_2\right)}{\Phi\left(-\beta_2\right)}}{\sqrt{1 - \rho_{12}^2 A_2\left(-\beta_2 + A_2\right)}}\right)$$
(6)

In which $A_2 = \frac{\varphi(-\beta_2)}{\Phi(-\beta_2)}$, β_1 and β_2 denotes the reliability index of compounding failure event of the first failure modes respectively, and it can be obtained from the solution failure probability P_{f1} and P_{f2} .

Using this way, the system failure probability composed of m failure modes can be carried as follows:

$$P_{fs} = P_{f12,\dots,m-1} + (1 - r_{12,\dots,m-1,m}) P_{fm} = P_{f1} + (1 - r_{12}) P_{f2} + (1 - r_{12,3}) P_{f3} + \dots + (1 - r_{12,\dots,m}) P_{fm}$$
(7)

In which $r_{12,...,m}$ denotes the relevancy between the new compounding event $M_{12,...,m-1}$ coupled by the first m-1 failure mode and the remaining M_m . All terms in Eq. (7) are known except $r_{12}, r_{123}, \ldots, r_{12,...,m}$, which can be calculated as follows:

$$r_{12,\dots,m-1,m} = r_{ij} \approx \Phi\left(\frac{-\beta_i - \mu_{ij}}{\sigma_{ij}}\right) = \Phi\left(\frac{-\beta_i + \rho_{ij}\frac{\varphi\left(-\beta_j\right)}{\Phi\left(-\beta_j\right)}}{\sqrt{1 - \rho_{ij}^2 A_j\left(-\beta_j + A_j\right)}}\right)$$
(8)

In which $A_j = \frac{\varphi(-\beta_j)}{\Phi(-\beta_j)}$, β_i denotes the reliability index of compounding failure event of

the first i(i = 1, 2, ..., m - 1) failure modes, and it can be obtained from the solution failure probability $P_{f1,2,...,n-1}$. β_j denotes the reliability index of the *j*th (j = i + 1) failure mode remaining in the system. The correlation coefficient ρ_{ij} is given by $\rho_{ij} = \max \{\rho_{1m}, \rho_{2m}, ..., \rho_{m-1,m}\}$.

It should be noted that, the two-dimensional standard normal distribution will not change due to the arrangement of the two quartiles theoretically. However, for the multidimensional conditional correlation considered in this paper, the order of the conditional quantile has important influence on the result. In the formula, the calculation result is more accurate with the smaller the quantile value as the condition.

From Eq. (8), it is known that the calculation of system failure probability has been represented as an add of n conditional probability terms, and each term is approximated by correlation coefficient, and it can be carried out iteratively according to rearranged failure modes until all terms have been combined sequentially. Comparing with PCM, only diagonal elements are required for reliability analysis, which reduces the difficulty and is helpful for sensitivity analysis.

3.2 Sensitivity Assessment

Failure probability sensitivity analysis can provide information about the importance ranking of each random variable. To measure the sensitivity we can compute the partial derivative of the failure probability with respect to the parameter distribution interest [35,36]. Using Eq. (7), the sensitivity of the failure probability with respect to the distribution parameter $\theta(\mu, \sigma)$ can be computed as Eq. (9), where $r_{12,...,n}$ and $\frac{\partial r_{12,...,n}}{\partial \theta}$ are computed by efficient procedures in Eq. (8).

$$\frac{\partial P_{fs}}{\partial \theta} = \frac{\partial P_{f1}}{\partial \theta} + (1 - r_{12}) \frac{\partial P_{f2}}{\partial \theta} + (-P_{f2}) \frac{\partial r_{12}}{\partial \theta} + (1 - r_{123}) \frac{\partial P_{f3}}{\partial \theta} + (-P_{f3}) \frac{\partial r_{123}}{\partial \theta} + (1 - r_{12,\dots,n}) \frac{\partial P_{fn}}{\partial \theta} + (-P_{fn}) \frac{\partial r_{12,\dots,n}}{\partial \theta}$$

$$(9)$$

As expressed in Eq. (9), sensitivity results are easy to obtain owing to the simplified form of reliability analysis. For simplicity, we consider a system composed of two failure modes for which system failure probability can be given as $P_{f12} = P_{f1} + P_{f2} - r_{12}P_{f2} = P_{f1} + (1 - r_{12})P_{f2}$.

