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Article

Fractional Analysis of Thin Film Flow of Non-Newtonian Fluid

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Abstract: Modeling and analysis of thin film flow with respect to magneto hydro dynamical effect has been an important theme in the field of fluid dynamics, due to its vast industrial applications. The analysis involves studying the behavior and response of governing equations on the basis of various parameters such as thickness of the film, film surface profile, shear stress, liquid velocity, volumetric flux, vorticity, gravity, viscosity among others, along with different boundary conditions. In this article, we extend this analysis in fractional space using a homotopy based scheme, considering the case of a Non-Newtonian Pseudo-Plastic fluid for lifting and drainage on a vertical wall. After applying similarity transformations, the given problems are reduced to highly non-linear and inhomogeneous ordinary differential equations. Moreover, fractional differential equations are obtained using basic definitions of fractional calculus. The Homotopy Perturbation Method (HPM), along with fractional calculus is used for obtaining approximate solutions. Physical quantities such as the velocity profile, volume flux and average velocity respectively for lift and drainage cases have been calculated. To the best of our knowledge, the given problems have not been attempted before in fractional space. Validity and convergence of the obtained solutions are confirmed by finding residual errors. From a physical perspective, a comprehensive study of the effects of various parameters on the velocity profile is also performed. Study reveals that Stokes number S_t, non-Newtonian parameter β and magnetic parameter M have inverse relationship with fluid velocity in lifting case. In the drainage case, Stokes number S_t and non-Newtonian parameter β have direct relationship with fluid velocity, but magnetic parameter M shows inverse relationship with velocity. The investigation also shows that the fractional parameter α has direct relationship with the fluid velocity in lifting case, while it has inverse relationship with velocity in the drainage case.

Keywords: Pseudo-plastic fluid; magneto hydro dynamic; fractional differential equation; homotopy perturbation method

1 Introduction

Free drainage refers to a phenomena where a fluid flows down a vertical object such that it adheres to the objects form and is subject to gravitational and viscous forces [1]. The fluid is bounded with a free surface

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(usually air), and there is no forced addition of liquids. The thickness δ of the drained fluid is much shorter than the length of the object such that the flow takes place along a longer dimension. The flow is considered to be one-dimensional as the flow velocity is much greater than the velocity perpendicular to the belt. Drainage can be observed in many naturally occurring phenomena such as movement of rain droplets down a window pane, eye tears, and lava flows. Industrial and engineering applications are found in oil refining processes, chip manufacturing, nuclear reactors, construction and civil works, irrigation, laser cutting, paint finishing, among others [2–5]. Since flow conditions can be significantly affected by both external and internal macroscopic instabilities, a significant amount of work has been performed to understand the underlying fundamental flow characteristics. A brief of this body of work is provided in the remainder of this section.

The initial work with respect to the problem was performed on the basis of Newtonian fluids [4], where acceleration terms were omitted and the resulting process was a balance between viscous and gravitational forces. Although the approach was valid for large temporal ranges, it was not sufficient for non-linear analysis required in non-Newtonian fluids such as pastes, gels, molten plastics, lubricants containing polymer additives, blood, and food items such as honey and ketchup [6]. As such, Siddiqui et al. [7,8] have approached the drainage problem with respect to Phan-Thein-Tanner fluids; a third grade fluid, flowing down an inclined plane. Siddiqui et al. [9] have also approached the problem using fourth-grade fluid on vertical cylinder. In terms of flow types, initial studies involved analysis of laminar flows were performed by Yih [10]. The analysis was extended to turbulent flows by Landau [11] and Stuart [12]. Stability analysis taking into account surface tension was performed by Nakaya [13] and Lin [14]. Ullah et al. [15] study thin film flow of a generalized Maxwell fluid along with slip conditions, confronting withdrawal and drainage on non-isothermal cylindrical surfaces. Ruan et al. [16] examine the dynamics of a thin film formed by a distributed liquid source on a vertical solid wall. Ahmad et al. [17] proposed a modified nanofluid model for homogeneous–heterogeneous reactions in a gravity driven liquid films.

In this article, we extend the study of fluid drainage and lifting to fractional space by representing the problem as Fractional Differential Equations (FDEs), and obtaining a solution using Homotopy Perturbation Method (HPM), proposed by He et al. [18–21]. The HPM essentially combines both homotopy and classical perturbation techniques, and has been successfully applied to solve many linear [18–21] and non-linear problems [22–36]. The FDEs are generalizations of ordinary differential equations to non-integer order. The usage of FDEs allow us to model and observe more complex spatial and temporal phenomena concerning the fluid flow by taking into account non-local relations [37]. To the best of our knowledge, the analysis of thin film flow in fractional space has not yet been performed.

2 Preliminaries

2.1 Fractional Calculus

FDEs have been the focus of many studies in physics, biology, engineering, signal processing, control theory, and finance due to its ability to capture complex non-linear phenomena not commonly captured by ordinary differential equations [11–13]. Before proceeding to the remainder sections describing the lifting and drainage model in fractional space, we present a few basic definitions and properties of fractional calculus that will be used in later sections.

***Definition 2.1.1*:** A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$ such that $h(t) = t^p h_1(t), \, where \, h_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if

$$h^n \in C_\mu, n \in N. \tag{1}$$

Definition 2.1.2: The fractional derivative D^α of $h(t)$ in the Caputo sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^n(\tau) d\tau, \tag{2}$$

For $n-1 < \alpha < n, n \in N, t > 0, h \in C_{-1}^n$.

The following are two basic properties of the Caputo fractional derivative [30]:

(1) Let $h \in C_{-1}^n, n \in N$. Then $D^\alpha h, 0 \le \alpha \le n$ is well defined and $D^\alpha h \in C_{-1}$.

(2) Let $n-1 \le \alpha \le n, n \in N$ and $h \in C_\mu^n, \mu \ge -1$. Then

$$J^\alpha D^\alpha h(t) = h(t) - \sum_{k=0}^{n-1} h^k(0^+) \frac{t^k}{k!}$$

2.2 Basic Theory of Homotopy Perturbation Method (HPM)

To illustrate the basic idea of the HPM for solving nonlinear differential equations, consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \boldsymbol{\Omega} \tag{3}$$

With boundary conditions

$$B\left(u, \frac{du}{dx}\right) = 0, r\varepsilon\boldsymbol{\Phi} \tag{4}$$

where A is the general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and $\boldsymbol{\Phi}$ is the boundary over the domain $\boldsymbol{\Omega}$. Generally A can be divided into a linear part L and a nonlinear part N. Eq. (3) can be written as

$$L(u) + N(u) - f(r) = 0, r \in \Omega \tag{5}$$

By the homotopy technique, we construct a homotopy $V(r, p)$: $\boldsymbol{\Omega} \times [0, 1] \to \Re$, which satisfies:

$$H(V, p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega \tag{6}$$

or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0 \tag{7}$$

where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (3), which satisfies the boundary conditions. From Eqs. (5) and (6) we have

$$H(V, 0) = L(V) - L(u_0)$$

$$H(V, 1) = A(V) - f(r) \tag{8}$$

Thus, the changing process of p from zero to unity is just that of $V(r, p)$ from $u_0(r)$ to $u(r)$.

According to HPM, we initially use the embedding parameter p as a small parameter and assume that the solution of (3) and (4) can be written as a power series in p.

$$V = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ \ldots \tag{9}$$

Setting $p = 1$ results in the approximate solution of (3)

$$u = \lim_{p \to 1} V = v_0 + v_1 + v_2 \ldots \tag{10}$$

2.3 Governing Equations

The basic equations governing the motion of an isothermal, homogeneous and incompressible fluid as [9] are given as:

$$div\mathbf{V} = 0, \tag{11}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{f} - gradP + div\mathbf{S} \tag{12}$$

where $\mathbf{V}$ is the velocity vector, ρ is the constant density and $\mathbf{f}$ is the body force per unit mass, P denotes the dynamic pressure and $\frac{D}{Dt}$ denotes the material derivative which is defined as

$$\frac{D(*)}{Dt} = \frac{\partial}{\partial t}(*) - (\mathbf{V}.\nabla)(*)$$

Here $\mathbf{S}$ is the extra stress tensor for pseudo-plastic fluid model [24], defined as

$$\mathbf{S} + \lambda_1 \mathbf{S}^{\nabla} + \frac{1}{2}(\lambda_1 - \mu_1)(\mathbf{A_1 S} + \mathbf{S A_1}) = \eta_0 \mathbf{A_1}, \tag{13}$$

where η_0 is the zero shear viscosity, λ_1 is the relaxation time and μ_1 is the material constant.

