Zubair Lomax Distribution: Properties and Estimation Based on Ranked Set Sampling

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Abstract: In this article, we offer a new adapted model with three parameters, called Zubair Lomax distribution. The new model can be very useful in analyzing and modeling real data and provides better fits than some others new models. Primary properties of the Zubair Lomax model are determined by moments, probability weighted moments, Renyi entropy, quantile function and stochastic ordering, among others. Maximum likelihood method is used to estimate the population parameters, owing to simple random sample and ranked set sampling schemes. The behavior of the maximum likelihood estimates for the model parameters is studied using Monte Carlo simulation. Criteria measures including biases, mean square errors and relative efficiencies are used to compare estimates. Regarding the simulation study, we observe that the estimates based on ranked set sampling are more efficient compared to the estimates based on simple random sample. The importance and flexibility of Zubair Lomax are validated empirically in modeling two types of lifetime data.

Keywords: Lomax distribution, Zubair-g family, moments, maximum likelihood estimation, ranked set sampling.

1 Introduction

Ranked set sampling (RSS) is a statistical procedure for data collection that generally leads to more structural alternative approach to simple random sample (SRS). The RSS is suitable in positions where the optical ordering of a set of units is done comfortably, while the exact measurement of the units is difficult or cost. The efficiency of RSS related to SRS in different statistical methods has been investigated by several researchers [Noughabi (2017)].

The RSS procedure is summarized as follows:

- Draw m random samples, each of size m, from the target population.
- Without taking any measurements, rank units within each raw depend on a criterion

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determined by the researcher.

- Choose a sample for actual judgment by involving the smallest ranked unit in the first set, the second smallest ranked unit in the second set, the operation is continues in this manner until the largest ranked unit is selected from the last set.
- To obtain a sample of size mk units from the RSS data, the cycle may be repeated k times.

Studies have been examined by various researches via RSS scheme. For instance; parameter estimation for the modified Weibull distribution was discussed by Al-Hadhrami [Al-Hadhrami (2010)]. Estimation of the Weibull parameters was studied by Helu et al. [Helu, Abu-Salih and Alkami (2010); Hassan (2012)] provided the goodness of fit tests for the exponentiated Pareto distribution via extreme RSS scheme. Hassan [Hassan (2013)] discussed the Bayesian and maximum likelihood (ML) estimators of generalized exponential. Hassan et al. [Hassan, Abd-Elfattah and Nagy (2013)] considered modified goodness of fit tests of Weibull distribution based on extreme RSS. Yousef et al. [Yousef and Al-Subh (2014)] discussed parameter estimation of Gumbel distribution. Hassan et al. [Hassan, Assar and Yahya (2014); Hassan, Assar and Yahiya (2015)] handled with stress strength reliability estimation when both populations are two independent Burr type XII distribution via some modifications of RSS. Bayesian parameter estimator of exponential distribution has been provided by Sadek et al. [Sadek, Sultan and Balakrishnan (2015)]. Khamnei et al. [Khamnei and Abusaleh (2017)] discussed estimators of generalized logistic distribution.

Lomax distribution is of a great importance and has applications in many areas, like actuarial science, economics, biological sciences and engineering. This distribution is regarded as a useful alternative to survival problems and life-testing in engineering [Hassan and Al-Ghamdi (2009); Hassan, Assar and Shelbaia (2016)]. The cumulative distribution function (cdf) of a Lomax distribution with shape parameter α and scale parameter β is given by:

$$G(x;\beta,\alpha)=1-\left(1+\frac{x}{\beta}\right)^{-\alpha},\ x,\beta,\alpha>0. \tag{1}$$

The probability density function (pdf) of Lomax distribution is as follows:

$$g(x;\beta,\alpha)=\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1},\ x,\beta,\alpha>0. \tag{2}$$

Generalized and extended forms of a Lomax distribution were provided by many authors; examples include Marshall Olkin extended Lomax by Ghitany et al. [Ghitany, Al-Awadhi and Alkhalfan (2007)]; Kumaraswamy Lomax (KwL), beta-Lomax (BL), McDonald-Lomax by Lemonte et al. [Lemonte and Cordeiro (2013)], exponentiated Lomax (EL) by Abdul-Moniem et al. [Abdul-Moniem and Abdel-Hameed (2012)]; gamma-Lomax (GL) by Cordeiro et al. [Cordeiro, Ortega and Popović (2014)]; Weibull-Lomax (WL) by Tahir et al. [Tahir, Cordeiro, Mansoor et al. (2015)]; transmuted WL (TWL) by Afify et al. [Afify, Nofal, Yousof et al. (2015c)]; Gumbel-Lomax by Tahir et al. [Tahir, Hussain, Corderio et al. (2016)]; Power Lomax (PL) by Rady et al. [Rady, Hassanein and

Elhaddad (2016)]; EL geometric by Hassan et al. [Hassan and Abd-Allah (2017)]; PL Poisson by Hassan et al. [Hassan and Nassr (2018)]; exponentiated WL by Hassan et al. [Hassan and Abd-Allah (2018)]; inverse PL distribution by Hassan et al. [Hassan and Abd-Allah (2019); inverted EL by Hassan et al. [Hassan and Mohamed (2019a)]; Weibull inverse Lomax by Hassan et al. [Hassan and Mohamed (2019 b)]; truncated PL by Hassan et al. [Hassan, Sabry and Elsehetry (2020)] among others.

The generated distributions have attracted several statisticians to develop new models. Recently, Tahir et al. [Tahir and Cordeiro (2016)] proposed complementary exponentiated-G Poisson (CEGP) family of distributions. Our interest here with Zubair-G family proposed by Ahmed [Ahmed (2020)] which is considered as special sub-class from CEGP. The cdf and pdf of ZL-G family are given by

$$F(x;\lambda,\zeta)=\frac{e^{\lambda G(x)^2}-1}{e^{\lambda}-1}, \tag{3}$$

and,

$$f\left(x;\lambda,\zeta\right)=\frac{2\lambda g(x)G(x)e^{\lambda G(x)^2}}{e^{\lambda}-1}\text{ , }\lambda>0, \tag{4}$$

where $G\left(x;\xi\right)$ be the cdf of the baseline model.

