Polynomials of Degree-Based Indices for Three-Dimensional Mesh Network

Ali N. A. Koam[1] and Ali Ahmad[2, *]

Abstract: In order to study the behavior and interconnection of network devices, graphs structures are used to formulate the properties in terms of mathematical models. Mesh network (meshnet) is a LAN topology in which devices are connected either directly or through some intermediate devices. These terminating and intermediate devices are considered as vertices of graph whereas wired or wireless connections among these devices are shown as edges of graph. Topological indices are used to reflect structural property of graphs in form of one real number. This structural invariant has revolutionized the field of chemistry to identify molecular descriptors of chemical compounds. These indices are extensively used for establishing relationships between the structure of nanotubes and their physico-chemical properties. In this paper a representation of sodium chloride (NaCl) is studied, because structure of NaCl is same as the Cartesian product of three paths of length exactly like a mesh network. In this way the general formula obtained in this paper can be used in chemistry as well as for any degree-based topological polynomials of three-dimensional mesh networks.

Keywords: Topological polynomials, degree-based index, three-dimensional mesh network, chemical compounds.

1 Introduction

During the exploratoery research of graph products some new structures and their associated problems are evolved or sometimes their optimum solutions are also obtained. That's the reason invariance and inheritance of graph parameters and their factors are of great interest. Among several graph products the Cartesian product given in Imrich et al. [Imrich and Klavžar, (2000)] refers the fact that many of the classical graph parameters are inherited additively. In this nomenclature many important classes of graphs are evolved due to Cartesian product like n-dimensional grid (Cartesian product of lower dimensional grids), Hypercubes and its recursive structures (m-dimensional and n-dimensional hypercubesas $(m+n)$-dimensional. The Cartesian product of graphs $\mathcal{G}$ and

[1] Department of Mathematics, College of Science, Jazan University, Jazan, 45142, Saudi Arabia.

[2] College of Computer Science and Information Technology, Jazan University, Jazan, 45142, Saudi Arabia.

[*] Corresponding Author: Ali Ahmad. Email: ahmad@gmail.com.

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$\mathcal{H}$ is the graph $\mathcal{G} \square \mathcal{H}$ with vertices $V(\mathcal{G} \square \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$, and $(\alpha, \gamma)(\beta, \zeta)$ is an edge, if $\alpha = \beta$ and $\gamma\zeta \in E(\mathcal{H})$ or $\alpha\beta \in E(\mathcal{G})$ and $\gamma = \zeta$.

Graph theory supports combinatorial structures that are used in the area of computer science, biological science and modern chemistry that helps in determining different solutions of computer network problems and to characterize the chemical compound with respect to their properties. Topological indices are used to reflect structural properties of graphs in form of one real number. In structural chemistry they are used to explain the chemical, physical and biological properties of the chemical compounds like melting and boiling point, temperature, pressure, heat formation and density [Brückler, Došlic′, Graovac et al. (2011); Gonzalez-Diaz, Vilar, Santana et al. (2007); Gutman (1994)]; Matamala and Estrada (2005); Rücker and Rücker (1999)]. In 1947 an American chemist named Wienere, introduced distance-based topological indices (as path number) for the paraffin's boiling point in his article [Wiener (1947)]. Gutman et al. [Gutman and Trinajsti (1972)] characterized the degree-based indices to determine the π-electrons energy of chemical molecules. Recently, Wasson introduced the idea of linker competition with a Metal-Organic Frameworks (MOFs) for topological insights in Wasson et al. [Wasson, Lyu, Islamoglu et al. (2019)] and lately Hong et al. [Hong, Gu, Javaid et al. (2020)] determined the degree-based topological indices of Metal-Organic Networks.

Moreover, topological indices have shown a significant role in the studies of quantitative structure activity or property relationships (QSAR/QSPR) to relate the different structures with a biological property or activity. This relation can be expressed mathematically as $\mathcal{P} = \phi(\mathcal{M})$, where $\mathcal{P}$ is a property or activity (value of a biological or a chemical measurement) and $\mathcal{M}$ is a chemical structure. Topological indices for hexagonal network are in limelight for almost 15 years. Initial studies included [Diudea, Stefu, Pârv et al. (2004)], since then hundreds of papers have been published on topological indices for different networks. Topological indices are often studied with the help of their polynomials. The first Zagreb polynomial and the second Zagreb polynomial for hexagonal nanotubes were explained in [Farahani (2012)]. The first and second Zagreb polynomial along with forgotten polynomial of generalized prisms and toroidal polyhex networks were computed in Ajmal et al. [Ajmal, Nazeer, Munir et al. (2017)]. Zagreb polynomials of nanostars were computed in Li et al. [Li, Liu, Farahani et al. (2016); Siddiqui, Imran and Ahmad (2016)]. The harmonic polynomial of polycyclic aromatic hydrocarbons was studied in Farahani et al. [Farahani, Gao, Kanna et al. (2016)]. Polynomials of various networks are studied in Saheli et al. [Saheli, Loghman and Diudea (2016); Seiberta and Zahrádka (2013)] and hexagonal nanotubes were investigated in Bača et al. [Bača, Horváthová, Mokrišová et al. (2015, 2019); Vetrík (2018)]. The topological indices of different networks such as hexagonal oxide, icosahedral honeycomb, fullerene, octahedral, carbon nanotubes, benzene ring and benzenoid are studied in Akhter et al. [Akhter and Imran (2016); Javaid, Liu, Rehman et al. (2017); Ahmad (2017, 2018)].

