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Article

On Modeling the Medical Care Insurance Data via a New Statistical Model

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Abstract: Proposing new statistical distributions which are more flexible than the existing distributions have become a recent trend in the practice of distribution theory. Actuaries often search for new and appropriate statistical models to address data related to financial and risk management problems. In the present study, an extension of the Lomax distribution is proposed via using the approach of the weighted T-*X* family of distributions. The mathematical properties along with the characterization of the new model via truncated moments are derived. The model parameters are estimated via a prominent approach called the maximum likelihood estimation method. A brief Monte Carlo simulation study to assess the performance of the model parameters is conducted. An application to medical care insurance data is provided to illustrate the potentials of the newly proposed extension of the Lomax distribution. The comparison of the proposed model is made with the (i) Two-parameter Lomax distribution, (ii) Three-parameter models called the half logistic Lomax and exponentiated Lomax distributions, and (iii) A four-parameter model called the Kumaraswamy Lomax distribution. The statistical analysis indicates that the proposed model performs better than the competitive models in analyzing data in financial and actuarial sciences.

Keywords: Lomax distribution; family of distributions; financial sciences; Monte Carlo simulation; estimation

1 Introduction

Statistical distributions play a vital role in modeling data in applied areas, particularly in the area of risk management problems, banking, economics, financial and actuarial sciences, among others. However, the quality of the approaches mainly depends upon the assumed probability model of the phenomenon under consideration. Among the applied areas, the insurance datasets are usually positive, right-skewed, unimodal shaped and with heavy tails [1–4]. The real-life data sets skewed to the right may be adequately modeled by the skewed distributions [5].

Among the right-skewed models, the Lomax distribution is one of the promising model offers data modeling in the areas of income and wealth inequality, financial and actuarial sciences, medical and biological sciences. A random variable X is said to have Lomax distribution, if its cumulative distribution function (CDF) is given by

$$F(x) = 1 - (1 + \theta x)^{-\alpha}, \quad x \geq 0, \alpha, \theta > 0, \tag{1}$$

where α is a shape parameter and θ is a scale parameter. The probability density function (pdf) corresponding to Eq. (1) is given by

$$f(x) = \alpha\theta(1 + \theta x)^{-\alpha-1}, \quad x > 0, \alpha, \theta > 0. \tag{2}$$

Due to the importance of the Lomax distribution in applied sciences, a number of extensions of the Lomax distribution have been proposed and studied; for detail we refer the interested reader to [6–14]. For more recent developments about distribution theory [15]. We further carry this branch of distribution theory and propose another useful extension of the Lomax distribution.

Recently, [16] proposed the weighted T-X (WTX) family of distributions via the cdf given by

$$G(x) = 1 - \left(\frac{1 - F(x)}{e^{F(x)}}\right), \quad x \in \mathbb{R}, \tag{3}$$

with pdf given by

$$g(x) = \frac{f(x)}{e^{F(x)}}[2 - F(x)], \quad x \in \mathbb{R}. \tag{4}$$

For the illustrative purposes, Ahmad [16] studied a special-case of the weighted T-X Weibull (WTX-W) distribution. This paper proposes a new probability model with a minimum number of parameters and capable of modeling financial data sets. Henceforth, another special sub-model of the WTX family is introduced by using the Eq. (1) in Eq. (3). The new model may be called the weighted T-X Lomax (WTX-Lomax) distribution.

The rest of this paper is organized as follows. In Section 2, we introduce the WTX-Lomax distribution and provide plots of its density and hazard rate functions. In Section 3, we investigate various mathematical properties of the WTX-Lomax distribution. The characterization of the proposed model is provided in Section 4. In Section 5, estimation of the parameters is provided via the maximum likelihood estimation (MLE) method. Simulation results on the behavior of the MLEs are presented in Section 6. A real data application to medical care insurance data is presented in Section 7. Finally, in Section 8, we conclude the paper.

2 The WTX-L Distribution

A random variable, say X, is said to follow the WTX-Lomax distribution, if its cdf is defined by

$$G(x) = 1 - \left(\frac{(1 + \theta x)^{-\alpha}}{e^{1-(1+\theta x)^{-\alpha}}}\right), \quad x \geq 0, \alpha, \theta > 0. \tag{5}$$

The density and hazard rate functions corresponding to Eq. (5) are respectively, given by

$$g(x) = \frac{\alpha\theta(1 + \theta x)^{-\alpha-1}}{e^{1-(1+\theta x)^{-\alpha}}}[1 + (1 + \theta x)^{-\alpha}], \quad x > 0. \tag{6}$$

$$h(x) = \frac{\alpha\theta}{(1 + \theta x)}[1 + (1 + \theta x)^{-\alpha}], \quad x > 0.$$

The plots for the pdf and hazard rate function (hrf) of the WTX-Lomax distribution are presented in Figs. 1 and 2, respectively.