The first Rivlin–Ericksen tensor $\mathbf{A_1}$ is defined as

$$\mathbf{A_1} = (grad\mathbf{V}) + (grad\mathbf{V})^T \tag{14}$$

The contra variant convicted derivative denoted by super imposed ∇ over $\mathbf{S}$ is defined as

$$\mathbf{S}^{\nabla} = \frac{D\mathbf{S}}{Dt} - \left[(grad\mathbf{V})^T \mathbf{S} + \mathbf{S}(grad\mathbf{V})\right] \tag{15}$$

3 Formulation of the Electrically Conducted Paint Film Flow in Lifting Case

We consider a container filled with an incompressible non-Newtonian Pseudo-Plastic fluid. A wide belt is passing through this container which moves vertically upward with a constant speed U_0. The belt take up a thin film of fluid having uniform thickness δ. The gravity tries to make the fluid drain down the belt. It is also assume that the flow is laminar and uniform, and the pressure is atmospheric pressure. Let x-axis be normal to the belt and the z-axis is along the belt which is in upward direction.

The boundary conditions for the problem are:

(i) There is no-slip at the wall

$$\text{at } x = 0, \quad w = U_0 \tag{16}$$

(ii) At the free surface, the shear is negligible, thus

$$\text{at } x = \delta, \quad S_{xz} = 0 \tag{17}$$

where S_{xz} is the shear stress component of the pseudo-plastic fluid.

The velocity field with magnetic effect for the stated problem defines as

$$\mathbf{V} = \left(0,\ 0,\ -\sigma B_0^2 w(x) \right) \tag{18}$$

and extra stress tensor as

$$\mathbf{S} = \mathbf{S}(x) \tag{19}$$

By inserting (18) in (11) and (12), the continuity Eq. (11) is identically satisfied and Eq. (12) takes the following form

$$0 = \frac{dS_{xx}}{dx} + \rho f_1 \tag{20}$$

$$0 = \frac{dS_{zx}}{dx} + \rho f_3 - \sigma B_0^2 w(x) \tag{21}$$

where f_1 and f_3 are components of body force in x and z directions, respectively.

Since the z-axis is in upward direction and gravity acts in the negative z-direction (downward), this means that $g_z = -g$, hence the above equations become

$$0 = \frac{dS_{xx}}{dx} \tag{22}$$

$$0 = \frac{dS_{zx}}{dx} - \rho g - \sigma B_0^2 w(x) \tag{23}$$

The non-zero components of **S** are obtained by inserting Eqs. (14), (15) and (18) in Eq. (13)

$$S_{xx} = \frac{-\eta_0(\lambda_1 - \mu_1)\left(\frac{dw}{dx}\right)^2}{1 + \left({\lambda_1}^2 - {\mu_1}^2\right)\left(\frac{dw}{dx}\right)^2}, S_{zz} = \frac{\eta_0(\lambda_1 + \mu_1)\left(\frac{dw}{dx}\right)^2}{1 + \left({\lambda_1}^2 - {\mu_1}^2\right)\left(\frac{dw}{dx}\right)^2}, S_{zx} = \frac{\eta_0 \frac{dw}{dx}}{1 + \left({\lambda_1}^2 - {\mu_1}^2\right)\left(\frac{dw}{dx}\right)^2} \tag{24}$$

Substituting the value of S_{zx} in Eq. (23), we get

$$\frac{d}{dx}\left[\frac{\eta_0 \frac{\mathrm{dw}}{\mathrm{dx}}}{1 + \left({\lambda_1}^2 - {\mu_1}^2\right)\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2}\right] = \rho g + \sigma B_0^2 w(x) \tag{25}$$

Applying the non-dimensional parameters $w^* = \dfrac{w}{U_0}$ and $x^* = \dfrac{x}{\delta}$ in Eq. (25), we get

$$\frac{d}{dx}\left[\frac{\frac{\mathrm{dw}}{\mathrm{dx}}}{1 + \beta\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2}\right] = S_t + M^2 w(x) \tag{26}$$

with

$$\frac{\mathrm{dw}}{\mathrm{dx}} = 0 \text{ at } x = 1 \,,$$

$w = 1$ at $x = 0$,

where $\beta = \frac{(\lambda_1{}^2 - \mu_1{}^2)U_0{}^2}{\delta^2}$ denotes the non-Newtonian parameter, $S_t = \frac{\rho g \delta^2}{\mu_{eff} U_0}$ represents the stokes number and $M^2 = \frac{\sigma B_0^2 \delta}{\mu_{eff}}$ is the MHD parameter.

Eq. (24) can be written as a second order differential equation as follows

$$\frac{d^2w}{dx^2} - \beta\left(\frac{dw}{dx}\right)^2 \frac{d^2w}{dx^2} - \beta^2 S_t \left(\frac{dw}{dx}\right)^4 - 2\beta S_t \left(\frac{dw}{dx}\right)^2 - M^2\beta^2 w(x)\left(\frac{dw}{dx}\right)^4 - 2\beta M^2 w(x)\left(\frac{dw}{dx}\right)^2 - M^2 w(x) = S_t, \tag{27}$$

with the wall and free surface boundary conditions, respectively;

$w = 1$ at $x = 0$,

$\frac{\mathrm{dw}}{\mathrm{dx}} = 0$ at $x = 1$.

After using basic definitions of fractional calculus discussed in the previous section, the fractional form of Eq. (27) is:

$$\frac{d^2w(x)}{dx^2} - \beta(D^\alpha w(x))^2 \frac{d^2w(x)}{dx^2} - \beta^2 S_t (D^\alpha w(x))^4 - 2\beta S_t (D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 - 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) = S_t, \tag{28}$$

with the following boundary conditions:

$$w(0) = 1, w'(1) = 0,\ 0 < \alpha < 1, t > 0. \tag{29}$$

4 Application of HPM to Fractional Differential Equation in Lifting Case

According to HPM, we can construct the following homotopy $\Omega \times [0, 1] \rightarrow R$ for Eq. (28)

$$(1-p)\frac{d^2w(x)}{dx^2} + p[\frac{d^2w(x)}{dx^2} - \beta(D^\alpha w(x))^2 \frac{d^2w}{dx^2} - \beta^2 S_t (D^\alpha w(x))^4 - 2\beta S_t (D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 - 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) - S_t] = 0 \tag{30}$$

Using (28) and (29) various order problems are as follows:

4.1 Zeroth-Order Problem

$$w_0{}''(x) = 0, w_0(0) = 1, w_0'(1) = 0 \tag{31}$$

4.2 First-Order Problem

$$-S_t - 2S_t\beta(D^\alpha w_0(x))^2 - S_t\beta^2(D^\alpha w_0(x))^4 - M^2 w_0(x) - 2M^2\beta(D^\alpha w_0(x))^2 w_0(x) - M^2\beta^2(D^\alpha w_0(x))^4 w_0(x) - \beta(D^\alpha w_0(x))^2 w_0{}''(x) + w_1{}''(x) = 0, w_1(0) = 0, w_1{}'(1) = 0 \tag{32}$$

4.3 Second-Order Problem

$$\begin{aligned}&-4S_t\beta(D^\alpha w_0(x))(D^\alpha w_1(x))-4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))-4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_0(x)-\\&4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_0(x)-M^2w_1(x)-2M^2\beta(D^\alpha w_0(x))^2w_1(x)-M^2\beta^2(D^\alpha w_0(x))^4w_1(x)-\\&2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_0{}''(x)-\beta(D^\alpha w_0(x))^2w_1{}''(x)+w_2{}''(x)=0,w_2(0)=0,w_2{}'(1)=0\end{aligned}\tag{33}$$