2 Degree-based indices and their polynomials

Let $\mathcal{G}$ be a graph with the vertex set $V(\mathcal{G})$ and the edge set $E(\mathcal{G})$. The degree d_υ of a vertex $\upsilon \in V(\mathcal{G})$ is the number of neighbours of υ. The most general indices based on degrees are the general Randic$'$ index of a graph $\mathcal{G}$,

$$R_\alpha(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} (d_\upsilon d_\nu)^\alpha \quad (1)$$

the general sum-connectivity index

$$\chi_\alpha(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} (d_\upsilon + d_\nu)^\alpha \quad (2)$$

and the generalized Zagreb index

$$GZ_{\alpha,\beta}(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} d_\upsilon^\alpha d_\nu^\beta + d_\nu^\alpha d_\upsilon^\beta \quad (3)$$

Note that the third redefined Zagreb index is defined as

$$ReZ(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} d_\upsilon d_\nu (d_\upsilon + d_\nu) \quad (4)$$

the harmonic index is defined as

$$H(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} \frac{2}{d_\upsilon + d_\nu} \quad (5)$$

the third Zagreb index

$$M_3(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} (d_\upsilon - d_\nu) \quad (6)$$

the fourth Zagreb index

$$M_4(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} d_\upsilon (d_\upsilon + d_\nu) \quad (7)$$

and the fifth Zagreb index

$$M_5(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} d_\nu (d_\upsilon + d_\nu) \quad (8)$$

Let us introduce a general invariant for polynomials of above mentioned topological indices.

$$P(\mathcal{G}, x) = \sum_{u\upsilon\in E(\mathcal{G})} x^{\varphi(d_\upsilon, d_\nu)} \quad (9)$$

where $\varphi(d_\upsilon, d_\nu)$ is a function of d_υ and d_ν such that $\varphi(d_\upsilon, d_\nu) = \varphi(d_\nu, d_\upsilon)$.

- If $\varphi(d_\upsilon, d_\nu) = (d_\upsilon d_\nu)^\alpha$, where α is a positive integer, then $P(\mathcal{G}, x)$ is the general Randic$'$ polynomial of $\mathcal{G}$. Moreover, $P(\mathcal{G}, x)$ is the second Zagreb polynomial if $\alpha = 1$.
- If $\varphi(d_\upsilon, d_\nu) = (d_\upsilon + d_\nu)^\alpha$, where α is a positive integer, then $P(\mathcal{G}, x)$ is the general sum-connectivity polynomial of G. Furthermore, $P(\mathcal{G}, x)$ is the first Zagreb polynomial for $\alpha = 1$ and the hyper-Zagreb polynomial for $\alpha = 2$.
- If $\varphi(d_\upsilon, d_\nu) = d_\upsilon^\alpha d_\nu^\beta + d_\nu^\alpha d_\upsilon^\beta$, where α is a positive integer and β is a non-negative integer, then $P(\mathcal{G}, x)$ is the generalized Zagreb polynomial of $\mathcal{G}$. Moreover, $P(\mathcal{G}, x)$ is the forgotten polynomial if $\alpha = 2$ and $\beta = 0$.
- If $\varphi(d_\upsilon, d_\nu) = d_\upsilon d_\nu (d_\upsilon + d_\nu)$, then $P(\mathcal{G}, x)$ is the third redefined Zagreb polynomial of $\mathcal{G}$.
- If $\varphi(d_\upsilon, d_\nu) = d_\upsilon + d_\nu - 1$, then $P(\mathcal{G}, x)$ is one half of the harmonic polynomial $H(\mathcal{G}, x)$ of $\mathcal{G}$. Note that the harmonic polynomial is defined differently from the other polynomials.
- If $\varphi(d_\upsilon, d_\nu) = |d_\upsilon - d_\nu|$, then $P(\mathcal{G}, x)$ is the third Zagreb polynomial of $\mathcal{G}$.