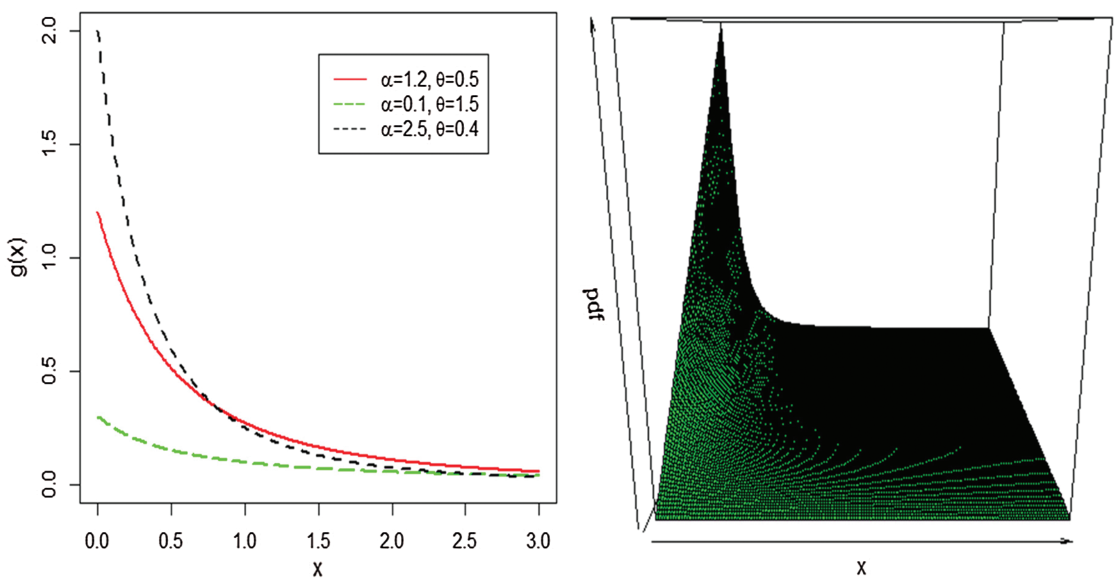

Figure 1: Plots of the WTX-Lomax pdf for some selected parameter values

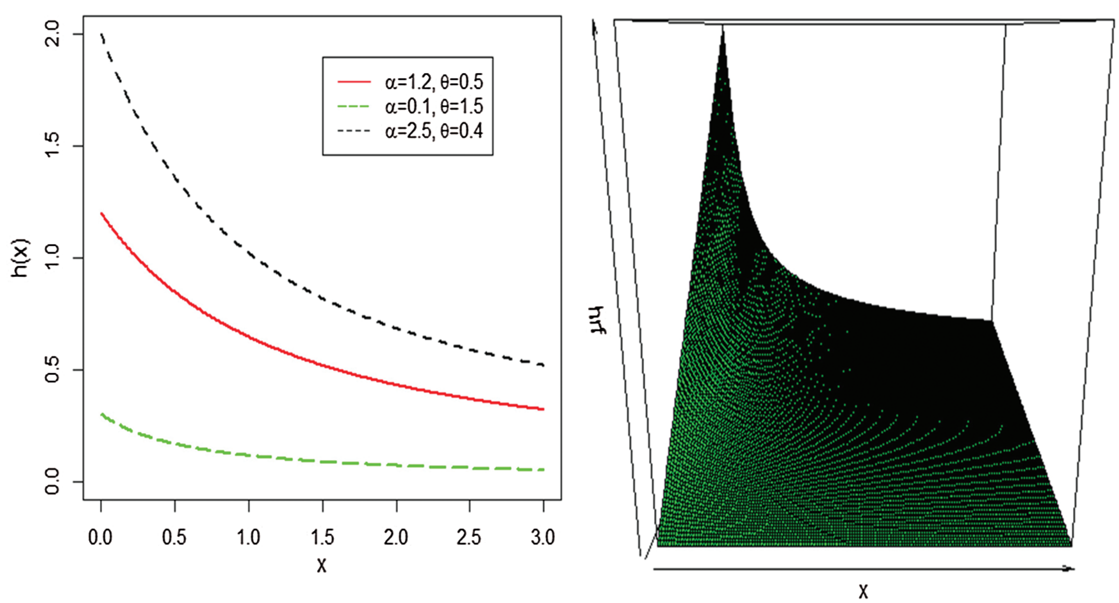

Figure 2: Plots of the WTX-Lomax hrf for some selected parameter values

3 Mathematical Properties

This section offers some mathematical properties of the WTX-Lomax distribution.

3.1 Quantile and Random Number Generation

The distribution function of the WTX-Lomax distribution is given by Eq. (5). Inverting the expression $G(x) = u$, we get

$$\log(1-u) + 1 - (1+\theta x)^{-\alpha} - \log(1+\theta x)^{-\alpha} = 0, \tag{7}$$

where $u \in (0,1)$. The Eq. (7) can be used to generate random numbers from the proposed model. Furthermore, the effects of the shape parameters on the skewness and kurtosis can be detected on quantile measures. We obtain skewness and kurtosis measures of the proposed family using Eq. (7). The Bowley's skewness of X is given by

$$Skewness = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)},$$

whereas, the Moor's kurtosis is

$$Kurtosis = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

These measures are less sensitive to outliers. Moreover, they do exist for distributions without moments.