4.4 Third-Order Problem

$$\begin{aligned}&-2S_t\beta(D^\alpha w_1(x))^2-6S_t\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2-4S_t\beta(D^\alpha w_0(x))(D^\alpha w_2(x))-\\&4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))-2M^2\beta(D^\alpha w_1(x))^2w_0(x)-6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2w_0(x)-\\&4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_0(x)-4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))w_0(x)-\\&4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_1(x)-4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_1(x)-M^2w_2(x)-\\&2M^2\beta(D^\alpha w_0(x))^2w_2(x)-M^2\beta^2(D^\alpha w_0(x))^4w_2(x)-\beta(D^\alpha w_1(x))^2w_0{}''(x)-\\&2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_0{}''(x)-2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_1{}''(x)-\beta(D^\alpha w_0(x))^2w_2{}''(x)+w_3{}''(x)=0,\\&w_3(0)=0,w_3{}'(1)=0\end{aligned}\tag{34}$$

4.5 Fourth-Order Problem

$$\begin{aligned}&-4S_t\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3-4S_t\beta(D^\alpha w_1(x))(D^\alpha w_2(x))-12S_t\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x))-\\&4S_t\beta(D^\alpha w_0(x))(D^\alpha w_3(x))-4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x))-4M^2\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3w_0(x)-\\&4M^2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))w_0(x)-12M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x))w_0(x)-\\&4M^2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))w_0(x)-4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x))w_0(x)-2M^2\beta(D^\alpha w_1(x))^2w_1(x)-\\&6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2w_1(x)-4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_1(x)-\\&4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))w_1(x)-4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_2(x)-\\&4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_2(x)-M^2w_3(x)-2M^2\beta(D^\alpha w_0(x))^2w_3(x)-\\&M^2\beta^2(D^\alpha w_0(x))^4w_3(x)-2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))w_0{}''(x)-2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))w_0{}''(x)-\\&\beta(D^\alpha w_1(x))^2w_1{}''(x)-2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_1{}''(x)-2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_2{}''(x)-\\&\beta(D^\alpha w_0(x))^2w_3{}''(x)+w_4{}''(x)=0,w_4(0)=0,w_4{}'(1)=0\end{aligned}\tag{35}$$

Using Caputo definition while keeping $\alpha = 0.99$, $\beta = 0.1$, $S_t = 0.1$ and $M = 0.\,1$ fixed, the approximate solution is

$$\begin{aligned}&W(x)=1-0.0000775435x+0.000124893x^{2.04}-2.56792\times10^{-6}x^3-0.0000908513x^{3.04}+\\&0.0000312582x^{4.04}+2.48889\times10^{-8}x^5-4.88682\times10^{-6}x^{5.04}+8.09076\times10^{-7}x^{6.04}-\\&1.77778\times10^{-9}x^7+2.22222\times10^{-10}x^8+1/2(-0.28x+0.14x^2)+\\&1/24(0.0448x-0.0224x^3+0.0056x^4)+(0.000105828(-3.63975x^{2.96}+4.99946x^4+0.117591x^{4.96}-\\&3.35361x^5+0.830102x^6-0.0176386x^{6.96}+0.00293977x^{7.96}))/x^{1.96}\end{aligned}\tag{36}$$

The residual of the problem is

$$R = \frac{d^2W(x)}{dx^2} - \beta(D^\alpha W(x))^2\frac{d^2W(x)}{dx^2} - \beta^2 S_t(D^\alpha W(x))^4 - 2\beta S_t(D^\alpha W(x))^2 - M^2\beta^2 w(x)(D^\alpha W(x))^4 - 2\beta M^2 W(x)(D^\alpha W(x))^2 - M^2 W(x) - S_t \tag{37}$$

5 Flow Rate and Average Velocity in Lifting Case

Flow rate per unit width is given by

$$Q = \int_0^1 W(x)dx, \tag{38}$$

$$\begin{aligned} Q = -945(-4+\alpha)\Big(&-18S^3(-3+\alpha)^2(-576+1172\alpha-925\alpha^2+343\alpha^3-60\alpha^4+4\alpha^5)\Big) \\ &+ M^8(-25344+69048\alpha-79018\alpha^2+48835\alpha^3-17582\alpha^4+3693\alpha^5-420\alpha^6 \\ &+ 20\alpha^7) + M^2S^2\Big(-42(-3+\alpha)^2(-576+1172\alpha-925\alpha^2+343\alpha^3-60\alpha^4+4\alpha^5) \\ &+ S(-103680+282048\alpha-321330\alpha^2+197121\alpha^3-70234\alpha^4+14555\alpha^5 \\ &- 1628\alpha^6+76\alpha^7)) + M^6\Big(-12(-3+\alpha)^2(-576+1172\alpha-925\alpha^2+343\alpha^3-60\alpha^4 \\ &+ 4\alpha^5) + S(-126720+341952\alpha-387650\alpha^2+237197\alpha^3-84458\alpha^4+17519\alpha^5 \\ &- 1964\alpha^6+92\alpha^7)) + M^4S\Big(-36(-3+\alpha)^2(-576+1172\alpha-925\alpha^2+343\alpha^3 \\ &- 60\alpha^4+4\alpha^5) + S(-205056+554952\alpha-629962\alpha^2+385483\alpha^3-137110\alpha^4 \\ &+ 28381\alpha^5-3172\alpha^6+148\alpha^7)))\beta + 4(62M^8-945(-3+S)+189M^2(-5+2S) \\ &- 9M^4(-42+17S)+M^6(-153+62S))(-315+286\alpha-84\alpha^2 \\ &+ 8\alpha^3)\Gamma[5-\alpha]^2\Big)/\Big(11340(-315+286\alpha-84\alpha^2+8\alpha^3)\Gamma[5-\alpha]^2 \end{aligned}$$

The average velocity $\bar{V}$ for lifting problem is given by

$$\bar{V} = \mathrm{Q}. \tag{39}$$

6 Formulation of the Electrically Conducted Paint Film Flow in Drainage Case

Now we consider the fluid falling on the stationary infinite stationary belt. Due to gravity the flow is in the downward direction, which means that $g_z = g$, so the Eq. (23) become:

$$0 = \frac{dS_{zx}}{dx} + \rho g - \sigma B_0^2 w(x) \tag{40}$$

From Eq. (24) substituting the value of S_{zx} in Eq. (40), we get

$$\frac{d}{dx}\left[\frac{\eta_0\frac{\mathrm{dw}}{\mathrm{dx}}}{1+\left(\lambda_1{}^2-\mu_1{}^2\right)\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2}\right] = -\rho g + \sigma B_0^2 w(x) \tag{41}$$

After applying the non-dimensional parameters $w^* = \dfrac{w}{U_0}$ and $x^* = \dfrac{x}{\delta}$ we get the following:

$$\frac{d}{dx}\left[\frac{\frac{\mathrm{dw}}{\mathrm{dx}}}{1+\beta\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2}\right] = -S_t + M^2 w(x) \tag{42}$$

with

$\dfrac{\mathrm{dw}}{\mathrm{dx}} = 0$ at $x = 1$,

$w = 0$ at $x = 0$,

where $\beta = \dfrac{\left(\lambda_1{}^2 - \mu_1{}^2\right)U_0{}^2}{\delta^2}$ denotes the non-Newtonian parameter, $S_t = \dfrac{\rho g \delta^2}{\mu_{eff} U_0}$ represents the stokes number and $M^2 = \dfrac{\sigma B_0^2 \delta}{\mu_{eff}}$ is the MHD parameter.

Eq. (42) can be written as a second order differential equation as follows

$$\frac{d^2w}{dx^2} - \beta\left(\frac{dw}{dx}\right)^2 \frac{d^2w}{dx^2} + S_t\left(\frac{dw}{dx}\right)^4 + 2\beta S_t\left(\frac{dw}{dx}\right)^2 - M^2\beta^2 w(x)\left(\frac{dw}{dx}\right)^4 - 2\beta M^2 w(x)\left(\frac{dw}{dx}\right)^2 - M^2 w(x) = -S_t, \tag{43}$$

with the wall and free surface boundary conditions, respectively;

$w = 0$ at $x = 0$,

$\dfrac{\mathrm{dw}}{\mathrm{dx}} = 0$ at $x = 1$.