• If $\varphi(d_\upsilon, d_\nu) = d_\upsilon(d_\upsilon + d_\nu)$, then $P(\mathcal{G}, x)$ is the fourth Zagreb polynomial of $\mathcal{G}$.
• If $\varphi(d_\upsilon, d_\nu) = d_\nu(d_\upsilon + d_\nu)$, then $P(\mathcal{G}, x)$ is the fifth Zagreb polynomial of $\mathcal{G}$.
So the general Randic$'$ polynomial of any graph $\mathcal{G}$ is defined as

$$R_\alpha(G, x) = \sum_{\upsilon\nu\in E(G)} x^{(d_\upsilon d_\nu)^\alpha} \quad (10)$$

the general sum-connectivity polynomial is

$$\chi_\alpha(\mathcal{G}, x) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{(d_\upsilon + d_\nu)^\alpha} \quad (11)$$

the generalized Zagreb polynomial of any graph $\mathcal{G}$,

$$GZ_{\alpha,\beta}(\mathcal{G}, x) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{d_\upsilon^\alpha d_\nu^\beta + d_\nu^\alpha d_\upsilon^\beta} \quad (12)$$

the third redefined Zagreb polynomial is defined as

$$ReZ(\mathcal{G}, x) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{d_\upsilon d_\nu (d_\upsilon + d_\nu)} \quad (13)$$

the harmonic polynomial is

$$H(\mathcal{G}, x) = 2\sum_{\upsilon\nu\in E(\mathcal{G})} x^{d_\upsilon + d_\nu - 1} \quad (14)$$

the third Zagreb polynomial

$$M_3(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{|d_\upsilon - d_\nu|} \quad (15)$$

the fourth Zagreb polynomial

$$M_4(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{d_\upsilon(d_\upsilon + d_\nu)} \quad (16)$$

and the fifth Zagreb polynomial

$$M_5(\mathcal{G}) = \sum_{\upsilon\nu\in E(\mathcal{G})} x^{d_\nu(d_\upsilon + d_\nu)} \quad (17)$$

3 Main results

A three dimensional mesh $\Pi_{p,q,r}$ is defined as the cartesian product $P_p \times P_q \times P_r$. In a three dimensional mesh there are pqr number of vertices and $(2pq - p - q)r + (r - p)pq$ number of edges. A representation of sodium chloride (NaCl) is same as the Cartesian product of three paths of length exactly like a mesh network. The three dimensional mesh network $\Pi_{3,3,3}$ is shown in Fig. 1.

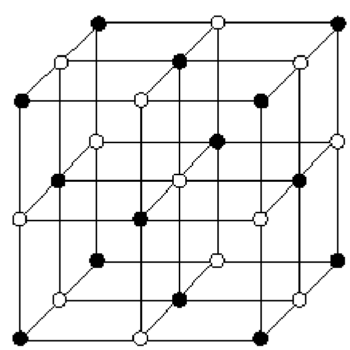

Figure 1: The three dimensional mesh network $\Pi_{3,3,3}$

We find that the unit cell representation of sodium chloride $NaCl$ is the same as the three dimensional mesh network $\Pi_{3,3,3}$. For more detail related to sodium chloride see [Al-Kandari, Manuel and Rajasingh (2011); Ahmad (2020)]. Let the vertex set of $\Pi_{p,q,r}$ be the set $V(\Pi_{p,q,r}) = \{v_1v_2v_3: 0 \leq v_1 \leq l-1, 0 \leq v_2 \leq w-1, 0 \leq v_3 \leq h-1\}$ and two vertices $v = v_1v_2v_3$ and $v' = v'_1v'_2v'_3$ are linked by an edge if $\sum_{i=1}^{3} |v_i - v'_i| = 1$.

We present results, which can be used to compute any degree-based topological polynomials. Our results generalize known results in the area. We give exact values of the most well-known degree-based polynomials for three dimensional mesh network $\Pi_{p,q,r}$. Vetrík [Vetrík (2019)] introduced a new method to calculate the topological indices and also in Ahmad et al. [Ahmad (2020)], we follow the same technique in this paper. Let us give a formula, which can be used to obtain any polynomial of indices based on degrees for three dimensional mesh network $\Pi_{p,q,r}$.

Lemma 3.1: *Let $\Pi_{p,q,r}$ be a three dimensional mesh network. Then* $P(\Pi_{p,q,r}, x) = 3pqrx^{\varphi(6,6)} + 4\{x^{\varphi(4,4)} + 2x^{\varphi(4,5)} - 5x^{\varphi(5,5)} - 2x^{\varphi(5,6)} + 4x^{\varphi(6,6)}\}(p+q+r) + \{4x^{\varphi(5,5)} + 2x^{\varphi(5,6)} - 7x^{\varphi(6,6)}\}(pq+qr+rp) + 12\{2x^{\varphi(3,4)} - 3x^{\varphi(4,4)} - 4x^{\varphi(4,5)} + 6x^{\varphi(5,5)} + 2x^{\varphi(5,6)} - 3x^{\varphi(6,6)}\}$.