3.2 Moments

Suppose X is a WTX-Lomax distributed random variable, then the rth moment of X is derived as

$$\mu_r^{/} = \int_0^{\infty} x^r \frac{\alpha\theta(1+\theta x)^{-\alpha-1}}{e^{1-(1+\theta x)^{-\alpha}}}[1 + (1+\theta x)^{-\alpha}]dx,$$

$$\mu_r^{/} = \alpha\theta \int_0^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^r (1+\theta x)^{-\alpha-1}(1-(1+\theta x)^{-\alpha})^i [1 + (1+\theta x)^{-\alpha}]dx,$$

$$\mu_r^{/} = \alpha\theta \int_0^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{i}{j} x^r (1+\theta x)^{-\alpha-1}(1+\theta x)^{-\alpha j}[1 + (1+\theta x)^{-\alpha}]dx,$$

$$\mu_r^{/} = \alpha\theta \int_0^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{i}{j} x^r (1+\theta x)^{-\alpha(j+1)-1}[1 + (1+\theta x)^{-\alpha}]dx. \tag{8}$$

Let $y = 1 + \theta x, \frac{dy}{\theta} = dx$. From Eq. (8), we get

$$\mu_r^{/} = \frac{\alpha}{\theta^r} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{i}{j} \eta_{r,\alpha,j}.$$

where

$$\eta_{r,\alpha,j} = \int_0^\infty y^{-\alpha(j+1)-1}(y-1)^r(1+y^{-\alpha})dy.$$

The effects of different values of the parameters α and θ on the mean, variance, skewness, and kurtosis of the WTX-Lomax distribution are illustrated in Figs. 3 and 4.