Combined Eq. (8) with Eq. (9), the sensitivity of failure probability with respect to $\theta(\mu, \sigma)$ can be obtained as

$$\frac{\partial P_{12}}{\partial \theta} = \frac{\partial P_1}{\partial \theta} + (1 - r_{12}) \frac{\partial P_2}{\partial \theta} + (-P_2) \frac{\partial r_{12}}{\partial \theta}$$
(10)

where $\frac{\partial P_1}{\partial \theta}$, $\frac{\partial P_2}{\partial \theta}$ can be calculated by moment method or Monte Carlo method. According to the definition of the correlation coefficient $r_{12} = \Phi(\beta_{12})$ and $\beta_{12} = \frac{-\beta_1 + \rho_{12}A}{\sqrt{1 - \rho_{12}^2 B}}$, $A = \frac{\phi(-\beta_2)}{\Phi(-\beta_2)}$,

$$B = A (-\beta_2 + A), \ \frac{\partial r_{12}}{\partial \theta} \text{ can be obtained by further derivation.}$$

$$\frac{\partial r_{12}}{\partial \theta} = \phi (\beta_{12}) \frac{\partial \beta_{12}}{\partial \theta}$$
(11)

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where

$$\frac{\partial \beta_{12}}{\partial \theta} = \frac{1}{\sqrt{1 - \rho_{12}^2 B}} \left(-\frac{\partial \beta_1}{\partial \theta} + \rho_{12} \frac{\partial A}{\partial \theta} \right) + \frac{-\beta_1 + \rho_{12} A}{2 \left(1 - \rho_{12}^2 B \right)^{\frac{3}{2}}} \rho_{12}^2 \frac{\partial B}{\partial \theta}$$

$$\frac{\partial A}{\partial \theta} = \frac{\phi' \left(-\beta_2 \right)}{\Phi \left(-\beta_2 \right)} \left(-\frac{\partial \beta_2}{\partial \theta} \right) - \frac{\phi \left(-\beta_2 \right)}{\Phi^2 \left(-\beta_2 \right)} \phi \left(-\beta_2 \right) \left(-\frac{\partial \beta_2}{\partial \theta} \right) = \frac{-\Phi \left(-\beta_2 \right) \beta_2 \phi \left(-\beta_2 \right) + \phi^2 \left(-\beta_2 \right)}{\Phi^2 \left(-\beta_2 \right)} \frac{\partial \beta_2}{\partial \theta}$$

$$\frac{\partial B}{\partial \theta} = \frac{\partial A}{\partial \theta} \left(-\beta_2 \right) - A \frac{\partial \beta_2}{\partial \theta} + 2A \frac{\partial A}{\partial \theta} = \left(2A - \beta_2 \right) \frac{\partial A}{\partial \theta} - A \frac{\partial \beta_2}{\partial \theta}$$

Similarly, for a system with terms of r_{123} , $\frac{\partial r_{123}}{\partial \theta}$, ..., $r_{12,...,n}$, and $\frac{\partial r_{12,...,n}}{\partial \theta}$ included in Eq. (9) can be solved successively by Eqs. (10) and (11). Therefore system sensitivity analysis is convenient to be realized without involving second-order or higher-order joint probability calculation. In essence, the result of reliability sensitivity analysis using Eq. (11) is accurate, and the calculation error depends only on the correlation coefficient and the solution error of the reliability of each failure mode. Therefore, as long as the single failure mode to ensure the reliability of sensitivity and the correlation accuracy of the approximate formula, it will ensure accuracy of system sensitivity analysis.

This paper combines the structural reliability algorithm and the proposed conditional reduction method based on correlation coefficient to calculate the structural system sensitivity as shown in Fig. 1 Among them, the structural reliability iterative algorithm is used to solve the reliability of the functional limit state function, and the reliability index β_i or failure probability P_f and sensitivity coefficient α_i are obtained.



Figure 1: Flowchart of system failure sensitivity based on correlation coefficient method

4 Numerical Examples

Three numerical examples are demonstrated the effectiveness of the proposed method. It should be noted that though the aim for failure probability sensitivity analysis is different from reliability itself, the calculation accuracy of sensitivity analysis depends on that of reliability analysis. Herein the first example is conducted to illustrate reliability calculation accuracy by comparing PCM and I-PCM, and the examples of Example 2 and Example 3 are

used to illustrate conducted to illustrate the effectiveness of the fast product of conditional reduction method.

4.1 Example 1: Analysis of Failure Probability of Series System

Consider a series system with 20 elements and each element having an identical element reliability index β and identical correlation coefficient ρ . The failure probability for the system with $\beta = 3.5$ is computed using exact integration, PCM, IPCM and the proposed method respectively. Fig. 2 shows the failure probability varies as ρ varies from 0.1 to 0.9. The variation of numerical error from different method with ρ is shown in Fig. 3 below.