Proof. The graph $\Pi_{p,q,r}$ contains pqr vertices and $3pqr - pq - qr - rp$ edges. Each vertex of $\Pi_{p,q,r}$ has degree 3,4,5 or 6, vertices of $\Pi_{p,q,r}$ can be partitioned according to their degrees. Let

$V_i = \{v \in V(\Pi_{p,q,r}): d_v = i\}$

This means that the set V_i contains the vertices of degree i. The set of vertices with respect to their degrees is as follows:

$V_3 = \{v \in V(\Pi_{p,q,r}): d_v = 3\}$

$V_4 = \{v \in V(\Pi_{p,q,r}): d_v = 4\}$

$V_5 = \{v \in V(\Pi_{p,q,r}): d_v = 5\}$

$V_6 = \{v \in V(\Pi_{p,q,r}): d_v = 6\}$

Since, $|V_3| = 8$, $|V_4| = 4(p+q+r-6)$, $|V_5| = 2(pq+qr+rp) - 8(p+q+r-3)$ and $|V_6| = (p-2)(q-2)(r-2)$. Let us divide the edges of $\Pi_{p,q,r}$ into partition sets according to the degree of its end vertices. Let

$\Xi_{3,4} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 3, d_\nu = 4\}$

$\Xi_{4,4} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 4, d_\nu = 4\}$

$\Xi_{4,5} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 4, d_\nu = 5\}$

$\Xi_{5,5} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 5, d_\nu = 5\}$

$\Xi_{5,6} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 5, d_\nu = 6\}$

$\Xi_{6,6} = \{\upsilon\nu \in E(\Pi_{p,q,r}): d_\upsilon = 6, d_\nu = 6\}$

Note that $E(\Pi_{p,q,r}) = \Xi_{3,4} \cup \Xi_{4,4} \cup \Xi_{4,5} \cup \Xi_{5,5} \cup \Xi_{5,6} \cup \Xi_{6,6}$. The number of edges incident to one vertex of degree 3 and other vertex of degree 4 is 24, so $|\Xi_{3,4}| = 24$. The number of edges incident to two vertices of degree 4 is $4(p+q+r-9)$, so $|\Xi_{4,4}| = 4(p+q+$

$r-9$). The number of edges incident to one vertex of degree 5 and other vertex of degree 4, 5 and 6 are $8(p+q+r-6)$, $4(pq+qr+rp)-20(p+q+r)+72$, &$2(pq+qr+rp)-8(p+q+r)+24$, respectively, so $|\Xi_{4,5}|=8(p+q+r-6)$, $|\Xi_{5,5}|=4(pq+qr+rp)-20(p+q+r)+72$, $|\Xi_{5,6}|=2(pq+qr+rp)-8(p+q+r)+24$.

Now, the remaining number of edges are those edges which are incident to two vertices of degree 6,

$\Xi_{6,6}=|E(\Pi_{p,q,r})|-\Xi_{3,4}-\Xi_{4,4}-\Xi_{4,5}-\Xi_{5,5}-\Xi_{5,6}=3pqr-7(pq+qr+rp)+16(p+q+r)-36$.

Hence,

$P(\Pi_{p,q,r},x)=\sum_{\upsilon\nu\in E(\Pi_{p,q,r})}x^{\varphi(d_\upsilon,d_\nu)}=\sum_{\upsilon\nu\in\Xi_{3,4}}x^{\varphi(3,4)}+\sum_{\upsilon\nu\in\Xi_{4,4}}x^{\varphi(4,4)}+\sum_{\upsilon\nu\in\Xi_{4,5}}x^{\varphi(4,5)}+\sum_{\upsilon\nu\in\Xi_{5,5}}x^{\varphi(5,5)}+\sum_{\upsilon\nu\in\Xi_{5,6}}x^{\varphi(5,6)}+\sum_{\upsilon\nu\in\Xi_{6,6}}x^{\varphi(6,6)}=24x^{\varphi(3,4)}+4(p+q+r-9)x^{\varphi(4,4)}+8(p+q+r-6)x^{\varphi(4,5)}+\{4(pq+qr+rp)-20(p+q+r)+72\}x^{\varphi(5,5)}+\{2(pq+qr+rp)-8(p+q+r)+24\}x^{\varphi(5,6)}+\{3pqr-7(pq+qr+rp)+16(p+q+r)-36\}x^{\varphi(6,6)}$.

After simplification, we get

$$P(\Pi_{p,q,r},x)=3pqrx^{\varphi(6,6)}+4\{x^{\varphi(4,4)}+2x^{\varphi(4,5)}-5x^{\varphi(5,5)}-2x^{\varphi(5,6)}+4x^{\varphi(6,6)}\}(p+q+r)+\{4x^{\varphi(5,5)}+2x^{\varphi(5,6)}-7x^{\varphi(6,6)}\}(pq+qr+rp)+12\{2x^{\varphi(3,4)}-3x^{\varphi(4,4)}-4x^{\varphi(4,5)}+6x^{\varphi(5,5)}+2x^{\varphi(5,6)}-3x^{\varphi(6,6)}\}.$$

Now we present polynomials of the best-known degree based polynomials of three dimensional mesh network in the following theorem.