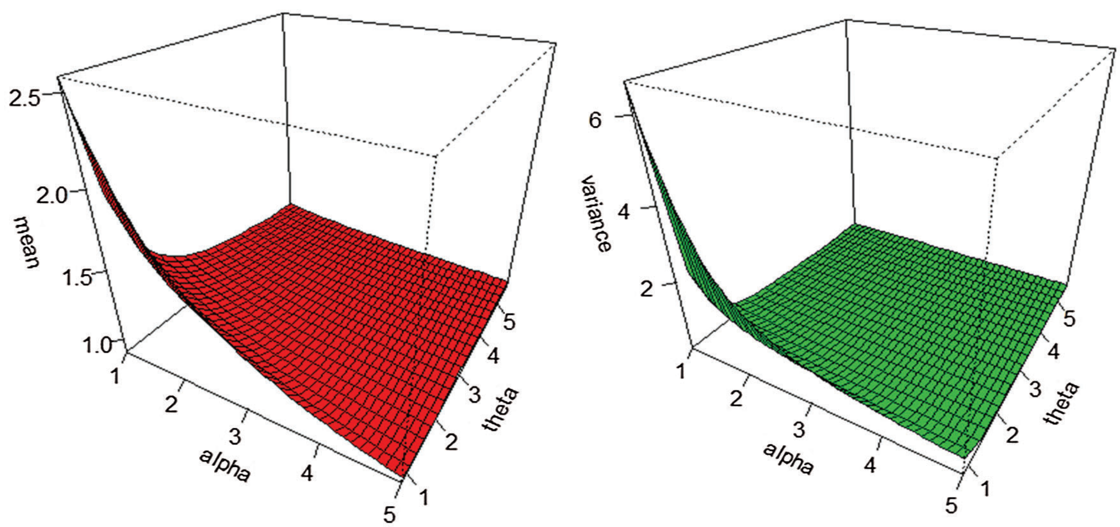

Figure 3: The mean and variance plots of the WTX-Lomax distribution

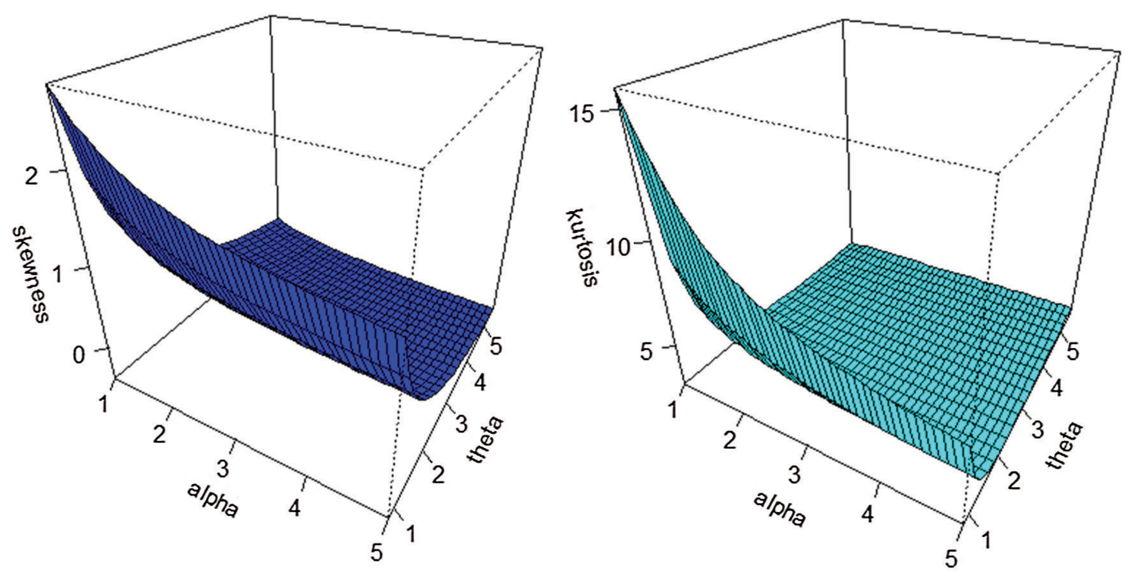

Figure 4: The skewness and kurtosis plots of the WTX-Lomax distribution

4 Characterization of the WTX-Lomax Distribution

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present a characterization of the WTX-Lomax distribution based on the conditional expectation (truncated moment) of certain function of a random variable.

4.1 Characterization Based on Two Truncated Moments

This subsection deals with the characterizations of WTX-Lomax distribution in terms of a simple relationship between two truncated moments. We will employ Theorem 1 given in the Appendix A. This characterization is stable in the sense of weak convergence.

Proposition 4.1.1. Let X be a continuous random variable and let $q_1(x) = \dfrac{e^{1-(1+\theta x)^{-\alpha}}}{[1-(1+\theta x)^{-\alpha}]}$ and $q_2(x) = q_1(x)(1+\theta x)^{-1}$ for $x > 0$. Then X has pdf given in Eq. (6) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{\alpha}{\alpha+1}(1+\theta x)^{-1}, \quad x > 0.$$

Proof. If X has pdf Eq. (6), then

$$(1-G(x))E[q_1(X)|X \geq x] = (1+\theta x)^{-\alpha}, \quad x > 0,$$

and

$$(1-G(x))E[q_2(X)|X \geq x] = \frac{\alpha}{\alpha+1}(1+\theta x)^{-\alpha-1}, \quad x > 0,$$

and hence

$$\xi(x) = \frac{\alpha}{\alpha+1}(1+\theta x)^{-1}, \quad x > 0.$$

We also have

$$\xi(x)q_1(x) - q_2(x) = -\frac{1}{\alpha+1}q_1(x)(1+\theta x)^{-1} < 0, \quad \textit{for } x > 0.$$

Conversely, if $\xi(x)$ is of the above form, then

$$s^{/}(x) = \frac{\xi^{/}(x)q_1(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{\alpha\theta}{(1+\theta x)}, \quad x > 0,$$

and

$s(x) = \log\{(1+\theta x)^{\alpha}\}.$

Now, according to Theorem 1, X has density provided in Eq. (6).

5 The Maximum Likelihood Estimation

In this section, we consider the estimation of the unknown parameters of the WTX-Lomax distribution from complete samples only via the method of maximum likelihood. Let $X_1, X_2, \ldots, X_n$ be a random sample from the WTX-Lomax distribution with observed values $x_1, x_2, \ldots, x_n$. The log-likelihood function is

$$LogL = n\log\alpha + n\log\theta - (\alpha+1)\sum_{i=1}^{n}\log(1+\theta x_i) + \sum_{i=1}^{n}\log(1+(1+\theta x_i)^{-\alpha}) - \sum_{i=1}^{n}[1-(1+\theta x_i)^{-\alpha}] \quad (9)$$

The nonlinear likelihood equations can be obtained by differentiating Eq. (9) as follows:

$$\frac{\partial LogL}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n}\log(1+\theta x_i) - \sum_{i=0}^{n}\frac{\log(1+\theta x_i)(1+\theta x_i)^{-\alpha}}{(1+(1+\theta x_i)^{-\alpha})} - \sum_{i=0}^{n} log(1+\theta x_i)(1+\theta x_i)^{-\alpha} = 0,$$

and

$$\frac{\partial LogL}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\frac{(\alpha+1)x_i}{(1+\theta x_i)} - \alpha\sum_{i=1}^{n}\frac{(1+\theta x_i)^{-\alpha-1}x_i}{(1+(1+\theta x_i)^{-\alpha})} - \sum_{i=1}^{n}(1+\theta x_i)^{-\alpha-1}x_i = 0.$$

Solving $\frac{\partial LogL}{\partial\alpha} = 0$ and $\frac{\partial LogL}{\partial\theta} = 0$ gives the maximum likelihood estimates of parameters α and θ, respectively. Meanwhile, the solution cannot be obtained analytically except numerically when data sets are available. Softwares like R, MATLAB, MAPLE, and so on could be used to obtain the estimates.