We aim to come with a new improvement of a Lomax distribution related to Zubair-G family. We are encouraged to study Zubair Lomax (ZL) distribution (i) to enhance the merit and flexibility of a Lomax distribution; (ii) to modify the Lomax distribution by inserting only one shape parameter, instead of two or more parameters; and (iii) to yield better fits than some another models with the same or higher number of parameters. We study main properties of ZL model, estimate the population parameter via SRS and RSS schemes. Further, an application to real data is utilized. This paper is outlined as follows. In Section 2, the pdf, cdf, reliability, and hazard rate function (hrf) for ZL distribution are defined. Properties of ZL including moments, probability weighted moments, quantile function, Rényi entropy, incomplete moments and stochastic ordering are derived in Section 3. The maximum likelihood estimators of the model parameters as well as simulation study are derived in case of SRS and RSS as provided in Section 4. The ZL model is shown to give better fit for real data sets as explained in Section 5. Eventually, some concluding remarks are given in Section 6.

2 Zubair lomax distribution

This section provides a new three-parameter Zubair Lomax distribution. We give the form of the pdf, cdf, survival function (sf) and hrf of ZL model.

Definition: A random variable X is said to have a three-parameter ZL distribution if its cdf is of the form

$$F(x;\phi)=\left(e^{\lambda}-1\right)^{-1}\left[\mathrm{e}^{\lambda\left(1-\left(1+x/\beta\right)^{-\alpha}\right)^{2}}-1\right], \tag{5}$$

where; $\phi = (\alpha, \theta, \beta)$ is a set of parameters. The pdf of the ZL distribution is given by

$$f(x;\phi) = \frac{2\alpha\lambda(e^{\lambda}-1)^{-1}}{\beta}\left(1+{x}/{\beta}\right)^{-\alpha-1}\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle e^{\lambda\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle^{2}} ;x,\alpha,\lambda,\beta > 0. \quad (6)$$

Also, the sf, say $\bar{F}(x;\varphi)$, and hrf, say $h(x;\varphi)$, of X are given, respectively, as follows:

$$\bar{F}(x;\phi) = \left(e^{\lambda}-1\right)^{-1}\left[e^{\lambda} - e^{\lambda\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle^{2}}\right], \quad (7)$$

and

$$h(x;\phi) = \frac{2\alpha\lambda}{\beta}\left(1+{x}/{\beta}\right)^{-\alpha-1}\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle e^{\lambda\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle^{2}}\left[e^{\lambda} - e^{\lambda\left\langle 1-\left(1+{x}/{\beta}\right)^{-\alpha}\right\rangle^{2}}\right]^{-1}. \quad (8)$$

A random variable X with pdf Eq. (6) will be denoted by X~ZL (φ). Plots of pdf and hrf of X are explained for specific choices of φ are given in Fig. 1.

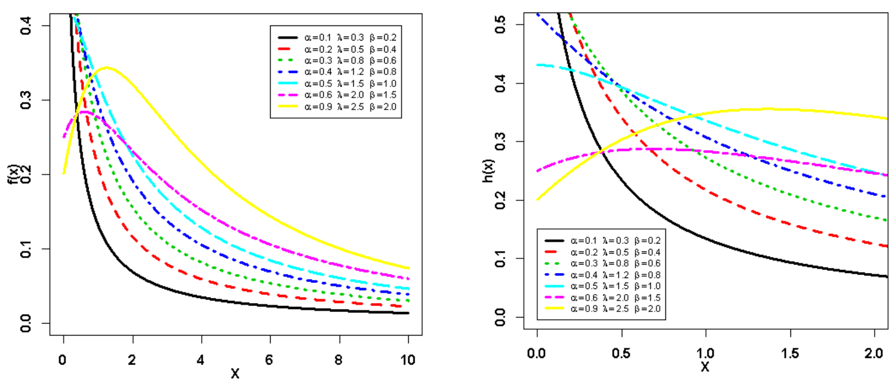


Figure 1: The pdf and hrf plots of ZL model

3 Fundamental properties

In this section, we obtain some important statistical properties of the ZL distribution such as; quantile function, moments and related statistics, incomplete moments, the probability weighted moments and Rényi entropy.

3.1 Quantile function

The ZL quantile function, say $Q(u) = F^{-1}(u)$, is simply to be worked out by inverting Eq. (5) as follows:

$$x_p = \beta\left\{\left\langle 1-\left(\left[\frac{1}{\lambda}\ln\left(u\left(e^{\lambda}-1\right)+1\right)\right]^{0.5}\right)\right\rangle^{-1/\alpha} - 1\right\}, \quad (9)$$

where, $x_p = Q(u)$. The ZL model can be generated from Eq. (9), where u has the uniform distribution $U(0,1)$.

3.2 Moments and related statistics

Many of the motivating aspect of any model can be examined via their moments. The s^{th} moment about origin of X has a ZL distribution is obtained as follows:

$$\mu_s' = \int_0^\infty x^s \frac{2\alpha\lambda(e^\lambda - 1)^{-1}}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle e^{\lambda\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle^2} dx. \tag{10}$$

By using exponential and binomial expansions in (10), we get

$$\mu_s' = \beta^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\int_0^\infty x^s\left(1+\frac{x}{\beta}\right)^{-\alpha-\alpha j-1} dx, \tag{11}$$

where, $N_{i,j} = \binom{2i+1}{j}\frac{2\alpha\lambda^{i+1}(-1)^j}{(e^\lambda - 1)i!}$.

After simplification, the s^{th} moment of ZL distribution is given by:

$$\mu_s' = \sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\beta^s N_{i,j}\,\mathrm{B}(s+1, \alpha+\alpha j - s), \tag{12}$$

where, B(.,.) is the beta function. The s^{th} central moment of ZL distribution is given from

$$\mu_s = E(X-\mu_1')^s = \sum_{i=0}^{s}(-1)^i\binom{s}{i}(\mu_1')^i\mu_{s-i}'. \tag{13}$$

The skewness and kurtosis can be obtained from Eq. (13).