Figure 2: Probability of failure of series system



Figure 3: Error versus correlation coefficient

It can be seen from Figs. 2 and 3 that the difference of errors from different method are all sensitive to the correlation coefficient ρ . The error is small when $\rho < 0.4$ and the absolute error are less than 5%. However the error behaves obvious difference when $\rho > 0.4$, where the error associated with PCM increases rapidly and the error of 60% is seen as ρ approaches 0.7. While

I-PCM method and the proposed method can both reduce the error effectively and the error associated with the proposed method is smaller than that associated with I-PCM. The numerical accuracy of the proposed method is considered acceptable with maximum error of about 20% in range of $0 < \rho < 0.9$. In addition, the proposed method involves much less iteration calculation compared with I-PCM method, and is more suitable for reliability analysis of large scale systems.

4.2 Example 2: Analysis of Reliability Sensitivities

Assume a series system contains three failure modes, the basic random variables obey normal distribution with $x_1 \sim N(3, 1)$, $x_2 \sim N(4, 1)$, $x_3 \sim N(6, 1)$. The limit state equation is given as follow.

$$\begin{cases} g_1(x) = 2x_1 - 3x_2 + x_3 + 8 = 0\\ g_2(x) = x_1 + 2x_2 + 4x_3 - 23 = 0\\ g_3(x) = -2x_1 - x_2 + x_3 + 10 = 0 \end{cases}$$
(12)

To compute the sensitivities according to Eq. (11) of Example 2, reliability analysis of each performance function g_i is applied. FORM is available to compute the failure probabilities of g_i and unit normal vectors $\boldsymbol{\alpha}_i$ at the MPP, and the correlation coefficient is calculated by $\rho_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j$. The results of reliability and sensitivity are given in Tabs. 2 and 3 respectively.

Performance function	Failure probability P_f	Unit normal vector $\boldsymbol{\alpha}_i$
g1 g2 g3	0.0163 0.0044 0.0072	(-0.5345, 0.8018, -0.2673) (-0.2182, -0.4364, -0.8729) (0.8165, 0.4082, -0.4082)

Table 2: Reliability results of Example 2

	$\partial P/\partial \mu_1$	$\partial P/\partial \mu_2$	$\partial P/\partial \mu_3$	$\partial P/\partial \sigma_1$	$\partial P/\partial \sigma_2$	$\partial P/\partial \sigma_3$
Monte Carlo method	-0.00822	0.03476	-0.03007	0.05848	0.06988	0.03991
The proposed method	-0.00831	0.03458	-0.02969	0.05784	0.06936	0.03928
Methods by Song et al. [37]	-0.007806	0.032610	-0.026837	0.058259	0.062748	0.03161

 Table 3: Comparison of sensitivity results of Example 2

4.3 Example 3: Analysis of Reliability Sensitivities of Simple Beam

The example is taken from Sues et al. [17], in which a simple beam under a uniform load. Three failure modes are considered composed of bending, shear and combined bending and shear. The three limit state functions for the three failure modes are assumed as follow respectively.

$$\begin{cases} g_1 = M_0 - \frac{1}{8} w_0 L^2 \\ g_2 = V_0 - \frac{1}{2} w_0 L \\ g_3 = 1 - \left(\frac{w_0 L^2}{8M_0} + \frac{w_0 M_0}{2V_0^2} \right) \end{cases}$$
(13)

where M_0 and V_0 is the maximum moment and maximum shear force the beam can bear, w_0 is uniform load applied on the beam, the length of the beam is L = 20. Random variables M_0 , V_0 and w_0 are all distributed normally, and the mean Value is {470, 159, 6} respectively, and variation coefficient is {0.10, 0.15, 0.25} respectively.

To compute the sensitivities according to Eq. (11) of Example 3, reliability analysis of performance function g_i is applied. FORM is available to compute the failure probabilities of g_i and unit normal vectors $\boldsymbol{\alpha}_i$ at the MPP, and the correlation coefficient is calculated by $\rho_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j$. The results of reliability and sensitivity are given in Tabs. 4 and 5 respectively.