6 Monte Carlo Simulation Study

The behavior of the maximum likelihood estimators of the WTX-Lomax distribution has been investigated by conducting the Monte Carlo simulation studies using R software. Data sets were generated from the WTX-Lomax distribution with a replication number $N = 500$, random samples of sizes $n = 25, 50, \ldots, 500$. The simulation is conducted for two different cases using varying parameter values. The selected true parameter values are: (i) Set 1, $\alpha = 0.6$ and $\theta = 1.2$ and (ii) Set 1, $\alpha = 1.2$ and $\theta = 0.8$. The simulation results are provided in Figs. 5–8, indicating that

i) The estimates are quite stable and, more importantly, are close to the true values for these sample sizes,
ii) The estimated biases decrease when the sample size n increases,
iii) The estimated MSEs decay toward zero when the sample size n increases.

7 An Application to Medical Care Insurance Data

The main applications of the heavy-tailed models are the so-called extreme value theory or insurance loss phenomena. In this section, we illustrate the potentiality of the proposed model via a real-life application taken from actuarial sciences. The data set representing the medical care insurances and is available at: https: instruction.bus.wisc.edujfreesjfreesbooksRegression.

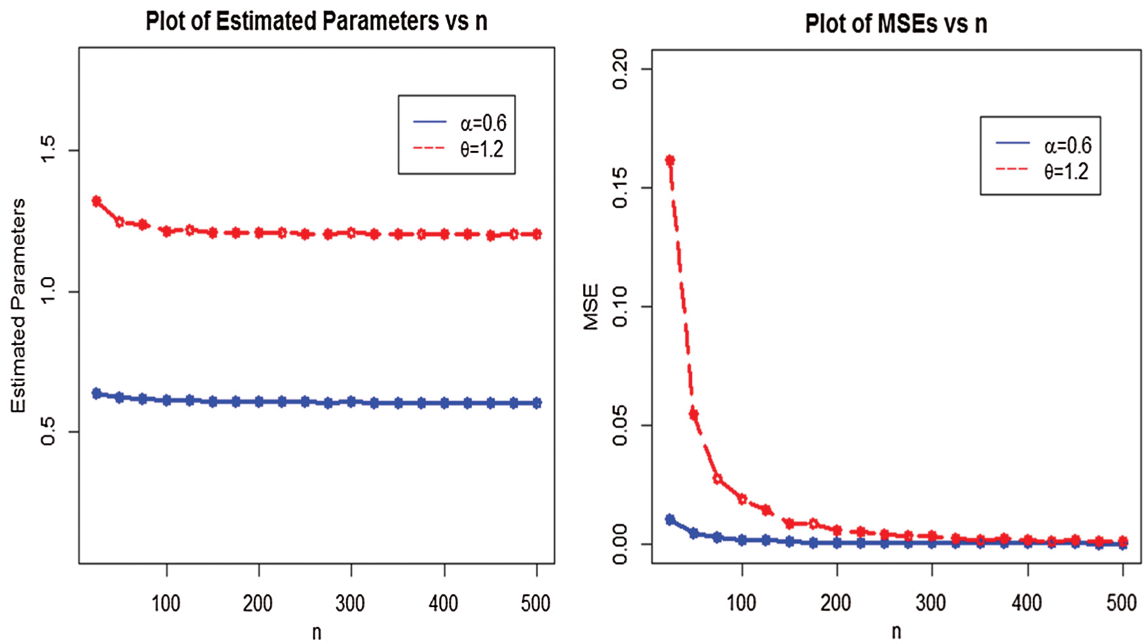

Figure 5: Plots of the estimated parameters and MSEs of the WTX-Lomax distribution