Theorem 3.2 *For the three dimensional mesh network $\Pi_{p,q,r}$, we have* the general Randic′ polynomial of $\Pi_{p,q,r}$

$$R_\alpha(\Pi_{p,q,r},x)=3pqrx^{(36)^\alpha}+4\{x^{(16)^\alpha}+2x^{(20)^\alpha}-5x^{(25)^\alpha}-2x^{(30)^\alpha}+4x^{(36)^\alpha}\}(p+q+r)+\{4x^{(25)^\alpha}+2x^{(30)^\alpha}-7x^{(36)^\alpha}\}(pq+qr+rp)+12\{2x^{(12)^\alpha}-3x^{(16)^\alpha}-4x^{(20)^\alpha}+6x^{(25)^\alpha}+2x^{(30)^\alpha}-3x^{(36)^\alpha}\},$$

the second Zagreb polynomial of $\Pi_{p,q,r}$

$$R_1(\Pi_{p,q,r},x)=3pqrx^{36}+4\{x^{16}+2x^{20}-5x^{25}-2x^{30}+4x^{36}\}(p+q+r)+\{4x^{25}+2x^{30}-7x^{36}\}(pq+qr+rp)+12\{2x^{12}-3x^{16}-4x^{20}+6x^{25}+2x^{30}-3x^{36}\}.$$

Proof. For $R_\alpha(\Pi_{p,q,r},\ x)$ which is the general Randic′ polynomial of $\Pi_{p,q,r}$ we have $\varphi(d_\upsilon,d_\nu)=(d_\upsilon d_\nu)^\alpha$, therefore $\varphi(3,4)=(12)^\alpha$, $\varphi(4,4)=(16)^\alpha$, $\varphi(4,5)=(20)^\alpha$, $\varphi(5,5)=(25)^\alpha$, $\varphi(5,6)=(30)^\alpha$ and $\varphi(6,6)=(36)^\alpha$.

Thus by Lemma 3.1,

$$R_\alpha(\Pi_{p,q,r},x)=3pqrx^{(36)^\alpha}+4\{x^{(16)^\alpha}+2x^{(20)^\alpha}-5x^{(25)^\alpha}-2x^{(30)^\alpha}+4x^{(36)^\alpha}\}(p+q+r)+\{4x^{(25)^\alpha}+2x^{(30)^\alpha}-7x^{(36)^\alpha}\}(pq+qr+rp)+12\{2x^{(12)^\alpha}-3x^{(16)^\alpha}-4x^{(20)^\alpha}+6x^{(25)^\alpha}+2x^{(30)^\alpha}-3x^{(36)^\alpha}\}.$$

For $\alpha=1$, the second Zagreb polynomial is

$$R_1(\Pi_{p,q,r},x) = 3pqrx^{36} + 4\{x^{16} + 2x^{20} - 5x^{25} - 2x^{30} + 4x^{36}\}(p+q+r) + \{4x^{25} + 2x^{30} - 7x^{36}\}(pq+qr+rp) + 12\{2x^{12} - 3x^{16} - 4x^{20} + 6x^{25} + 2x^{30} - 3x^{36}\}.$$

In the next theorem, we determined general sum-connectivity polynomial, first Zagreb polynomial and hyper-Zagreb polynomial of the three dimensional mesh network $\Pi_{p,q,r}$

Theorem 3.3 *For the three dimensional mesh network $\Pi_{p,q,r}$ we have*

the general sum-connectivity polynomial of $\Pi_{p,q,r}$

$\chi_\alpha(\Pi_{p,q,r},x) = 3pqrx^{12^\alpha} + 4\{x^{8^\alpha} + 2x^{9^\alpha} - 5x^{10^\alpha} - 2x^{11^\alpha} + 4x^{12^\alpha}\}(p+q+r) + \{4x^{10^\alpha} + 2x^{11^\alpha} - 7x^{12^\alpha}\}(pq+qr+rp) + 12\{2x^{7^\alpha} - 3x^{8^\alpha} - 4x^{9^\alpha} + 6x^{10^\alpha} + 2x^{11^\alpha} - 3x^{12^\alpha}\}$the first Zagreb polynomial of $\Pi_{p,q,r}$,

$$\chi_1(\Pi_{p,q,r},x) = 3pqrx^{12} + 4\{x^8 + 2x^9 - 5x^{10} - 2x^{11} + 4x^{12}\}(p+q+r) + \{4x^{10} + 2x^{11} - 7x^{12}\}(pq+qr+rp) + 12\{2x^7 - 3x^8 - 4x^9 + 6x^{10} + 2x^{11} - 3x^{12}\}.$$

the hyper-Zagreb polynomial of $\Pi_{p,q,r}$

$$\chi_2(\Pi_{p,q,r},x) = 3pqrx^{144} + 4\{x^{64} + 2x^{81} - 5x^{100} - 2x^{121} + 4x^{144}\}(p+q+r) + \{4x^{100} + 2x^{121} - 7x^{144}\}(pq+qr+rp) + 12\{2x^{49} - 3x^{64} - 4x^{81} + 6x^{100} + 2x^{121} - 3x^{144}\}.$$