3.3 Incomplete moments

The s^{th} lower incomplete moment, say $\varpi_s(z)$, of ZL model, is given by

$$\varpi_s(z) = \int_0^z \frac{2\alpha\lambda(e^\lambda - 1)^{-1}x^s}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle e^{\lambda\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle^2} dx. \tag{14}$$

As mentioned above, we employ the exponential and binomial expansions, and then the s^{th} incomplete moment is as follows

$$\varpi_s(\mathrm{z}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\beta^s N_{i,j}\,\mathrm{B}(s+1, \alpha+\alpha j - s, z/(\beta+z)), \tag{15}$$

where, B(.,.,z) denotes incomplete beta function. Bonferroni and Lorenz curves are considerable applications of $\varpi_1(t)$, which are benefit in many fields of research. Also, one has the mean residual life (MRL) and mean waiting time (MWT) as another applications. Hence the MRL of ZL model is given by

$$M(t)=(\bar{F}(t))^{-1}\left[E(T)-\varpi_1(t)\right]-t$$
$$=\left[\bar{F}(t;\phi)\right]^{-1}\left[\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\beta N_{i,j}\mathrm{B}(2,\alpha+\alpha j-1)-\\ \sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\beta N_{i,j}\mathrm{B}(2,\alpha+\alpha j-1,(t/\beta+t))\end{array}\right]-t \tag{16}$$

where $\bar{F}(t;\phi)$ follows from Eq. (7). Also, the MWT of ZL model is given by

$$\bar{M}(t)=t-\varpi_1(t)\left\{F(t)\right\}^{-1}$$
$$=t-\left\{F(t;\phi)\right\}^{-1}\left\langle\sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\beta N_{i,j}\mathrm{B}(2,\alpha+\alpha j-1,(t/\beta+t))\right\rangle, \tag{17}$$

where, $F(t;\phi)$ follows from Eq. (5).

3.4 The probability weighted moments

Probability weighted moments (PWM) is employed to obtain estimators of parameters and quantiles of distributions. The PWM of a random variable X, for positive integers, r and h is defined as follows:

$$\omega_{r,h}=E[X^r F(x)^h]=\int_{-\infty}^{\infty}x^r f(x)(F(x))^h\,dx. \tag{18}$$

Based on (18), the PWM of ZL distribution is given by

$$\omega_{r,h}=\sum_{m=0}^{h}\binom{h}{m}(-1)^{h-m}\int_0^{\infty}\frac{2\alpha\lambda x^r}{(e^{\lambda}-1)^{h+1}\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle\left[\mathrm{e}^{\lambda(m+1)\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\rangle^2}\right]. \tag{19}$$

Utilizing the exponential and binomial expansions in previous equation, then the PWM of ZL model is given by:

$$\omega_{r,h}=\sum_{m=0}^{h}\sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}\binom{2i+1}{j}\binom{h}{m}\frac{2(-1)^{j+h-m}\lambda^{i+1}\alpha(m+1)^i}{i!(e^{\lambda}-1)^{h+1}\beta}\int_0^{\infty}x^r\left(1+\frac{x}{\beta}\right)^{-\alpha-\alpha j-1}dx$$
$$=\sum_{m=0}^{h}\sum_{i=0}^{\infty}\sum_{j=0}^{2i+1}L_{m,i,j}\beta^r\mathrm{B}(r+1,\alpha+\alpha j-r), \tag{20}$$

where, $L_{m,i,j}=\binom{2i+1}{j}\binom{h}{m}\frac{2(-1)^{j+h-m}\alpha\lambda^{i+1}(m+1)^i}{i!(e^{\lambda}-1)^{h+1}}.$

3.5 Rényi entropy

The entropy of a random variable has been utilized in different fields. It is a measure of variation of uncertainty. The Rényi entropy of a random variable X is defined by:

$$I_R(X) = (1-\vartheta)^{-1}\log\int_{-\infty}^{\infty} f(x)^{\vartheta}dx, \quad \vartheta > 0 \text{ and } \vartheta \neq 1. \tag{21}$$

To obtain the Rényi entropy of ZL, firstly, we obtain the pdf $f(x)^{\vartheta}$ as follows:

$$(f(x))^{\vartheta} = \sum_{i.j=0}^{\infty}\frac{(-1)^j}{i!}\binom{2i+\vartheta}{j}\left(\frac{2\alpha}{e^{\lambda}-1}\right)^{\vartheta}\frac{\vartheta^i\lambda^{\vartheta+i}}{\beta^{\vartheta}}\left(1+\frac{x}{\beta}\right)^{-\vartheta(\alpha+1)-\alpha j}. \tag{22}$$

Therefore, the Rényi entropy of ZL distribution is given by

$$I_R(X) = \frac{1}{1-\vartheta}\log\sum_{i,j=0}^{\infty}\frac{(-1)^j\lambda^{\vartheta+i}\vartheta^i}{i!\left(\vartheta(\alpha+1)+\alpha j-1\right)\beta^{\vartheta-1}}\binom{2i+\vartheta}{j}\left(\frac{2\alpha}{e^{\lambda}-1}\right)^{\vartheta}. \tag{23}$$

3.6 Stochastic ordering

Let X and Y are independent random variables with cdfs F_X and F_Y respectively, then according to Shaked et al. [Shaked and Shanthikumar (2007)], X is said to be smaller than Y if the following ordering holds;

Stochastic order $(X \leq_{sr} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.

Likelihood ratio order $(X \leq_{lr} Y)$ if $f_X(x)/f_Y(x)$ is decreasing in x.

Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.

Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x.

We have the following chain of implications among the various partial orderings mentioned above:

$$\begin{array}{ccc} X \leq_{lr} Y \Rightarrow X & \leq_{hr} & Y \Rightarrow X \leq_{mrl} Y \\ & \Downarrow & \\ & X \leq_{sr} Y & \end{array}$$

Theorem: Let $X \sim ZL(\alpha_1,\lambda_1,\beta_1)$ and $Y \sim ZL(\alpha_1,\lambda_1,\beta_1)$. If $\alpha_1 > \alpha_2, \lambda_1 > \lambda_2$ and $\beta_1 = \beta_2 = \beta$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$, and $X \leq_{sr} Y$.