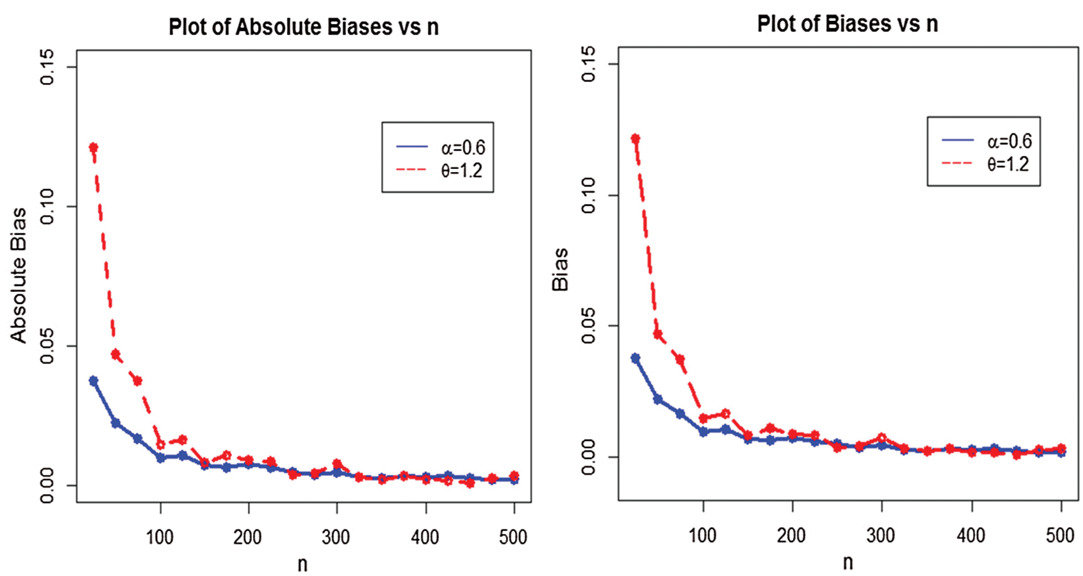

Figure 6: Plots of absolute biases and biases for WTX-Lomax distribution

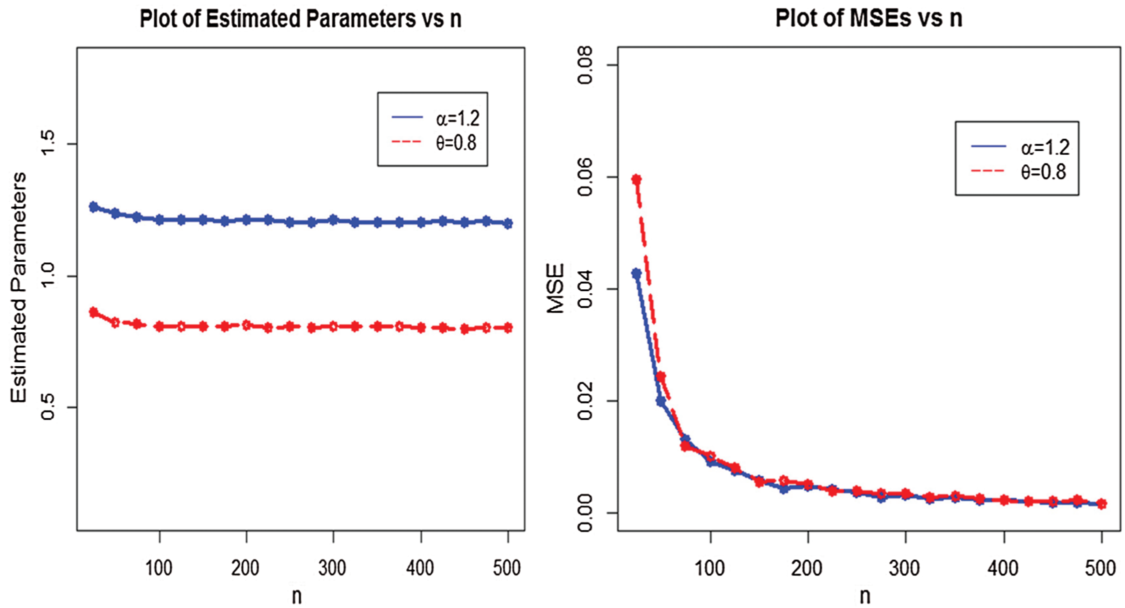

Figure 7: Plots of the estimated parameters and MSEs of the WTX-Lomax distribution