Proof. For $\chi_\alpha(\Pi_{p,q,r},x)$ which is the general sum-connectivity polynomial of $\Pi_{p,q,r}$ we have $\varphi(d_v,d_v) = (d_v + d_v)^\alpha$, therefore $\varphi(3,4) = (7)^\alpha, \varphi(4,4) = (8)^\alpha, \varphi(4,5) = (9)^\alpha, \varphi(5,5) = (10)^\alpha, \varphi(5,6) = (11)^\alpha$ and $\varphi(6,6) = (12)^\alpha$. Thus by Lemma 3.1,

$$\chi_\alpha(\Pi_{p,q,r},x) = 3pqrx^{12^\alpha} + 4\{x^{8^\alpha} + 2x^{9^\alpha} - 5x^{10^\alpha} - 2x^{11^\alpha} + 4x^{12^\alpha}\}(p+q+r) + \{4x^{10^\alpha} + 2x^{11^\alpha} - 7x^{12^\alpha}\}(pq+qr+rp) + 12\{2x^{7^\alpha} - 3x^{8^\alpha} - 4x^{9^\alpha} + 6x^{10^\alpha} + 2x^{11^\alpha} - 3x^{12^\alpha}\}.$$

For $\alpha = 1$, the first Zagreb polynomial is

$$\chi_1(\Pi_{p,q,r},x) = 3pqrx^{12} + 4\{x^8 + 2x^9 - 5x^{10} - 2x^{11} + 4x^{12}\}(p+q+r) + \{4x^{10} + 2x^{11} - 7x^{12}\}(pq+qr+rp) + 12\{2x^7 - 3x^8 - 4x^9 + 6x^{10} + 2x^{11} - 3x^{12}\}.$$

For $\alpha = 2$, the hyper-Zagreb polynomial is

$$\chi_2(\Pi_{p,q,r},x) = 3pqrx^{12^2} + 4\{x^{8^2} + 2x^{9^2} - 5x^{10^2} - 2x^{11^2} + 4x^{12^2}\}(p+q+r) + \{4x^{10^2} + 2x^{11^2} - 7x^{12^2}\}(pq+qr+rp) + 12\{2x^{7^2} - 3x^{8^2} - 4x^{9^2} + 6x^{10^2} + 2x^{11^2} - 3x^{12^2}\}$$

$$= 3pqrx^{144} + 4\{x^{64} + 2x^{81} - 5x^{100} - 2x^{121} + 4x^{144}\}(p+q+r) + \{4x^{100} + 2x^{121} - 7x^{144}\}(pq+qr+rp) + 12\{2x^{49} - 3x^{64} - 4x^{81} + 6x^{100} + 2x^{121} - 3x^{144}\}.$$

In the following theorem, we determined generalized Zagreb polynomial and forgotten polynomial of the three dimensional mesh network $\Pi_{p,q,r}$.

Theorem 3.4 *For the three dimensional mesh network* $\Pi_{p,q,r}$ *we have* the generalized Zagreb polynomial of $\Pi_{p,q,r}$

$$GZ_{\alpha,\beta}(\Pi_{p,q,r},x) = 3pqrx^{2(6^{\alpha+\beta})} + 4\{x^{2^{2\alpha+2\beta+1}} + 2x^{4^{\alpha}5^{\beta}+5^{\alpha}4^{\beta}} - 5x^{2(5^{\alpha+\beta})} - 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} + 4x^{2(6^{\alpha+\beta})}\}(p+q+r) + \{4x^{2(5^{\alpha+\beta})} + 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} - 7x^{2(6^{\alpha+\beta})}\}(pq+qr+rp) + 12\{2x^{3^{\alpha}4^{\beta}+4^{\alpha}3^{\beta}} - 3x^{2^{2\alpha+2\beta+1}} - 4x^{4^{\alpha}5^{\beta}+5^{\alpha}4^{\beta}} + 6x^{2(5^{\alpha+\beta})} + 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} - 3x^{2(6^{\alpha+\beta})}\},$$

the forgotten polynomial of $\Pi_{p,q,r}$

$$GZ_{2,0}(\Pi_{p,q,r},x) = 3pqrx^{72} + 4\{x^{32} + 2x^{41} - 5x^{50} - 2x^{61} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{61} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{25} - 3x^{32} - 4x^{41} + 6x^{50} + 2x^{61} - 3x^{72}\}.$$