Proof

It is sufficient to show $\frac{f_X(x)}{f_Y(x)}$ is a decreasing function of x; the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\lambda_1\left(1+x/\beta\right)^{-\alpha_1-1}\left\langle 1-\left(1+x/\beta\right)^{-\alpha_1-1}\right\rangle\left(e^{\lambda_2}-1\right)\left[\exp\lambda_1\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha_1}\right\rangle^2\right]}{\alpha_2\lambda_2\left(1+x/\beta\right)^{-\alpha_2-1}\left\langle 1-\left(1+x/\beta\right)^{-\alpha_2-1}\right\rangle\left(e^{\lambda_1}-1\right)\left[\exp\lambda_2\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha_2}\right\rangle^2\right]}, \tag{24}$$

therefore,

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{-(\alpha_1+1)}{\beta+x} + \frac{2\lambda_1\alpha_1}{\beta}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha_1}\right\rangle\left(1+\frac{x}{\beta}\right)^{-\alpha_1-1} + \frac{(\alpha_2+1)}{\beta+x}$$
$$-\frac{(\alpha_2+1)}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha_2-2}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha_2}\right\rangle^{-1} - \frac{2\lambda_2\alpha_2}{\beta}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha_2}\right]\left(1+\frac{x}{\beta}\right)^{-\alpha_2-1} \quad (25)$$
$$+\frac{(\alpha_1+1)}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha_1-2}\left\langle 1-\left(1+\frac{x}{\beta}\right)^{-\alpha_1}\right\rangle^{-1} < 0$$

Thus, $f_X(x)/f_Y(x)$ is decreasing in x and hence $X \leq_{lr} Y$. Similarly, we can conclude for $X \leq_{hr} Y$, $X \leq_{mrl} Y$, and $X \leq_{sr} Y$.

4 Parameter estimation

Here, ML estimator of ZL parameters is derived based on SRS and RSS. Further, simulation illustration is done to compare the estimator performances for both schemes.

4.1 ML estimators via SRS

Estimators for the ZL parameters depending on the ML method are derived. Let $X_1, X_2, ..., X_n$ be a SRS from ZL distribution with observed values $x_1, x_2, ..., x_n$. The log likelihood function of ZL model, denoted by $\ln\ell$, is obtained as follows

$$\ln\ell = n\ln(2\alpha) + n\ln\lambda - n\ln\beta - n\ln\left(e^{\lambda}-1\right) - (\alpha+1)\sum_{i=1}^{n}\ln\left(1+{x_i}/{\beta}\right) + \sum_{i=1}^{n}\ln\left[1-z_i\right]$$
$$+\lambda\sum_{i=1}^{n}\left(1-z_i\right)^2, \quad (26)$$

where, $z_i = \left(1+{x_i}/{\beta}\right)^{-\alpha}$. The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by

$$\frac{\partial\ln\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right) + \sum_{i=1}^{n} z_i \ln\left(1+\frac{x_i}{\beta}\right)\left[\left(1-z_i\right)^{-1} + 2\lambda\left(1-z_i\right)\right], \quad (27)$$

$$\frac{\partial\ln\ell}{\partial\lambda} = \frac{n}{\lambda} - \frac{n}{(1-e^{-\lambda})} + \sum_{i=1}^{n}\left(1-z_i\right)^2, \quad (28)$$

and,

$$\frac{\partial\ln\ell}{\partial\beta} = \frac{-n}{\beta} + (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{(\beta^2+\beta x_i)} - \frac{2\lambda\alpha}{\beta^2}\sum_{i=1}^{n} x_i\left(1-z_i\right)\left(1+\frac{x_i}{\beta}\right)^{-\alpha-1}$$
$$+\sum_{i=1}^{n}\frac{\alpha x_i}{\beta^2}\left(1-z_i\right)^{-1}\left(1+\frac{x_i}{\beta}\right)^{-\alpha-1}. \quad (29)$$

Solving non-linear equations: $\partial \ln \ell / \partial \alpha = 0, \partial \ln \ell / \partial \beta = 0$ and $\partial \ln \ell / \partial \lambda = 0,$ numerically, then we get the ML estimators of the population parameters.

4.2 ML Estimators via RSS

Suppose that $X_{(i)ic}$, $i = 1 \ldots m$, $c = 1 \ldots k$ is a RSS from ZL distribution, with sample size $n = m\,k$, m is the set size and k is the number of cycles. For simplicity, let $Y_{ic} = X_{(i)ic}$ then for fixed c, Y_{ic} are independent with pdf equal to pdf of i^{th} order statistics. The likelihood function of the sample $y_{1c}, y_{2c}, ..., y_{mc}$.

$$\begin{aligned}\ell_1 &= \prod_{c=1}^{k}\prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!}[F(y_{ic})]^{i-1} f(y_{ic})[1-F(y_{ic})]^{m-i} \\ &= \prod_{c=1}^{k}\prod_{i=1}^{m} \frac{2\alpha\lambda m!\mathrm{e}^{\lambda(1-z_{ic})^2}}{(i-1)!(m-i)!} \frac{1}{\beta(e^{\lambda}-1)^m} \left(1+\frac{y_{ic}}{\beta}\right)^{-\alpha-1} \left\langle 1-\left(1+\frac{y_{ic}}{\beta}\right)^{-\alpha}\right\rangle \\ &\quad \times \left[\mathrm{e}^{\lambda(1-z_{ic})^2}-1\right]^{i-1} \left[\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}\right]^{m-i},\end{aligned} \tag{30}$$

where, $z_{ic} = \left(1+\frac{y_{ic}}{\beta}\right)^{-\alpha}$. The log-likelihood function, denoted by $\ln \ell_1$, of ZL distribution under RSS is given by

$$\begin{aligned}\ln \ell_1 &= \ln c + mk \ln 2\alpha\lambda - mk \ln \beta - m^2\, \mathrm{k} \ln(e^{\lambda}-1) + \sum_{c=1}^{k}\sum_{i=1}^{m} \ln(1-z_{ic}) + \lambda \sum_{c=1}^{k}\sum_{i=1}^{m}(1-z_{ic})^2 \\ &\quad -(\alpha+1)\sum_{c=1}^{k}\sum_{i=1}^{m} \ln\left(1+\frac{y_{ic}}{\beta}\right) + \sum_{c=1}^{k}\sum_{i=1}^{m} (\mathrm{i}-1)\ln\left[\mathrm{e}^{\lambda(1-z_{ic})^2}-1\right] \\ &\quad + \sum_{c=1}^{k}\sum_{i=1}^{m}(m-i)\ln\left[\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}\right].\end{aligned} \tag{31}$$