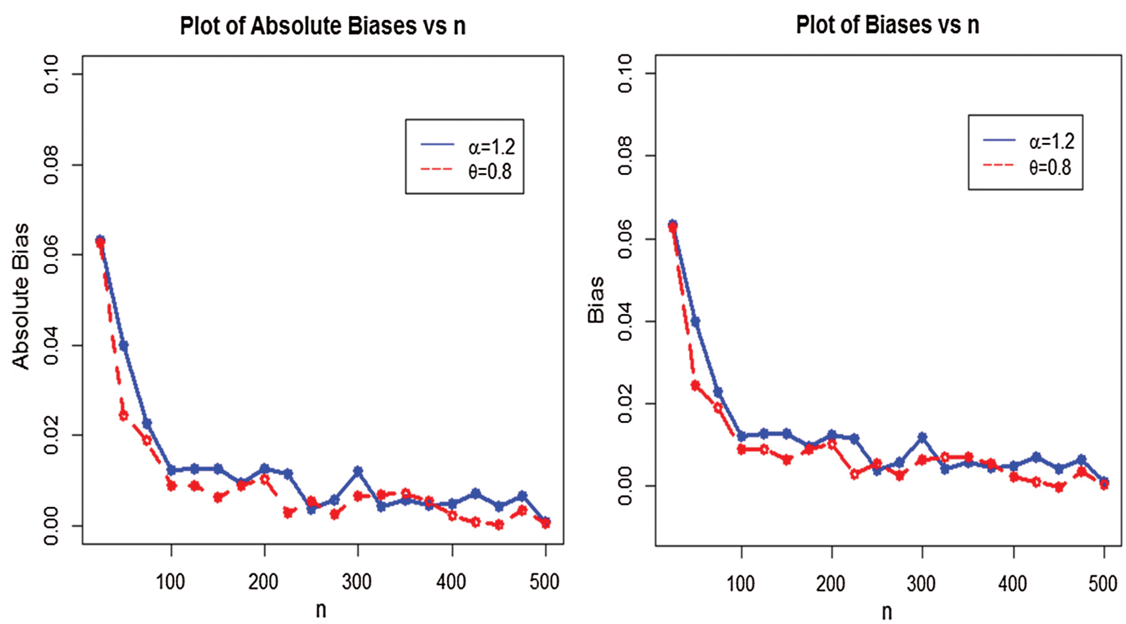

Figure 8: Plots of absolute biases and biases for WTX-Lomax distribution

The comparison of the WTX-Lomax distribution is made with two parameters, three parameters and four parameters models. The density functions of the competitive distributions are:

- Lomax distribution

$F(x) = 1 - (1 + \theta x)^{-\alpha}, \quad x \geq 0, \alpha, \theta > 0.$

- Kumaraswamy Lomax (Ku-Lomax) distribution

$$F(x) = 1 - \left[1 - \left(1 - (1+\theta x)^{-\alpha}\right)^{a}\right]^{b}, \quad x \geq 0, a, b, \alpha, \theta > 0.$$

- Exponentiated Lomax (E-Lomax) distribution

$$F(x) = \left[1 - (1+\theta x)^{-\alpha}\right]^{a}, \quad x \geq 0, a, \alpha, \theta > 0.$$

- Half Logistic Lomax (HL-Lomax) distribution

$$F(x) = \frac{\left[1 - (1+\theta x)^{-\alpha}\right]^{a}}{\left[1 + (1+\theta x)^{-\alpha}\right]^{a}}, \quad x \geq 0, a, \alpha, \theta > 0.$$

To decide about the goodness of fit between the proposed and competing distributions, we consider certain statistical measures. In this regard, we took (i) four discrimination measures such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC) and Consistent Akaike Information Criterion (CAIC) and (ii) two goodness of fit procedure including the Cramer–Von Messes (CM) test statistic and Anderson Darling (AD) test statistic.

The proposed WTX-Lomax and the competing distributions are applied to this data set. The maximum likelihood estimates of the models for the medical care insurance data are presented in Tab. 1, whereas the analytical and goodness of fit measures of the proposed and other competitive models are provided in Tabs. 2 and 3, respectively.

Table 1: The estimated values of the parameters of the fitted distributions

Distributions	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
WTX-Lomax	0.457	1.064	–	–
Lomax	0.925	2.185	–	–
E-Lomax	0.856	1.385	0.809	–
Ku-Lomax	0.753	1.249	0.7452	0.935
HL-Lomax	1.967	2.397	1.064	–

Table 2: The discrimination measures of the fitted models

Distributions	AIC	BIC	CAIC	HQIC
WTX-Lomax	2874.756	2902.709	2875.653	2892.632
Lomax	3002.658	3098.604	3003.835	3065.796
E-Lomax	2886.734	2961.087	2887.549	2945.473
Ku-Lomax	2893.905	2975.763	2894.009	2958.876
HL-Lomax	2986.308	3067.245	2987.738	3032.058

Table 3: The goodness of fit measures of the fitted models

Distributions	AD	CM
WTX-Lomax	0.498	0.398
Lomax	1.409	1.386
E-Lomax	0.693	0.607
Ku-Lomax	0.784	0.695
HL-Lomax	0.966	0.895

A distribution with lower values of these measures is considered a good candidate model among the applied distributions for the data under consideration. Form Tabs. 2 and 3, it is well clear that the by considering the above statistical tools, we observed that the WTX-Lomax distribution provides the best fit compared to the other competitors since the values of all selected criteria of goodness of fit are significantly smaller for the proposed distribution.

Furthermore, the fitted cdf and Kaplan–Meier survival plots of the proposed model are plotted in Fig. 9, whereas the probability–probability (PP) plot of the proposed model are sketched in Fig. 10. From Fig. 9, it is clear that the proposed model fits the estimated cdf and Kaplan–Meier survival very closely. From Fig. 10, we can easily detect that the proposed model is closely followed the PP-plot which is an empirical tool for finding a best candidate model.