Proof. For $GZ_{\alpha,\beta}(\Pi_{p,q,r},x)$ which is the generalized Zagreb polynomial of $\Pi_{p,q,r}$, we have $\varphi(d_v,d_v) = d_u^{\alpha}d_v^{\beta} + d_v^{\alpha}d_u^{\beta}$, therefore $\varphi(3,4) = 3^{\alpha}4^{\beta} + 4^{\alpha}3^{\beta}$, $\varphi(4,4) = 4^{\alpha}4^{\beta} + 4^{\alpha}4^{\beta} = 2^{2\alpha+2\beta+1}$, $\varphi(4,5) = 4^{\alpha}5^{\beta} + 5^{\alpha}4^{\beta}$, $\varphi(5,5) = 5^{\alpha}5^{\beta} + 5^{\alpha}5^{\beta} = 2(5^{\alpha+\beta})$, $\varphi(5,6) = 5^{\alpha}6^{\beta} + 6^{\alpha}5^{\beta}$ and $\varphi(6,6) = 6^{\alpha}6^{\beta} + 6^{\alpha}6^{\beta} = 2(6^{\alpha+\beta})$. Thus by Lemma 3.1,

$$GZ_{\alpha,\beta}(\Pi_{p,q,r},x) = 3pqrx^{2(6^{\alpha+\beta})} + 4\{x^{2^{2\alpha+2\beta+1}} + 2x^{4^{\alpha}5^{\beta}+5^{\alpha}4^{\beta}} - 5x^{2(5^{\alpha+\beta})} - 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} + 4x^{2(6^{\alpha+\beta})}\}(p+q+r) + \{4x^{2(5^{\alpha+\beta})} + 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} - 7x^{2(6^{\alpha+\beta})}\}(pq+qr+rp) + 12\{2x^{3^{\alpha}4^{\beta}+4^{\alpha}3^{\beta}} - 3x^{2^{2\alpha+2\beta+1}} - 4x^{4^{\alpha}5^{\beta}+5^{\alpha}4^{\beta}} + 6x^{2(5^{\alpha+\beta})} + 2x^{5^{\alpha}6^{\beta}+6^{\alpha}5^{\beta}} - 3x^{2(6^{\alpha+\beta})}\}.$$

For $\alpha = 2, \beta = 0$, the forgotten polynomial is

$$GZ_{2,0}(\Pi_{p,q,r},x) = 3pqrx^{2(6^2)} + 4\{x^{2^5} + 2x^{4^2+5^2} - 5x^{2(5^2)} - 2x^{5^2+6^2} + 4x^{2(6^2)}\}(p+q+r) + \{4x^{2(5^2)} + 2x^{5^2+6^2} - 7x^{2(6^2)}\}(pq+qr+rp) + 12\{2x^{3^2+4^2} - 3x^{2^5} - 4x^{4^2+5^2} + 6x^{2(5^2)} + 2x^{5^2+6^2} - 3x^{2(6^2)}\}$$
$$= 3pqrx^{72} + 4\{x^{32} + 2x^{41} - 5x^{50} - 2x^{61} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{61} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{25} - 3x^{32} - 4x^{41} + 6x^{50} + 2x^{61} - 3x^{72}\}.$$

In the following theorem, we determined third redefined Zagreb polynomial and harmonic polynomial of the three dimensional mesh network $\Pi_{p,q,r}$.

Theorem 3.5 *For the three dimensional mesh network* $\Pi_{p,q,r}$, *we have*

the third redefined Zagreb polynomial of $\Pi_{p,q,r}$,

$$ReZ(\Pi_{p,q,r},x) = 3pqrx^{432} + 4\{x^{128} + 2x^{180} - 5x^{250} - 2x^{330} + 4x^{432}\}(p+q+r) + \{4x^{250} + 2x^{330} - 7x^{432}\}(pq+qr+rp) + 12\{2x^{84} - 3x^{128} - 4x^{180} + 6x^{250} + 2x^{330} - 3x^{432}\}.$$

and the harmonic polynomial of $\Pi_{p,q,r}$,

$$H(\Pi_{p,q,r},x) = 6pqrx^{11} + 8\{x^7 + 2x^8 - 5x^9 - 2x^{10} + 4x^{11}\}(p+q+r) + 2\{4x^9 + 2x^{10} - 7x^{11}\}(pq+qr+rp) + 24\{2x^6 - 3x^7 - 4x^8 + 6x^9 + 2x^{10} - 3x^{11}\}.$$

Proof. For $ReZ(\Pi_{p,q,r},x)$ which is the third redefined Zagreb polynomial of $\Pi_{p,q,r}$ we have $\varphi(d_v,d_v) = d_v d_v(d_v + d_v)$,

therefore

$\varphi(3,4) = 12(7) = 84, \varphi(4,4) = 16(8) = 128, \varphi(4,5) = 20(9) = 180, \varphi(5,5) = 25(10) = 250, \varphi(5,6) = 30(11) = 330$ and $\varphi(6,6) = 36(12) = 432$.

Thus by Lemma 3.1,

$$ReZ(\Pi_{p,q,r},x) = 3pqrx^{432} + 4\{x^{128} + 2x^{180} - 5x^{250} - 2x^{330} + 4x^{432}\}(p+q+r) + \{4x^{250} + 2x^{330} - 7x^{432}\}(pq+qr+rp) + 12\{2x^{84} - 3x^{128} - 4x^{180} + 6x^{250} + 2x^{330} - 3x^{432}\}.$$

For $H(\Pi_{p,q,r},x)$ which is the harmonic polynomial of $\Pi_{p,q,r}$, we have $\varphi(d_v,d_v) = d_v + d_v - 1$,

therefore

$\varphi(3,4) = 6, \varphi(4,4) = 7, \varphi(4,5) = 8, \varphi(5,5) = 9, \varphi(5,6) = 10$ and $\varphi(6,6) = 11$.