After using basic definitions of fractional calculus discussed in previous section, Eq. (43) change to the following fractional differential equation:

$$\frac{d^2w(x)}{dx^2} - \beta(D^\alpha w(x))^2 \frac{d^2w(x)}{dx^2} + \beta^2 S_t (D^\alpha w(x))^4 + 2\beta S_t (D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 - 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) = -S_t \tag{44}$$

with the following boundary conditions:

$$w(0) = 0, w'(1) = 0, \quad 0 < \alpha < 1 \tag{45}$$

7 Application of HPM to Fractional Differential Equation in Drainage Case

We construct the following homotopy $\Omega \times [0, 1] \rightarrow R$ for Eq. (44)

$$(1-p)\frac{d^2w}{dx^2} + p\left[\frac{d^2w}{dx^2} - \beta(D^\alpha w(x))^2 \frac{d^2w}{dx^2} + \beta^2 S_t (D^\alpha w(x))^4 + 2\beta S_t (D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 - 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) + S_t\right] = 0. \tag{46}$$

Using (44) and (45) various order problems are as follows:

7.1 Zeroth-Order Problem

$$w_0{''}(\mathrm{x}) = 0, w_0(0) = 0, \;\; w_0'(1) = 0. \tag{47}$$

7.2 First-Order Problem

$$\begin{aligned} &S_t + 2S_t\beta(D^\alpha w_0(\mathrm{x}))^2 + S_t\beta^2(D^\alpha w_0(\mathrm{x}))^4 - M^2 w_0(\mathrm{x}) - 2M^2\beta(D^\alpha w_0(\mathrm{x}))^2 w_0(\mathrm{x}) - \\ &M^2\beta^2(D^\alpha w_0(\mathrm{x}))^4 w_0(\mathrm{x}) - \beta(D^\alpha w_0(\mathrm{x}))^2 w_0{''}(\mathrm{x}) + w_1{''}(\mathrm{x}) = 0,\; w_1(0) = 0, w_1{'}(1) = 0. \end{aligned} \tag{48}$$

7.3 Second-Order Problem

$$\begin{aligned} &4S_t\beta(D^\alpha w_0(x))(D^\alpha w_1(x)) + 4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_0(x) - \\ &4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_0(x) - M^2 w_1(x) - 2M^2\beta(D^\alpha w_0(x))^2 w_1(x) - M^2\beta^2(D^\alpha w_0(x))^4 w_1(x) - \\ &2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_0''(x) - \beta(D^\alpha w_0(x))^2 w_1''(x) + w_2''(x) = 0, w_2(0) = 0, w_2'(1) = 0. \end{aligned} \tag{49}$$

7.4 Third-Order Problem

$$\begin{aligned} &2S_t\beta(D^\alpha w_1(x))^2 + 6S_t\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2 + 4S_t\beta(D^\alpha w_0(x))(D^\alpha w_2(x)) + \\ &4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x)) - 2M^2\beta(D^\alpha w_1(x))^2 w_0(x) - 6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2 w_0(x) - \\ &4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_0(x) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))w_0(x) - \\ &4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_1(x) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_1(x) - M^2 w_2(x) - \\ &2M^2\beta(D^\alpha w_0(x))^2 w_2(x) - M^2\beta^2(D^\alpha w_0(x))^4 w_2(x) - \beta(D^\alpha w_1(x))^2 w_0''(x) - \\ &2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_0''(x) - 2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_1''(x) - \beta(D^\alpha w_0(x))^2 w_2''(x) + w_3''(x) = 0, \\ &w_3(0) = 0, w_3{'}(1) = 0 \end{aligned} \tag{50}$$

7.5 Fourth-Order Problem

$$\begin{aligned} &4S_t\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3 + 4S_t\beta(D^\alpha w_1(x))(D^\alpha w_2(x)) + 12S_t\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x)) + \\ &4S_t\beta(D^\alpha w_0(x))(D^\alpha w_3(x)) + 4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x)) - 4M^2\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3 w_0(x) - \\ &4M^2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))w_0(x) - 12M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x))w_0(x) - \\ &4M^2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))w_0(x) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x))w_0(x) - 2M^2\beta(D^\alpha w_1(x))^2 w_1(x) - \\ &6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2 w_1(x) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_1(x) - \\ &4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))w_1(x) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_2(x) - \\ &4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))w_2(x) - M^2 w_3(x) - 2M^2\beta(D^\alpha w_0(x))^2 w_3(x) - \\ &M^2\beta^2(D^\alpha w_0(x))^4 w_3(x) - 2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))w_0''(x) - 2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))w_0''(x) - \\ &\beta(D^\alpha w_1(x))^2 w_1''(x) - 2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))w_1''(x) - 2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))w_2''(x) - \\ &\beta(D^\alpha w_0(x))^2 w_3''(x) + w_4''(x) = 0, w_4(0) = 0, w_4'(1) = 0. \end{aligned} \tag{51}$$

Using Caputo definition while keeping $\alpha = 0.99$, $\beta = 0.1$, $S_t = 0.1$ and $M = 0.\,1$ fixed, the approximate solution in drainage case is

$$
\begin{aligned}
W(x) &= 0.000029643x - 0.0000484352x^{2.02} + 1.67006\times10^{-6}x^{3} + 0.0000329444x^{3.02} - \\
&8.20867\times10^{-7}x^{4.01} - 8.7445\times10^{-6}x^{4.02} - 2.77778\times10^{-10}x^{5} + 1.63846\times10^{-7}x^{5.01} + \\
&3.28098\times10^{-7}x^{5.02} - 5.45014\times10^{-8}x^{6.02} + 1.98413\times10^{-11}x^{7} - 2.48016\times10^{-12}x^{8} + \\
&1/2(0.2x - 0.1x^{2}) + 1/24(-0.008x + 0.004x^{3} - 0.001x^{4}) - (0.0000228589(-43.8358x^{1.99} + \\
&43.344x^{3} + 0.0243037x^{3.99} - 14.4x^{4} - 0.00364556x^{5.99} + 0.000607593x^{6.99}))/x^{0.99}
\end{aligned} \tag{52}
$$

The residual of the problem is

$$
\begin{aligned}
R &= \frac{d^{2}W(x)}{dx^{2}} - \beta(D^{\alpha}W(x))^{2}\frac{d^{2}W(x)}{dx^{2}} + \beta^{2}S_{t}(D^{\alpha}W(x))^{4} + \\
&2\beta S_{t}(D^{\alpha}W(x))^{2} - M^{2}\beta^{2}W(x)(D^{\alpha}W(x))^{4} - 2\beta M^{2}W(x)(D^{\alpha}W(x))^{2} - M^{2}W(x) + S_{t}
\end{aligned} \tag{53}
$$

8 Flow Rate and Average Velocity in Drainage Case

Flow rate per unit width is given by

$$
Q = \int_{0}^{1} w(x)dx \tag{54}
$$

$$
\begin{aligned}
Q &= S(11340S(-70560 + 153524\alpha - 135635\alpha^{2} + 63636\alpha^{3} - 17227\alpha^{4} + 2702\alpha^{5} - 228\alpha^{6} + 8\alpha^{7}) \\
&\beta\Gamma[7-\alpha]^{2} + (-5+\alpha)\Gamma[5-\alpha](1890S^{2}(28 - 11\alpha + \alpha^{2})(30 - 11\alpha + \alpha^{2})^{2}(-3(-3+\alpha)^{2} \\
&(-576 + 1172\alpha - 925\alpha^{2} + 343\alpha^{3} - 60\alpha^{4} + 4\alpha^{5}) + M^{2}(-39168 + 106500\alpha - 121156\alpha^{2} + \\
&74143\alpha^{3} - 26326\alpha^{4} + 5431\alpha^{5} - 604\alpha^{6} + 28\alpha^{7}))\beta - 945M^{2}S(2056320 - 4851948\alpha + 4762928\alpha^{2} - \\
&2561589\alpha^{3} + 830250\alpha^{4} - 166755\alpha^{5} + 20334\alpha^{6} - 1380\alpha^{7} + 40\alpha^{8})\beta\Gamma[7-\alpha] - 4(-945 + 378M^{2} - \\
&153M^{4} + 62M^{6})(2205 - 2317\alpha + 874\alpha^{2} - 140\alpha^{3} + 8\alpha^{4})\Gamma[7-\alpha]^{2})))/(11340(-7+\alpha)(-5+\alpha) \\
&(-9+2\alpha)(35 - 24\alpha + 4\alpha^{2})\Gamma[5-\alpha]\Gamma[7-\alpha]^{2}
\end{aligned} \tag{55}
$$