The partial derivatives of $\ln \ell_1$, with respect to α, λ and β are as follows

$$\begin{aligned}\frac{\partial \ln \ell_1}{\partial \alpha} &= \frac{mk}{\alpha} + 2\lambda \sum_{c=1}^{k}\sum_{i=1}^{m} (z_{ic} - z_{ic}^2) \ln\left(1+y_{ic}/\beta\right) \\ &\quad + \sum_{c=1}^{k}\sum_{i=1}^{m} \frac{2\lambda(\mathrm{i}-1)\mathrm{e}^{\lambda(1-z_{ic})^2}(\mathrm{z}_{ic}-z_{ic}^2)\ln\left(1+y_{ic}/\beta\right)}{\left\{\mathrm{e}^{\lambda(1-z_{ic})^2}-1\right\}} \\ &\quad + \sum_{c=1}^{k}\sum_{i=1}^{m} \frac{z_{ic}}{1-z_{ic}} \ln\left(1+y_{ic}/\beta\right) - \sum_{c=1}^{k}\sum_{i=1}^{m} \ln\left(1+y_{ic}/\beta\right) \\ &\quad - \sum_{c=1}^{k}\sum_{i=1}^{m} \frac{2\lambda(m-i)\mathrm{e}^{\lambda(1-z_{ic})^2}(z_{ic}-z_{ic}^2)\ln\left(1+y_{ic}/\beta\right)}{\left\{\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}\right\}},\end{aligned} \tag{32}$$

$$\frac{\partial \ln \ell_1}{\partial \lambda}=\frac{mk}{\lambda}-\frac{m^2k}{(1-e^{-\lambda})}+\sum_{c=1}^{k}\sum_{i=1}^{m}(1-z_{ic})^2+\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{(i-1)\mathrm{e}^{\lambda(1-z_{ic})^2}(1-z_{ic})^2}{\left[\mathrm{e}^{\lambda(1-z_{ic})^2}-1\right]}$$
$$+\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{(m-i)(\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}(1-z_{ic})^2)}{\left[\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}\right]}, \tag{33}$$

and,

$$\frac{\partial \ln \ell_1}{\partial \beta}=\frac{-mk}{\beta}-\frac{\alpha}{\beta^2}\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{y_{ic}}{(1-z_{ic})}\left(1+y_{ic}/\beta\right)^{-\alpha-1}$$
$$-\frac{2\lambda\alpha}{\beta^2}\sum_{c=1}^{k}\sum_{i=1}^{m}(1-z_{ic})\left(1+y_{ic}/\beta\right)^{-\alpha-1}y_{ic}$$
$$+\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{2(i-1)\lambda\alpha(1-z_{ic})y_{ic}\left(1+y_{ic}/\beta\right)^{-\alpha-1}}{\beta^2\left\{1-\mathrm{e}^{-\lambda(1-z_{ic})^2}\right\}}+\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{(\alpha+1)y_{ic}}{\left(\beta^2+\beta y_{ic}\right)}$$
$$+\sum_{c=1}^{k}\sum_{i=1}^{m}\frac{2(m-i)\lambda\alpha(1-z_{ic})y_{ic}\left(1+y_{ic}/\beta\right)^{-\alpha-1}\mathrm{e}^{\lambda(1-z_{ic})^2}}{\beta^2\left\{\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda(1-z_{ic})^2}\right\}}. \tag{34}$$

The ML estimators of the population parameters are the solution of Eqs. (32)-(34). It is difficult to obtain a closed form solution for these equations, so numerical procedure is required to solve them.

4.3 Simulation procedure

This subsection gives the numerical study to obtain the ML estimates of the population parameters for the ZL distribution based on RSS and SRS. A comparison study between estimates is performed relative to mean square errors (MSEs), biases and relative efficiency. The simulation studies are as follows:

Step 1: A random sample of size *n*=100, 200 and 300 with set size $m=n$, number of cycles *k*=*n*, where $n^2=m\times k$ is generated from ZL distribution then ranking one observation from each cycle.

Step 2: The parameters values are selected as (α=0.5, λ=1.2, β=0.5), (α=0.5, λ=1.5, β=0.5), (α=0.5, λ=2, β=0.5) ,(α=0.3, λ=1.5, β=1) ,(α=1.5, λ=0.7, β=0.05) and (α=2, λ=1.5, β=0.03).

Step 3: For the chosen set of parameters and *n*, the ML estimate (MLE) are computed under SRS and RSS.

Step 4: Repeat the pervious steps from 1 to 3 N times representing different samples, where N=1000. Then, the biases, MSEs and relative efficiency (RE) = MSE(RSS) / MSE(SRS) of the estimates are computed.

Step 5: Numerical outcomes are given in Tabs. 1 to 6.

From Tab. 1 to Tab. 6, the following observations can be detected as follows:

- Based on SRS, biases and MSEs for the estimates are greater than the corresponding in RSS.
- For both sampling schemes, it is clear that the biases and MSEs decrease as set sizes increase in most of situations.
- The efficiency of estimates increases as the sample sizes increase for most of situations.
- The MSE of estimates based on RSS are smaller than the MSE of the corresponding based on SRS.