Thus by Lemma 3.1,

$$H(\Pi_{p,q,r},x) = 6pqrx^{11} + 8\{x^7 + 2x^8 - 5x^9 - 2x^{10} + 4x^{11}\}(p+q+r) + 2\{4x^9 + 2x^{10} - 7x^{11}\}(pq+qr+rp) + 24\{2x^6 - 3x^7 - 4x^8 + 6x^9 + 2x^{10} - 3x^{11}\}.$$

In the following theorem, we determined third Zagreb polynomial, fourth Zagreb polynomial and fifth Zagreb polynomial of the three dimensional mesh network $\Pi_{p,q,r}$.

Theorem 3.6 *For the three dimensional mesh network $\Pi_{p,q,r}$, we have*

the third Zagreb polynomial of $\Pi_{p,q,r}$,

$$M_3(\Pi_{p,q,r},x) = 3pqr + (2x-3)(pq+qr+rp).$$

the fourth Zagreb polynomial of $\Pi_{p,q,r}$,

$$M_4(\Pi_{p,q,r},x) = 3pqrx^{72} + 4\{x^{32} + 2x^{36} - 5x^{50} - 2x^{55} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{55} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{21} - 3x^{32} - 4x^{36} + 6x^{50} + 2x^{55} - 3x^{72}\}.$$

the fifth Zagreb polynomial of $\Pi_{p,q,r}$,

$$M_5(\Pi_{p,q,r},x) = 3pqrx^{72} + 4\{x^{32} + 2x^{45} - 5x^{50} - 2x^{66} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{66} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{28} - 3x^{32} - 4x^{45} + 6x^{50} + 2x^{66} - 3x^{72}\}.$$

Proof. For $M_3(\Pi_{p,q,r},x)$ which is the third Zagreb polynomial of $\Pi_{p,q,r}$, we have $\varphi(d_v,d_v) = |d_v - d_v|$, therefore $\varphi(3,4) = 1, \varphi(4,4) = 0, \varphi(4,5) = 1, \varphi(5,5) = 0, \varphi(5,6) = 1$ and $\varphi(6,6) = 0$.

Thus by Lemma 3.1,

$$M_3(\Pi_{p,q,r},x) = 3pqrx^0 + 4\{x^0 + 2x^1 - 5x^0 - 2x^1 + 4x^0\}(p+q+r) + \{4x^0 + 2x^1 - 7x^0\}(pq+qr+rp) + 12\{2x^1 - 3x^0 - 4x^1 + 6x^0 + 2x^1 - 3x^0\} = 3pqr + (2x-3)(pq+qr+rp).$$

For $M_4(\Pi_{p,q,r},x)$ which is the fourth Zagreb polynomial of $\Pi_{p,q,r}$, we have $\varphi(d_v,d_v) = d_v(d_v+d_v)$, therefore $\varphi(3,4) = 21, \varphi(4,4) = 32, \varphi(4,5) = 36, \varphi(5,5) = 50, \varphi(5,6) = 55$ and $\varphi(6,6) = 72$.

Thus by Lemma 3.1,

$$M_4(\Pi_{p,q,r},x) = 3pqrx^{72} + 4\{x^{32} + 2x^{36} - 5x^{50} - 2x^{55} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{55} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{21} - 3x^{32} - 4x^{36} + 6x^{50} + 2x^{55} - 3x^{72}\}.$$

For $M_5(\Pi_{p,q,r},x)$ which is the fifth Zagreb polynomial of $\Pi_{p,q,r}$, we have $\varphi(d_v,d_v) = d_v(d_v+d_v)$, therefore $\varphi(3,4) = 28, \varphi(4,4) = 32, \varphi(4,5) = 45, \varphi(5,5) = 50, \varphi(5,6) = 66$ and $\varphi(6,6) = 72$. Thus by Lemma 3.1,

$$M_5(\Pi_{p,q,r},x) = 3pqrx^{72} + 4\{x^{32} + 2x^{45} - 5x^{50} - 2x^{66} + 4x^{72}\}(p+q+r) + \{4x^{50} + 2x^{66} - 7x^{72}\}(pq+qr+rp) + 12\{2x^{28} - 3x^{32} - 4x^{45} + 6x^{50} + 2x^{66} - 3x^{72}\}.$$

4 Conclusion

Topological indices are often studied with the help of their polynomials. Formulae for these topological polynomials for three dimensional mesh network can lead future research to design some incipient architectures or networks in different fields of chemistry and computer science. This structural invariant has revolutionized the field of chemistry to identify molecular indices of chemical compounds. These indices are extensively used for establishing relationships between the structure of nanotubes and their physico-chemical properties. They can likewise be useful in creating productive physical structure in mechanics as well as for different computer network problems.

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