The average velocity $\bar{V}$ for drainage problem is given by

$\bar{V} = Q.$

9 Result and Discussion

In this article homotopy based fractional analysis of thin film flow of pseudo-plastic fluid for lifting and drainage on a vertical wall has been performed. Different parameters like fractional parameter α, non Newtonian parameter β, Stokes number S_t and MHD parameter M are involved in the fractional differential equations. We present our discussion of results based on these parameters and their different compositions. The problems have been solved for various values of fluid parameters and results are presented in Tabs. 1–4 in lifting case while Tabs. 5–8 in drainage case. Tabs. 1 and 5 show the solutions along with residual errors for various values of fractional parameter α, keeping other parameters fixed in lifting and drainage cases respectively. Tabs. 2 and 6 show the solutions along with residual errors for various values of non¯Newtonian parameter β, keeping other parameters fixed in lifting and drainage cases respectively. Similarly, Tabs. 3 and 7 indicate solutions along with residual errors for various values of Stoke number S_t, keeping other parameters fixed in lifting and drainage cases. Likewise, Tabs. 4 and 8 represent solutions along with residual errors for various values of magnetic parameter M keeping other parameters fixed in lifting and drainage cases respectively. Residual errors in each table indicate that the

Table 1: Solution along with residual error for various α keeping $\beta = 0.1,\ S_t = 0.01,\ M = 0.1$ fixed in lifting case

x	$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 0.99$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	0.998107	-4.07394×10^{-8}	0.998107	-2.32826×10^{-7}	0.998107	-4.60032×10^{-7}
0.2	0.996413	-8.12241×10^{-8}	0.996413	-2.52818×10^{-7}	0.996413	-3.67883×10^{-7}
0.3	0.994919	-1.13542×10^{-7}	0.994919	-2.39928×10^{-7}	0.994919	-2.83792×10^{-7}
0.4	0.993625	-1.36232×10^{-7}	0.993625	-2.11985×10^{-7}	0.993625	-2.09925×10^{-7}
0.5	0.992529	-1.49245×10^{-7}	0.992529	-1.76944×10^{-7}	0.992529	-1.4692×10^{-7}
0.6	0.991634	-1.5319×10^{-7}	0.991634	-1.39493×10^{-7}	0.991634	-9.50203×10^{-8}
0.7	0.990937	-1.49052×10^{-7}	0.990937	-1.02817×10^{-7}	0.990937	-5.43252×10^{-8}
0.8	0.990439	-1.38069×10^{-7}	0.990439	-6.92838×10^{-8}	0.990439	-2.48839×10^{-8}
0.9	0.990141	-1.21661×10^{-7}	0.990141	-4.07647×10^{-8}	0.990141	-6.73152×10^{-9}
1.	0.990041	-1.01389×10^{-7}	0.990041	-1.88132×10^{-8}	0.990041	9.12055×10^{-11}

Table 2: Solution along with residual error for various β keeping $\alpha = 0.98,\ S_t = 0.01,\ M = 0.1$ fixed in lifting case

x	$\beta = 0.1$		$\beta = 0.7$		$\beta = 0.9$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	0.998107	-4.45173×10^{-7}	0.998106	-3.12238×10^{-6}	0.998106	-4.01713×10^{-6}
0.2	0.996413	-3.61947×10^{-7}	0.996412	-2.53771×10^{-6}	0.996412	-3.26452×10^{-6}
0.3	0.994919	-2.8246×10^{-7}	0.994918	-1.97972×10^{-6}	0.994918	-2.54642×10^{-6}
0.4	0.993625	-2.11108×10^{-7}	0.993624	-1.47915×10^{-6}	0.993623	-1.90238×10^{-6}
0.5	0.992529	-1.49353×10^{-7}	0.992529	-1.04617×10^{-6}	0.992528	-1.34539×10^{-6}
0.6	0.991634	-9.78623×10^{-7}	0.991633	-6.85328×10^{-7}	0.991632	-8.81274×10^{-7}
0.7	0.990937	-5.69938×10^{-8}	0.990936	-3.9904×10^{-7}	0.990936	-5.13101×10^{-7}
0.8	0.990439	-2.69702×10^{-8}	0.990438	-1.88786×10^{-7}	0.990438	-2.42738×10^{-7}
0.9	0.990141	-7.95239×10^{-9}	0.99014	-5.56335×10^{-8}	0.99014	-7.15301×10^{-8}
1.	0.990041	-7.66525×10^{-11}	0.99004	-4.96159×10^{-10}	0.99004	-6.36304×10^{-10}

obtained solutions are valid and consistent. Furthermore, effects of various parameters on the velocity profiles have been investigated graphically. Figs. 1–4 show the effect of various parameter on the velocity profile in lifting case. Fig. 1 indicates the effect of α on the fluid velocity. It has been observed that increase in α increases the fluid velocity. Figs. 2–4 show the effect of β, S_t and M on the velocity profile respectively. It is seen that in all three cases velocity profile decreases with the increase in values of parameters. Similarly, Figs. 5–10 show the effect of various fluid parameters and their compositions on the velocity profiles in drainage case. Fig. 5 presents the effect of α on the velocity profile. It has been observed that fluid velocity decreases with an increase in α. Figs. 6 and 7 present the effect of β and S_t on the velocity profile. In both the cases it is seen that fluid velocity increases with an increase in fluid

Table 3: Solution along with residual error for various S_t keeping $\alpha = 0.99,\ \beta = 0.1,\ M = 0.1$ fixed in lifting case

x	$S_t = 0.001$		$S_t = 0.1$		$S_t = 0.2$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	0.998959	-1.01087×10^{-7}	0.989576	-1.80123×10^{-5}	0.980043	-8.54622×10^{-5}
0.2	0.998027	-8.08669×10^{-8}	0.980253	-1.41422×10^{-5}	0.962206	-6.35375×10^{-5}
0.3	0.997205	-6.24053×10^{-8}	0.972031	-1.07202×10^{-5}	0.946481	-4.56017×10^{-5}
0.4	0.996494	-4.61809×10^{-8}	0.964908	-7.79753×10^{-6}	0.932865	-3.14378×10^{-5}
0.5	0.995891	-3.23373×10^{-8}	0.958883	-5.36609×10^{-6}	0.921351	-2.05334×10^{-5}
0.6	0.995399	-2.093×10^{-8}	0.953955	-3.40574×10^{-6}	0.911936	-1.23685×10^{-5}
0.7	0.995015	-1.19832×10^{-8}	0.950123	-1.89559×10^{-6}	0.904617	-6.48547×10^{-6}
0.8	0.994742	$-5.50929^{*} \times 10^{-9}$	0.947386	-8.17842×10^{-7}	0.899391	-2.51417×10^{-6}
0.9	0.994578	-1.51721×10^{-9}	0.945744	-1.59423×10^{-7}	0.896256	-1.83059×10^{-7}
1.	0.994523	-1.66439×10^{-11}	0.945197	8.70732×10^{-8}	0.895212	6.74562×10^{-7}