Table 1: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=0.5, λ=1.2, β=0.5)

n	**Parameter**	**SRS**			**RSS**			**RE**
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	0.5019	0.0019	0.0068	0.4860	-0.0140	0.0041	0.6034
	λ	1.5112	0.3112	2.9362	1.4842	0.2842	2.7172	0.9254
	β	0.5421	0.0421	0.2121	0.4563	-0.0437	0.0444	0.2091
200	α	0.5062	0.0062	0.0020	0.5023	0.0023	0.0002	0.0844
	λ	1.6302	0.4302	1.6615	1.2982	0.0982	0.4179	0.2515
	β	0.4810	-0.0190	0.0610	0.5011	0.0011	0.0140	0.2291
300	α	0.4992	-0.0008	0.0014	0.5007	0.0007	0.0002	0.1207
	λ	1.5517	0.3517	1.3828	1.2398	0.0397	0.1166	0.0843
	β	0.4742	-0.0258	0.0543	0.4942	-0.0058	0.0056	0.1027

Table 2: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=0.5, λ=1.5, β=0.5)

n	**Parameter**	**SRS**			**RSS**			**RE**
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	0.5244	0.0244	0.0035	0.4994	-0.0006	0.0004	0.1250
	λ	2.2285	0.7285	1.9216	1.8401	0.3401	1.2607	0.6561
	β	0.4486	-0.0514	0.0760	0.4683	-0.0317	0.0408	0.5372
200	α	0.5004	0.0004	0.0020	0.5004	0.0004	0.0003	0.1278
	λ	1.9368	0.4368	1.5665	1.7860	0.2860	1.0256	0.6547
	β	0.5051	0.0051	0.0722	0.4745	-0.0255	0.0312	0.4322
300	α	0.5076	0.0076	0.0015	0.5036	0.0036	0.0001	0.0724
	λ	2.1150	0.6150	1.3516	1.5806	0.0806	0.1159	0.0858
	β	0.4431	-0.0569	0.0517	0.4925	-0.0075	0.0064	0.1236

Table 3: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=0.5, λ=2, β=0.5)

n	**Parameter**	**SRS**			**RSS**			**RE**
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	0.4752	-0.0248	0.0064	0.4989	-0.0011	0.0001	0.0222
	λ	1.6461	-0.3539	2.3019	2.5619	0.5619	2.0993	0.9120
	β	0.6638	0.1638	0.1775	0.4519	-0.0481	0.0563	0.3170
200	α	0.4817	-0.0183	0.0047	0.4999	-0.0001	0.0001	0.0249
	λ	2.0574	0.0574	2.1103	2.6692	0.6692	1.5391	0.7293
	β	0.5359	0.0359	0.1021	0.3969	-0.1031	0.0405	0.3963
300	α	0.4965	-0.0035	0.0013	0.4962	-0.0038	0.0001	0.0863
	λ	2.2580	0.2580	1.4138	1.8661	-0.1339	0.1793	0.1268
	β	0.5134	0.0134	0.0834	0.5423	0.0423	0.0126	0.1513

Table 4: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=0.3, λ=1.5, β=1)

n	**Parameter**	**SRS**			**RSS**			**RE**
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	0.2918	-0.0082	0.0035	0.3021	0.0021	0.0003	0.0724
	λ	2.1406	0.6406	4.8567	2.1182	0.6182	1.5444	0.3180
	β	0.8648	-0.1352	0.5824	0.7352	-0.2648	0.2230	0.3829
200	α	0.2879	-0.0121	0.0021	0.3035	0.0035	0.0001	0.0413
	λ	2.1928	0.6928	3.8697	1.8563	0.3563	0.6226	0.1609
	β	0.7690	-0.2310	0.4988	0.8441	-0.1559	0.0860	0.1723
300	α	0.2964	-0.0036	0.0006	0.2991	-0.0009	0.0000	0.0182
	λ	1.8836	0.3836	1.3340	1.4765	-0.0235	0.0196	0.0147
	β	0.8340	-0.1660	0.1825	1.0040	0.0040	0.0117	0.0640

Table 5: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=1.5, λ=0.7, β=0.05)

n	**Parameter**	**SRS**			**RSS**			RE
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	1.6202	0.1202	0.1414	1.5270	0.0270	0.0200	0.1418
	λ	0.1222	-0.5778	1.0341	0.4799	-0.2201	0.8113	0.7845
	β	0.0726	0.0226	0.0018	0.0582	0.0082	0.0004	0.2103
200	α	1.4297	-0.0703	0.0356	1.5040	0.0040	0.0030	0.0855
	λ	0.7047	0.0047	0.9098	0.6337	-0.0663	0.1502	0.1651
	β	0.0478	-0.0022	0.0001	0.0519	0.0019	0.0001	0.6916

300	α	1.3739	-0.1261	0.0315	1.4950	-0.0050	0.0013	0.0397
	λ	0.8398	0.1398	0.3405	0.8312	0.1312	0.3246	0.9535
	β	0.0447	-0.0053	0.0002	0.0486	-0.0014	0.0001	0.5702

Table 6: ML estimates, Biases, MSE, RE of ZL distribution under SRS and RSS for (α=2, λ=1.5, β=0.03)

n	**Parameter**	**SRS**			**RSS**			**RE**
		MLE	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**	
100	α	2.4555	0.4554	0.6944	2.0083	0.0083	0.0519	0.0747
	λ	0.9345	-0.5655	1.2366	1.3596	-0.1404	1.4150	1.1443
	β	0.0532	0.0232	0.0016	0.0361	0.0061	0.0005	0.3176
200	α	2.2038	0.2038	0.2042	2.1129	0.1129	0.0423	0.2070
	λ	1.3208	-0.1792	1.2331	0.9068	-0.5932	1.0902	0.8841
	β	0.0399	0.0099	0.0006	0.0414	0.0114	0.0004	0.6684
300	α	2.0993	0.0993	0.0528	1.9608	-0.0392	0.0121	0.2286
	λ	1.2467	-0.2533	0.9212	1.5963	0.0963	0.1996	0.2167
	β	0.0377	0.0077	0.0002	0.0289	-0.0011	0.0000	0.1665

5 Applications to real data

The validity of ZL distribution depending on two data sets is provided. We provide a formative judgment of the goodness of-fit of the models and make comparison with other models. The suggested measures are Akaike information criterion (AIC), Consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A*) and Cramér- von Mises (W*). However, the better model fit the data takes the smaller values of these statistics.