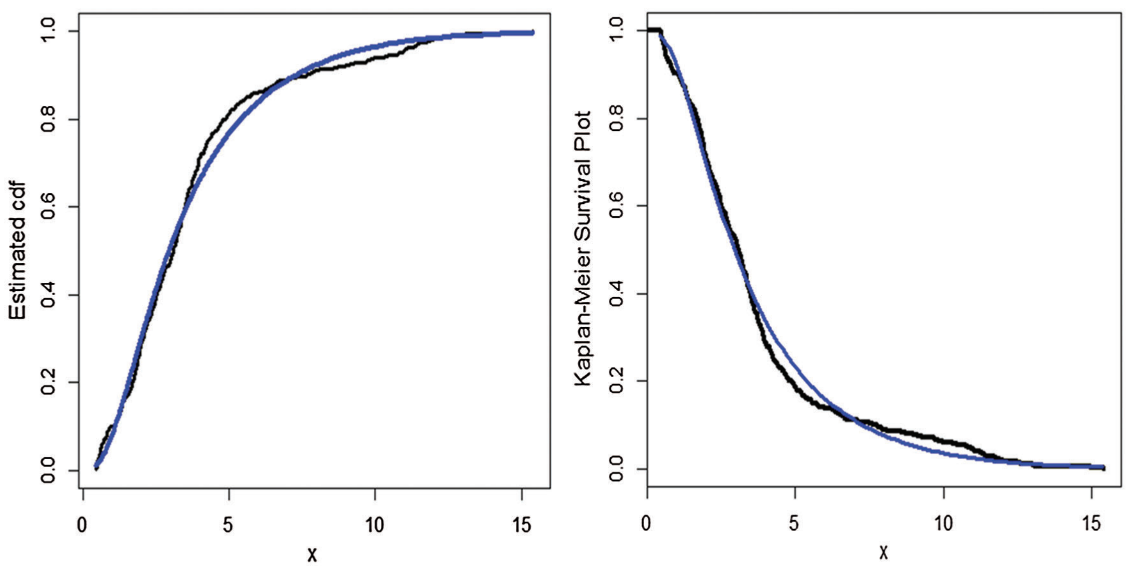

Figure 9: The estimated cdf and Kaplan–Meier survival plots of the WTX-Lomax distribution

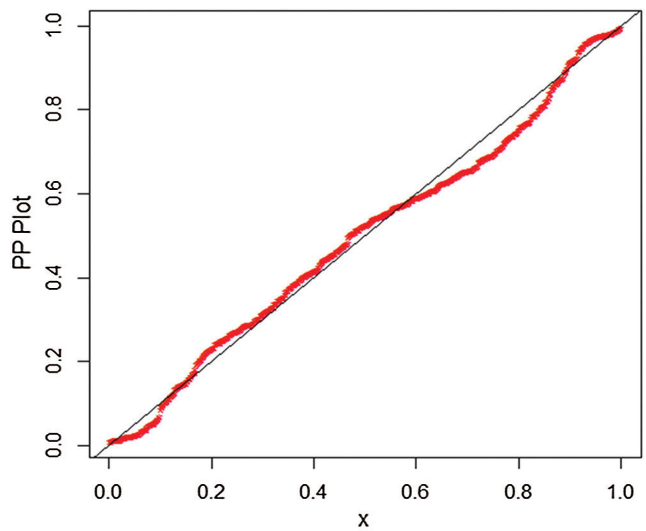

Figure 10: The PP plot of the WTX-Lomax distribution for the medical care insurance data

8 Concluding Remarks

Over the past couple of decades, the Lomax distribution and its various extensions have been used successfully to model real phenomena in applied areas, particularly in finance, banking, accounting and actuarial sciences. In this article, a new extension of the Lomax distribution, called weighted T-*X* Lomax distribution has been proposed. Some mathematical properties are derived and maximum likelihood estimates of the model parameters are obtained. The Monte Carlo simulation conducted shows the maximum likelihood estimators of the proposed model are stable enough and the MSEs and biases decreased as the sample size increased. A real-life application from insurances representing medical care insurance data is analyzed showing that the WTX-Lomax distribution provides better fit than some of the other well-known statistical models.

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Appendix A

Theorem 1. Let *(Ω, F, P)* be a given probability space and let *H = [a,b]* be an interval for some $a < b(a = -\infty, b = \infty$ might as well be allowed). Let $X{:}\Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E(q_2(X)|X \geq x) = E(q_1(X)|X \geq x)\xi(x), \quad x \in H,$$

is defined with some real function ξ. Assume that $q_1, q_2 \in C^1(H), \xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ particularly

$$F(x) = \int_a^x C\left|\frac{\xi^{/}(u)}{\xi(u)q_1(u) - q_2(u)}\right|\exp(-s(u))du,$$

where the function $s(u)$ is a solution of the differential equation $s^{/}(u) = \dfrac{\xi^{/} q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Note: The goal is to have the function $\xi(x)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and $\xi_n (n \in N)$ satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, and $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x) = \frac{E[q_2(X)|X \geq x]}{E[q_1(X)|X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is rejected by corresponding convergence of the functions q_1, q_2 and ξ, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations rejecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1 \equiv 1$, which reduces the condition of Theorem 1 to $E[q_2(X)|X \geq x] = \xi(x), x \in H$. We, however, believe that employing three functions q_1, q_2 and ξ will enhance the domain of applicability of Theorem 1.