Table 4: Solution along with residual error for various M keeping $\alpha = 0.95,\ \beta = 0.2,\ S_t = 0.01$ fixed in lifting case

x	$M = 0.1$		$M = 0.2$		$M = 0.4$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	0.998959	-1.76926×10^{-7}	0.996158	-8.57303×10^{-6}	0.985464	-3.83351×10^{-4}
0.2	0.998027	-1.51174×10^{-7}	0.992724	-7.22586×10^{-6}	0.972526	-3.0003×10^{-4}
0.3	0.997205	-1.22117×10^{-7}	0.989698	-5.76626×10^{-6}	0.961164	-2.23198×10^{-4}
0.4	0.996494	-9.41009×10^{-8}	0.987078	-4.39662×10^{-6}	0.951358	-1.59816×10^{-4}
0.5	0.995891	-6.87105×10^{-8}	0.984862	-3.18221×10^{-6}	0.943091	-1.09947×10^{-4}
0.6	0.995399	-4.67421×10^{-8}	0.983051	-2.15019×10^{-6}	0.936348	-7.20354×10^{-5}
0.7	0.995015	-2.86698×10^{-8}	0.981643	-1.31335×10^{-6}	0.931115	-4.42125×10^{-5}
0.8	0.994742	-1.48104×10^{-8}	0.980638	-6.78498×10^{-7}	0.927385	-2.47737×10^{-5}
0.9	0.994578	-5.39594×10^{-9}	0.980035	-2.50285×10^{-7}	0.92515	-1.23902×10^{-5}
1.	0.994523	-6.10912×10^{-10}	0.979834	-3.31801×10^{-8}	0.924405	-6.22141×10^{-6}

parameters. Fig. 8 shows the effect of M on the velocity profile. It has been observed that velocity profile decreases with an increase in M. Fig. 9 shows the effect of increasing S_t and M simultaneously. It has been observed that velocity profiles increases with an increase in be S_t and M. Comparison of the effects of S_t and M shows that S_t is more dominant parameter as compared to M in this case. Fig. 10 indicates the effect of increasing β and M simultineousely. It is seen that velocity profile decreases with an increase in β and M. Analysis of the effects of β and M shows that M is more dominant parameter as compared to β. Furthermore, the effect of increasing values of S_t, β and M simultaneously on the velocity profile while keeping α fixed in lifting and drainage case has been presented in Figs. 11 and 12 respectively.

Table 5: Solution along with residual error for various α keeping $\beta = 0.1,\ S_t = 0.01,\ M = 0.1$ fixed in drainage case

x	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.99$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	9.47543×10^{-4}	-5.25718×10^{-6}	9.47608×10^{-4}	-8.29812×10^{-6}	9.47604×10^{-4}	-1.75534×10^{-5}
0.2	1.79513×10^{-3}	-7.42904×10^{-6}	1.79523×10^{-3}	-1.0098×10^{-5}	1.79512×10^{-3}	-1.57478×10^{-5}
0.3	2.54282×10^{-3}	-8.79233×10^{-6}	2.54293×10^{-3}	-1.0882×10^{-5}	2.54266×10^{-3}	-1.38731×10^{-5}
0.4	3.19067×10^{-3}	-9.64093×10^{-6}	3.19077×10^{-3}	-1.1098×10^{-5}	3.19031×10^{-3}	-1.19651×10^{-5}
0.5	3.73875×10^{-3}	-1.01013×10^{-5}	3.73882×10^{-3}	-1.09218×10^{-5}	3.73816×10^{-3}	-1.00358×10^{-5}
0.6	4.1871×10^{-3}	-1.02438×10^{-5}	4.18714×10^{-3}	-1.04452×10^{-5}	4.18629×10^{-3}	-8.09066×10^{-6}
0.7	4.53576×10^{-3}	-1.01135×10^{-5}	4.53576×10^{-3}	-9.7244×10^{-6}	4.53474×10^{-3}	-6.13279×10^{-6}
0.8	4.78478×10^{-3}	-9.74146×10^{-6}	4.78474×10^{-3}	-8.79663×10^{-6}	7.8359×10^{-3}	-4.16411×10^{-6}
0.9	4.93417×10^{-3}	-9.15068×10^{-6}	4.93411×10^{-3}	-7.68844×10^{-6}	4.93288×10^{-3}	-2.18589×10^{-6}
1.	4.98397×10^{-3}	-8.35861×10^{-6}	4.9839×10^{-3}	-6.41959×10^{-6}	4.98263×10^{-3}	-1.99057×10^{-7}

Table 6: Solution along with residual error for various β keeping $\alpha = 0.99,\ S_t = 0.01,\ M = 0.3$ fixed in drainage case

x	$\beta = 0.1$		$\beta = 0.3$		$\beta = 0.5$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	9.22069×10^{-4}	-1.75523×10^{-5}	9.23843×10^{-4}	-5.26607×10^{-5}	9.25616×10^{-3}	-8.77648×10^{-5}
0.2	1.74478×10^{-3}	-1.57411×10^{-5}	1.74798×10^{-3}	-4.7235×10^{-5}	1.75117×10^{-3}	-7.87259×10^{-5}
0.3	2.4689×10^{-3}	-1.38608×10^{-5}	2.4732×10^{-3}	-4.16014×10^{-5}	2.4775×10^{-3}	-6.934×10^{-5}
0.4	3.09509×10^{-3}	-1.19476×10^{-5}	3.10022×10^{-3}	-3.58684×10^{-5}	3.10535×10^{-3}	-5.97879×10^{-5}
0.5	3.62395×10^{-3}	-1.00135×10^{-5}	3.62966×10^{-3}	-3.00719×10^{-5}	3.63538×10^{-3}	-5.01296×10^{-5}
0.6	4.05595×10^{-3}	-8.06428×10^{-6}	4.06206×10^{-3}	-2.4229×10^{-5}	4.06818×10^{-3}	-4.03934×10^{-5}
0.7	4.39152×10^{-3}	-6.10308×10^{-6}	4.39787×10^{-3}	-1.83493×10^{-5}	4.40422×10^{-3}	-3.05953×10^{-5}
0.8	4.63097×10^{-3}	-4.13194×10^{-6}	4.63744×10^{-3}	-1.24386×10^{-5}	4.64391×10^{-3}	-2.07453×10^{-5}
0.9	4.77453×10^{-3}	-2.15222×10^{-6}	4.78105×10^{-3}	-6.50116×10^{-6}	4.78757×10^{-3}	-1.08501×10^{-5}
1.	4.82237×10^{-3}	-1.64864×10^{-7}	4.82889×10^{-3}	-5.39668×10^{-7}	4.83541×10^{-3}	-9.1447×10^{-7}

Analysis shows that increasing these parameters simultaneously has opposite effect in lifting and drainage cases. Beside the above mentioned findings, different physical quantities such as volume flux and average velocities have been calculated in lifting and drainage cases.

Table 7: Solution along with residual error for various S_t keeping $\alpha = 0.98$, $\beta = 0.1$, $M = 0.1$ fixed in drainage case

x	$S_t = 0.001$		$S_t = 0.01$		$S_t = 0.1$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	9.46787×10^{-5}	-1.74065×10^{-7}	9.47607×10^{-4}	-1.727×10^{-5}	9.56054×10^{-3}	-1.58533×10^{-3}
0.2	1.79365×10^{-4}	-1.57324×10^{-7}	1.79513×10^{-3}	-1.5621×10^{-5}	1.81039×10^{-2}	-1.44697×10^{-3}
0.3	2.54068×10^{-4}	-1.39275×10^{-7}	2.54268×10^{-3}	-1.38403×10^{-5}	2.56327×10^{-2}	-1.29428×10^{-3}
0.4	3.18794×10^{-4}	-1.20634×10^{-7}	3.19034×10^{-3}	-1.19981×10^{-5}	3.21497×10^{-2}	-1.13284×10^{-3}
0.5	3.73551×10^{-4}	-1.01639×10^{-7}	3.7382×10^{-3}	-1.01176×10^{-5}	3.76574×10^{-2}	-9.64475×10^{-4}
0.6	4.18345×10^{-4}	-8.24003×10^{-8}	4.18633×10^{-3}	-8.20969×10^{-6}	4.21585×10^{-2}	-7.90064×10^{-4}
0.7	4.53179×10^{-4}	-6.29806×10^{-8}	4.53479×10^{-3}	-6.2804×10^{-6}	4.56554×10^{-2}	-6.101×10^{-4}
0.8	4.78058×10^{-4}	-4.34186×10^{-8}	4.78365×10^{-3}	-4.33352×10^{-6}	4.81505×10^{-2}	-4.24902×10^{-4}
0.9	4.92984×10^{-4}	-2.37402×10^{-8}	4.93293×10^{-3}	-2.3716×10^{-6}	4.96459×10^{-2}	-2.34695×10^{-4}
1.	4.97959×10^{-4}	-3.96393×10^{-9}	4.98269×10^{-3}	-3.96428×10^{-7}	5.01439×10^{-2}	-3.96463×10^{-5}