5.1 Aircraft windshield data

The first data represent the failure times for a particular windshield device which was studied by Murthy et al. [Murthy, Xie and Jiang (2004)]. The fits of the ZL model for these data are compared distribution with; GL, BL, KwL, TWL, beta Weibull (BW) by Lee et al. [Lee, Famoye and Olumolade (2007)], Kumaraswamy Weibull (KwW) by Cordeiro et al. [Cordeiro, Ortega and Nadarajah (2010)], exponentiated transmuted generalized Rayleigh (ETGR) by Afify et al. [Afify, Nofal and Ebraheim (2015a)], McDonald Weibull (McW) by Cordeiro et al. [Cordeiro, Hashimoto and Ortega (2014)], and transmuted Marshall-Olkin Fréchet (TMOFr) by Afify et al. [Afify, Hamedani, Ghosh et al. (2015b)]. The MLE of these distributions and the corresponding standard error (SE) for windshield data are given in Tab. 7. The statistics AIC, CAIC, HQIC, A* and W* are listed in Tab. 8. The estimated pdf, cdf, sf and PP plots for Aircraft Windshield data of the fitted models are represented in Fig. 2.

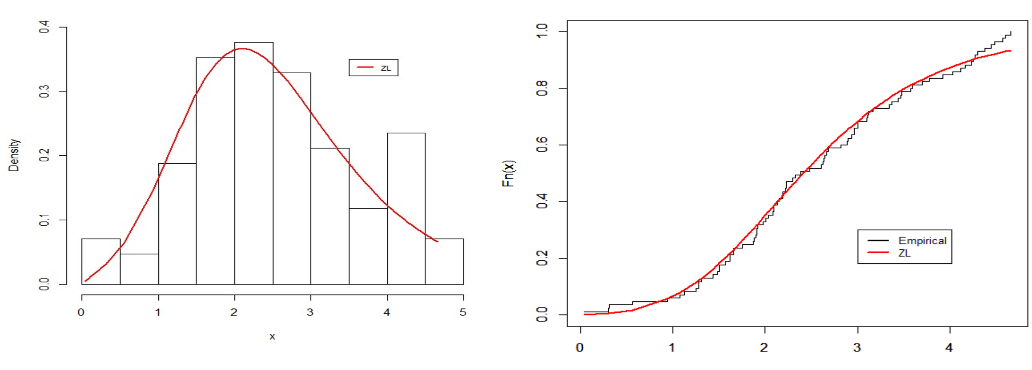
Table 7: MLE and SE for Aircraft Windshield data

Distribution	Estimates				
ZL (α, β, λ)	826.196	785.944	4.471		
	(3505)	(3343)	(0.886)		
KwL (a, b, α, β)	2.615	100.276	5.277	78.677	
	(1.343)	(404.095)	(37.988)	(799.338)	
BL (a, b, α, β)	3.6036	33.6387	4.8307	118.8374	
	(0.6187)	(63.7145)	(9.2382)	(428.9269)	
ETGR (α, β, λ, δ)	0.034	0.379	−0.354	26.430	
	(0.048)	(0.025)	(0.815)	(40.252)	
KwW (a, b, α, β)	34.660	81.846	14.433	0.204	
	(17.527)	(52.014)	(27.095)	(0.042)	
McW (a, b, α, β, c)	17.686	33.639	1.940	0.306	16.721
	(6.222)	(19.994)	(1.011)	(0.045)	(9.622)
BW (a, b, α, β)	34.180	11.496	1.360	0.298	
	(14.838)	(6.730)	(1.002)	(0.060)	
TMOFr (α, β, σ, λ)	200.747	1.952	0.102	-0.869	
	(87.275)	(0.125)	(0.017)	(0.101)	

Table 8: The AIC, CAIC, HQIC, A* and W* statistics for Aircraft Windshield data

Distribution	AIC	CAIC	BIC	HQIC	A*	W*
ZL	268.515	269.022	268.288	271.447	0.708	0.072
KwL	270.296	270.802	274.204	280.019	0.868	0.097
BL	285.435	285.935	295.206	289.365	1.408	0.168
ETGR	269.975	270.481	273.883	279.700	0.786	0.085
KwW	281.434	281.941	285.343	291.158	1.506	0.185
McW	283.899	284.669	288.785	296.053	1.591	0.199
BW	305.028	305.534	308.937	314.751	3.220	0.465
TMOFr	309.472	309.978	313.380	319.195	2.404	0.320

From Tab. 8, we conclude that the ZL takes the smallest values of statistics measures, so ZL distribution produces better fit than the other models.



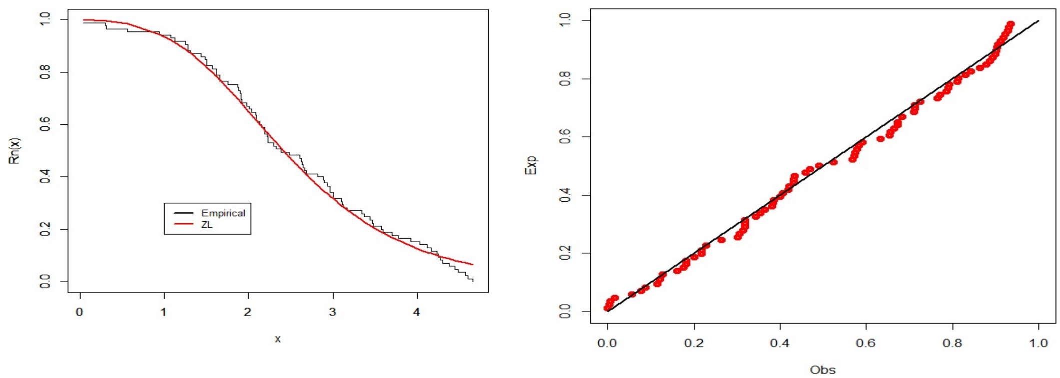


Figure 2: Plots of estimated pdf, cdf, sf and pp plots for Aircraft Windshield data