Performance function	Failure probability P_f	Unit normal vector $\boldsymbol{\alpha}_i$
g1 g2	0.02739 0.00022	(-0.5310, 0, 0.8474) (0, -0.8465, 0.5324)
g3	0.05786	(-0.4407, -0.1095, 0.8910)

Table 4: Reliability results of Example 3

 Table 5: Comparison of sensitivity results of Example 3

	$\frac{\partial P_f}{\partial \mu_{M_0}} (\times 10^{-3})$	$\frac{\partial P_f}{\partial \mu_{V_0}}$ (×10 ⁻⁴)	$\frac{\partial P_f}{\partial \mu_{w_0}}$ (×10 ⁻²)	$\frac{\partial P_f}{\partial \sigma_{M_0}} (imes 10^{-4})$	$\frac{\partial P_f}{\partial \sigma_{V_0}} (\times 10^{-4})$	$\frac{\partial P_f}{\partial \sigma_{w_0}}$ (×10 ⁻²)
Monte Carlo method	-1.1342	-6.7147	7.2365	7.982	4.7732	9.9634
The proposed method	-1.1531	-5.4635	7.0956	8.0624	1.9236	9.9107
Methods by Sues et al. [17]	-1.0872	-5.4088	6.88	7.553	1.342	9.593

In *Example 2* and *Example 3*, the reliability sensitivities of P_f with respect to distribution parameter using Monte Carlo method can be estimated based on sampling with $\frac{\partial P_f}{\partial \mu_i} = 1 \sum_{i=1}^{M} \partial P_{ik} \partial P_{ik} = 1 \sum_{i=1}^{M} \partial P_{ik}$

$$\frac{1}{M}\sum_{k=1}^{N}\frac{\partial Y_{jk}}{\partial \mu_{i}}, \frac{\partial Y_{jk}}{\partial \sigma_{i}} = \frac{1}{M}\sum_{k=1}^{N}\frac{\partial Y_{jk}}{\partial \sigma_{i}}.$$
 Considering normal independent variables, Reliability sensi-

tivities of P_{fk} with respect to μ_i and σ_i is given by $\frac{\partial P_{fk}}{\partial \mu_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{x_{ji} - \mu_i}{\sigma_i}, \quad \frac{\partial P_{fk}}{\partial \sigma_i} = \frac{1}{N} \frac{P_{fk}}{\sigma_i} \sum_{j=1}^N \frac{P_{fk}}{\sigma_i}$

 $\frac{1}{N} \frac{P_{f_k}}{\sigma_i} \sum_{j=1}^{N} \left[\left(\frac{x_{ji} - \mu_i}{\sigma_i} \right)^2 - 1 \right].$ The Monte Carlo method is used to generate sample points in the

failure domain that obey the density function, and then the expectation of samples is used to replace the population mean to estimate the reliability sensitivity.

4.4 Discussion

The failure probabilities of *Example 3* for the individual limit states obtained using FORM are 0.027, 0.0023 and 0.058, respectively. The system failure probability using the proposed method is 0.0587 and the result from MCS is 0.061. This result is reasonable because the system failure is dominated by a single failure mode and all three failure modes are highly correlated. Tabs. 3 and 5 compares sensitivity results of the proposed method with those from MCS, in which the estimate results of MCS as the reference value are performed with 200,0000 simulations. The results of the importance ranking of each random variable from the proposed approach are well agreement with those from MCS. That is the load, w_0 , and moment capacity, M_0 , are the two most important variables. Treating shear capacity, V_0 , as a deterministic variable would have very little effect on the probabilistic results.

These comparisons show that proposed approach given by Eqs. (10) and (11) is theoretically correct based on fully considering the contribution of each failure mode to failure probability. The calculation accuracy is slightly higher than the result by Sues et al. [17] and Song et al. [37] in which the method combining linear expansion of each failure mode with Monte Carlo method for sensitivity estimation. Note that the basic random variables are normal distribution in above examples. For correlated non-normal variables, Nataf transformation are often used to convert them into independent standard normal variables, and the analytical derivatives of the Nataf transformation with respect to distribution parameters are presented by Hesam et al. [33]. The sensitivities verifies reasonably accurate although some results of the sensitivities in Tab. 5 are underestimated because of approximation of correlation coefficient. The major advantage of the proposed method is that the system sensitivity is computed in closed form expressions with adding less calculation effort and without resorting to finite difference approximations. It does not even depend on the analytical expression of the limit state equation, and has a wide range of application.

5 Conclusion

This paper develops a fast product of conditional reduction method based on conditional probability for reliability sensitivity analysis. The method combines explicit iteration algorithm and new correlation coefficient, in which the new correlation coefficient is suggested based on two-dimensional standard normal distribution considering the effects of the conditional quantile. The calculation result shows that it is more accurate with the smaller the quantile value as the condition. The method is demonstrated to be efficient through comparison with PCM, and a wider value range of correlation coefficient for reliability analysis is confirmed. Also the method is convenient to implement because it does not involve second-order or higher-order joint probability calculation.

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