Table 8: Solution along with residual error for various M keeping $\alpha = 0.99$, $\beta = 0.1$, $S_t = 0.01$ fixed in drainage case

x	$M = 0.1$		$M = 0.3$		$M = 0.5$	
	Solution	Residual Error	Solution	Residual Error	Solution	Residual Error
0.1	9.47604×10^{-4}	-1.75534×10^{-5}	9.22069×10^{-4}	-1.75523×10^{-5}	8.75377×10^{-4}	-1.73369×10^{-5}
0.2	1.79512×10^{-3}	-1.57478×10^{-5}	1.74478×10^{-3}	-1.57411×10^{-5}	1.65275×10^{-3}	-1.53105×10^{-5}
0.3	2.54266×10^{-3}	-1.38731×10^{-5}	2.4689×10^{-3}	-1.38608×10^{-5}	2.33408×10^{-3}	-1.32259×10^{-5}
0.4	3.19031×10^{-3}	-1.19651×10^{-5}	3.09509×10^{-3}	-1.19476×10^{-5}	2.92108×10^{-3}	-1.11242×10^{-5}
0.5	3.73816×10^{-3}	-1.00358×10^{-5}	3.62395×10^{-3}	-1.00135×10^{-5}	3.41527×10^{-3}	-9.02234×10^{-6}
0.6	4.18629×10^{-3}	-8.09066×10^{-6}	4.05595×10^{-3}	-8.06428×10^{-6}	3.81788×10^{-3}	-6.93001×10^{-6}
0.7	4.53474×10^{-3}	-6.13279×10^{-6}	4.39152×10^{-3}	-6.10308×10^{-6}	4.12994×10^{-3}	-4.85386×10^{-6}
0.8	4.78359×10^{-3}	-4.16411×10^{-6}	4.63097×10^{-3}	-4.13194×10^{-6}	4.35226×10^{-3}	-2.79863×10^{-6}
0.9	4.93288×10^{-3}	-2.18589×10^{-6}	4.77453×10^{-3}	-2.15222×10^{-6}	4.48542×10^{-3}	-7.67659×10^{-7}
1.	4.98263×10^{-3}	-1.99057×10^{-7}	4.82237×10^{-3}	-1.64864×10^{-7}	4.52976×10^{-3}	1.23692×10^{-6}

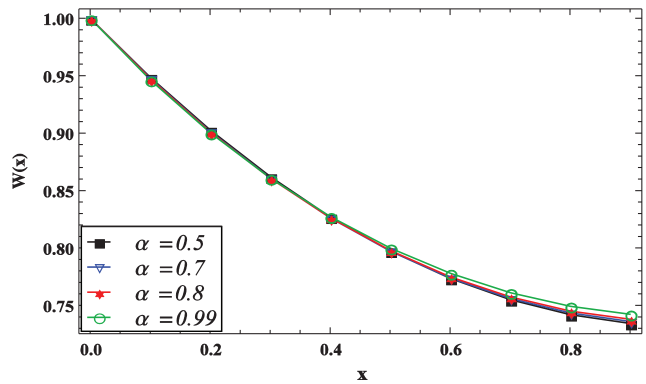

Figure 1: Effect of α on the velocity profile keeping $\beta = 1$, $S_t = 0.4$, $M = 0.3$ fixed in lifting case

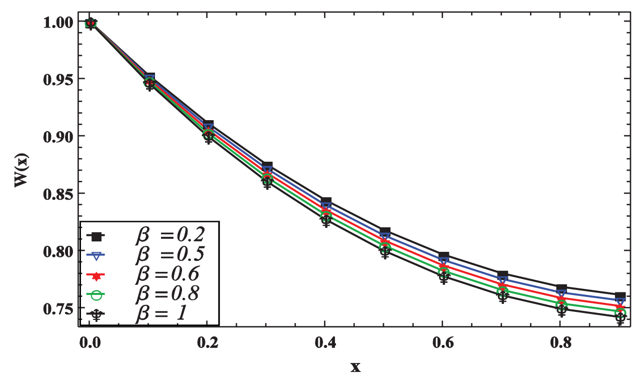

Figure 2: Effect of β on the velocity profile keeping $\alpha = 0.98$, $S_t = 0.4$, $M = 0.3$ fixed in lifting case

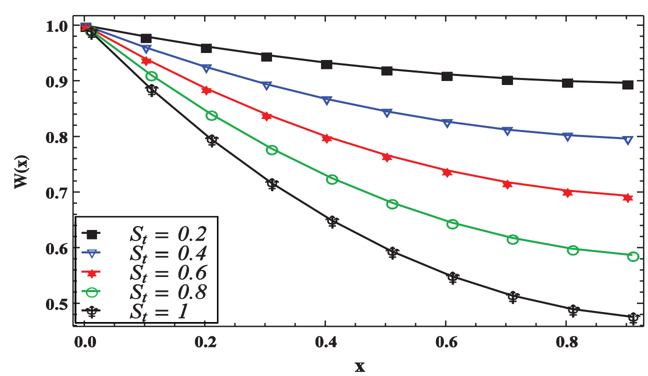

Figure 3: Effect of S_t on the velocity profile keeping $\alpha = 0.99$, $\beta = 0.1$, $M = 0.1$ fixed in lifting case

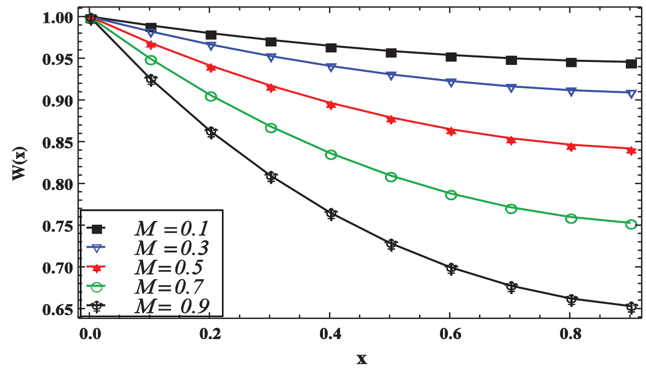

Figure 4: The effect of M on the velocity profile keeping $\alpha = 0.95$, $\beta = 0.2$, $S_t = 0.1$ fixed in lifting case

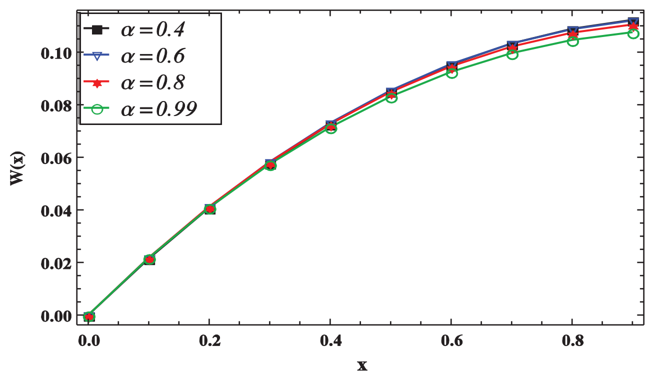

Figure 5: Effect of α on the velocity profile keeping $\beta = 0.9$, $S_t = 0.2$, $M = 0.3$ fixed in drainage case