Cancer Patient Data

Lee et al. [Lee and Wang (2003)] discussed the mission times (in months) of 128 bladder cancer patients data. The fits of the ZL model for these data are compared distribution with; beta Fréchet (BFr) by Nadarajah et al. [Nadarajah and Gupta (2004)], transmuted modified Weibull (TMW) distribution by Khan et al. [Khan and King (2013)], transmuted additive Weibull (TAW) distribution by Elbatal et al. [Elbatal and Aryal (2013)], beta exponentiated Burr XII (BEBXII) by Mead [Mead (2014)], generalized inverse gamma by Mead [Mead (2015)], and ETGR. The MLE and SE are recorded in Tab. 9. On the other hand, Tab. 10 includes statistics measures of the fitted models. From Tab. 10, we conclude that the ZL takes the smallest values of statistics measures, so ZL distribution performs better fit than the other models. More information can be found in Fig. 3.

Table 9: Estimates and SEs for cancer data

Distribution	Estimates				
ZL (α, β, λ)	2.677	7.235	1.32		
	(1.005)	(7.009)	(1.855)		
BEBXII (a, b, c, β, k)	22.186	20.277	0.224	1.780	1.306
	(21.956)	(17.296)	(0.144)	(1.076)	(1.079)
GIG (a, b, c, β, k)	2.327	0.0002	17.931	0.543	0.001
	(0.369)	(0.0002)	(7.385)	(0.042)	(0.0003)
BFr (a, b, α, β)	12.526	33.342	27.753	0.169	
	(24.469)	(36.348)	(71.507)	(0.104)	
ETGR (α, β, λ, δ)	7.376	0.047	0.118	0.049	
	(5.389)	(0.004)	(0.260)	(0.036)	
TMW (a, α, β, λ)	0.0002	0.1208	0.8955	0.407	
	(0.011)	(0.024)	(0.626)	(0.407)	
TAW (a, b, α, β, λ)	0.00003	1.0065	0.1139	0.9722	−0.1630
	(0.0061)	(0.035)	(0.032)	(0.125)	(0.280)

Table 10: The AIC, CAIC, HQIC, A* and W* statistics for cancer data

Distribution	AIC	CAIC	HQIC	A*	W*
ZL	827.465	827.659	830.942	0.340	0.048
BEBXII	841.268	841.760	855.528	0.900	0.134
GIG	839.824	840.316	854.085	2.618	0.410
BFr	842.965	843.290	854.373	1.121	0.168
ETGR	866.350	866.675	877.758	2.361	0.398
TMW	836.450	836.775	847.858	0.125	0.760
TAW	838.478	838.970	852.739	0.113	0.703

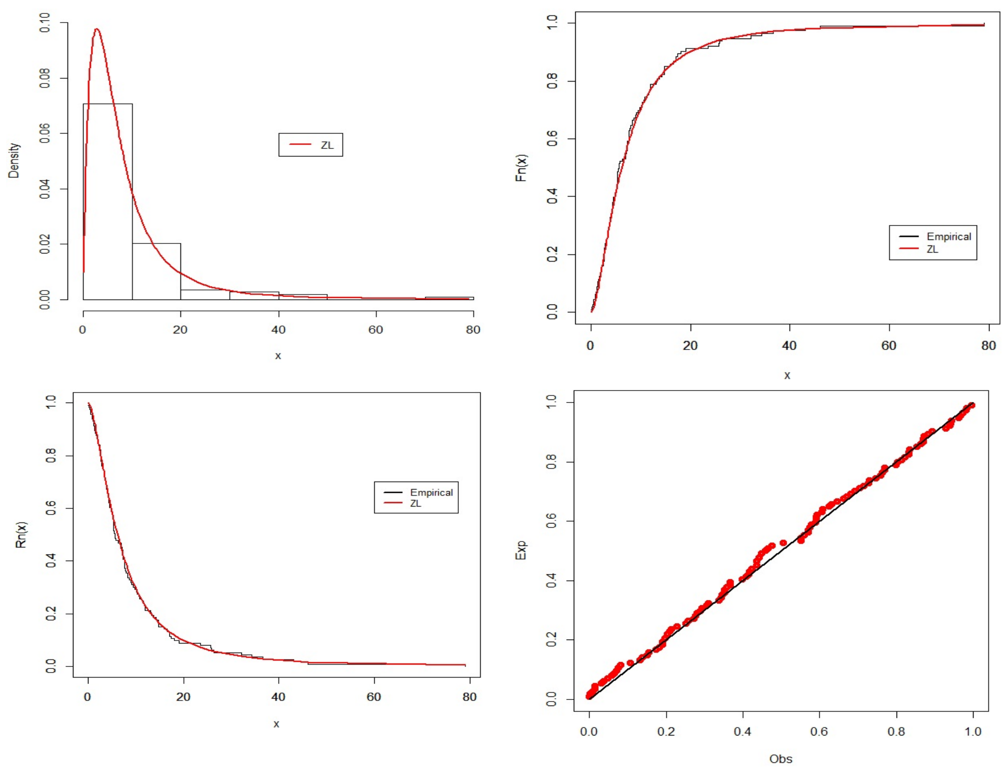


Figure 3: Plots of estimated pdf, cdf, sf and PP plots for cancer data

6 Concluding remarks

A new three-parameter lifetime model, called Zubair Lomax is proposed. Considerable properties of the Zubair Lomax like; moments, probability weighted moments, Rényi entropy, quantile function, stochastic ordering, mean residual life and mean waiting time are derived. Maximum likelihood estimators of parameters are achieved under simple random sample and ranked set sampling. Eventually, two real data sets are employed to confirm the flexibility of the proposed distribution in modeling real data applications.

Results of comparison showed that the Zubair Lomax distribution preforms better than some other distributions based on some criteria measures.

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Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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