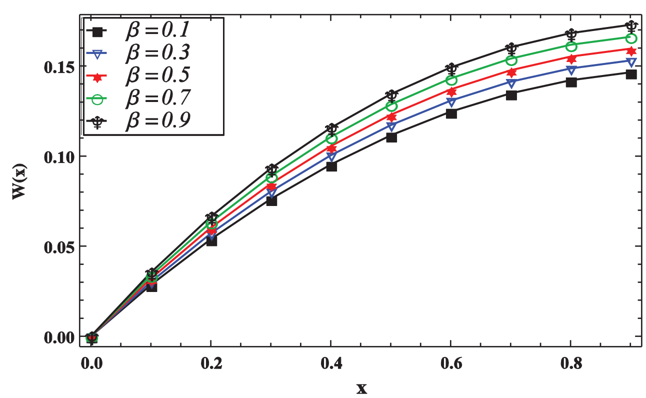

Figure 6: Effect of β on the velocity profile keeping $\alpha = 0.95$, $S_t = 0.3$, $M = 0.3$ fixed in drainage case

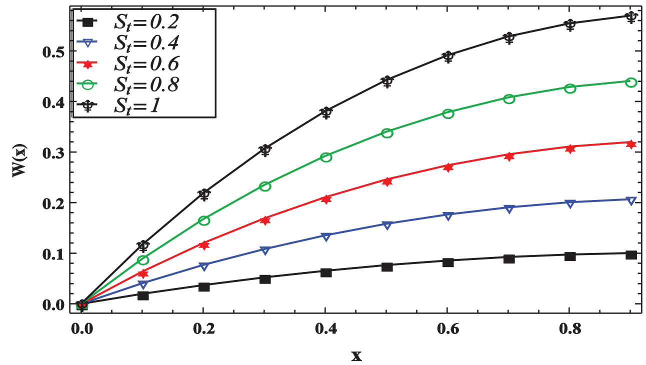

Figure 7: The effect of S_t on the velocity profile keeping $\alpha = 0.98,\ \beta = 0.2,\ M = 0.2$ fixed in drainage case

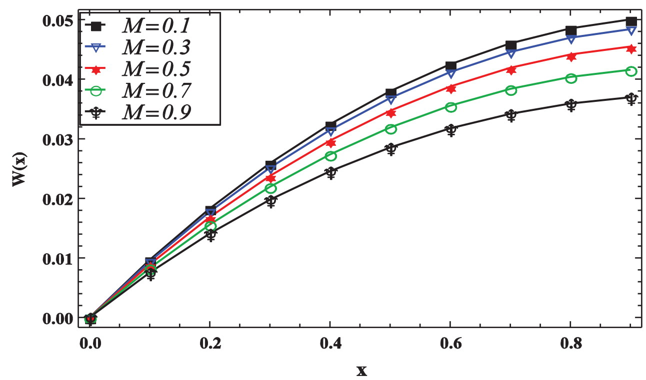

Figure 8: The effect of M on the velocity profile keeping $\alpha = 0.95,\ \beta = 0.2,\ S_t = 0.1$ fixed in drainage case

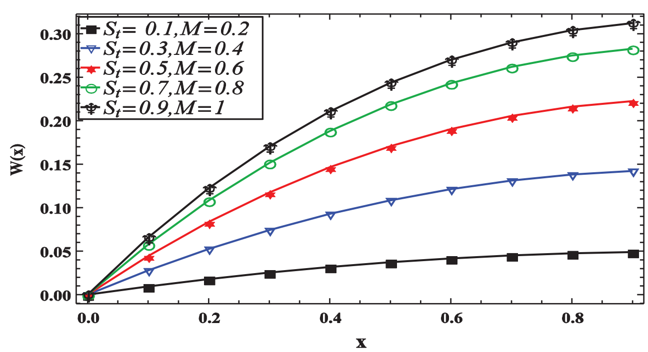

Figure 9: The effect of increasing S_t and M simultaneously on the velocity profile while keeping $\alpha = 0.98,\ \beta = 0.1$ fixed in drainage case

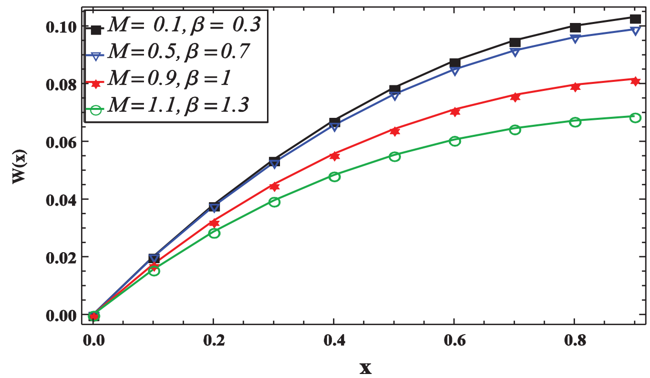

Figure 10: The effect of increasing M and β simultaneously on the velocity profile while keeping $\alpha = 0.95,\ S = 0.2$ fixed in drainage case

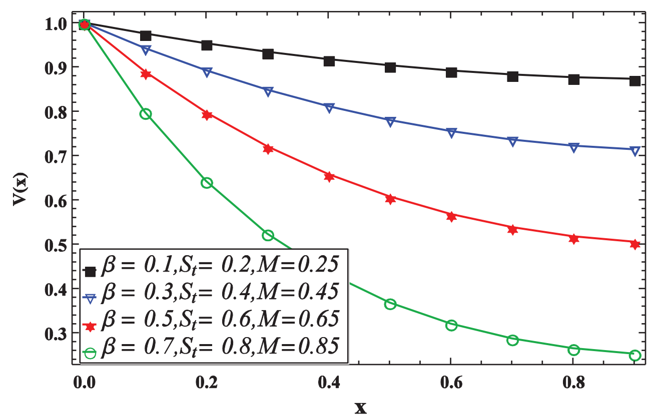

Figure 11: The effect of increasing S_t, β and M simultaneously on the velocity profile while keeping $\alpha = 0.99$ fixed in lifting case

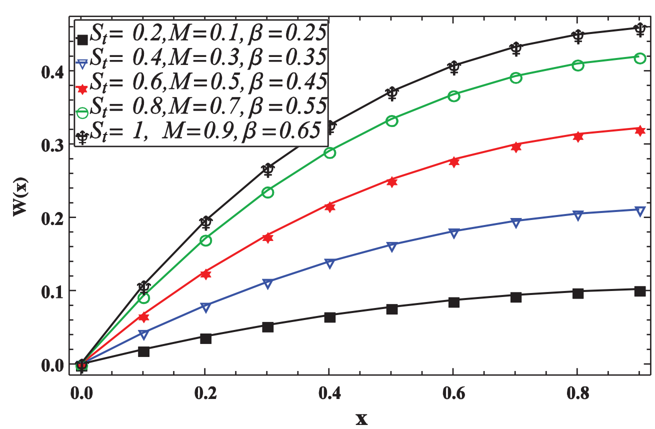

Figure 12: The effect of increasing S_t, β and M simultaneously on the velocity profile while keeping $\alpha = 0.98$ fixed in drainage case

10 Conclusions

In this article, we find homotopy based solutions of thin film flow of pseudo-plastic fluid in fractional space for lifting and drainage cases. Validity and convergence of obtained approximate solutions have been confirmed by finding residual errors. Some key findings related to effect of various parameters on the velocity profile in fractional environment has been observed. It is found that fractional parameter α has direct relationship with velocity profile in lifting case while it has inverse relationship with fluid velocity in drainage case. Consequently it is concluded that α is showing opposite behavior on the velocity profile in lifting and drainage scenarios. It has also been observed that magnetic parameter M has shown similar effect on the velocity profile in both lifting and drainage cases. Investigation also reveals that non-Newtonian parameter β is showing opposite effects on the velocity profiles in lifting and drainage cases. Furthermore, stokes number S_t is showing opposite behavior in lifting and drainage cases. We also try different compositions of fluid parameters in drainage case and found that S_t is more influenced parameter as compared to M(See Fig. 9), while M is more influenced than β (See Fig. 10).

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