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Delay-dependent Stability of Recurrent Neural Networks with Time-varying Delay

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ABSTRACT
This paper investigates the delay-dependent stability problem of recurrent neural networks with time-varying delay. A new and less conservative stability criterion is derived through constructing a new augmented Lyapunov-Krasovskii functional (LKF) and employing the linear matrix inequality method. A new augmented LKF that considers more information of the slope of neuron activation functions is developed for further reducing the conservatism of stability results. To deal with the derivative of the LKF, several commonly used techniques, including the integral inequality, reciprocally convex combination, and free-weighting matrix method, are applied. Moreover, it is found that the obtained stability criterion has a lower computational burden than some recent existing ones. Finally, two numerical examples are considered to demonstrate the effectiveness of the presented stability results.

KEY WORDS: Neural networks, Lyapunov-Krasovskii functional, Stability criterion, Time-varying delay, Lower computational burden

1 INTRODUCTION

DELAY-DEPENDENT stability criteria for neural networks with constant time or time-varying delays have received considerable attention in recent years (Zhang, et al. 2013; Li, et al. 2013; Kwon, et al. 2013; Chen & Zheng, 2013; Zhang, et al. 2014; Zhang, et al. 2014; Ge, et al. 2014; Wang, et al. 2015). On the one hand, the reason is that neural networks have received considerable attention due to their extensive applications, such as optimization (Nissinen, et al. 1999), automatic control (Benallegue & Meddah 2001), and others. On the other hand, delay-dependent stability results, which consider the information of time delays, are less conservative than the delay-independent stability results, especially when the time delays are small. Since the time delays are frequently encountered in electronic implementations of neural networks due to the finite switching speed of amplifiers and the inherent communication time between neurons, which may cause hidden oscillations, divergence, chaos, instability, or other poor performance behaviors (Zhang, et al. 2014; Ge, et al. 2014). Therefore, it is more effective to solve the delay-dependent stability problem of neural networks with time-varying delay from two perspectives of less conservatism and lower computational burden.

Most of delay-dependent stability criteria are derived via the Lyapunov stability theory, so the appropriate choice of Lyapunov-Krasovskii functional (LKF) is crucial for deriving less conservative stability criteria. To reduce the conservatism, recently, several commonly used techniques have been applied in the estimation of the derivative of LKF, such as free-weighting matrix (Zhang, et al. 2014; Zuo, et al. 2010; Hua, et al. 2011; Zhang, et al. 2013; Chen & Zhao, 2015), integral inequality (Li & Ye, 2010; Wu, et al. 2010; Stojanovic, 2016; Zeng, et al. 2011; Tian & Zhong, 2012; Zhang, et al. 2010; Wu, et al. 2012; Lakshmanan, et al. 2013; Zhang & Han, 2011; Wang, et al. 2011; Zheng, et al. 2010), Leibniz-Newton formula (Zhang, et al. 2014; Hua, et al. 2011; Shao & Han, 2011; Tian, & Zhong, 2012), reciprocally convex

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combination (Liu, 2013; Yang & Zhang, 2014; Farnam, et al. 2016), and their combinations (Zhang, et al. 2014). For the derivative of LKF, it is necessary to estimate the derivative for deriving stability criteria in terms of linear matrix inequalities (LMIs). It is usually difficult to deal with the integral terms of the derivative of LKF such that the derivative is enlarged. Although numerous techniques have been developed for estimating the time derivative of LKF, there still exists room for further study. It can be summarized as follows: 1) For the augmented LKF introduced in Kwon, et al. (2013), Zheng, et al. (2010), Kwon, et al. (2013), Rakkiyappan, et al. (2016), and Yang, et al. (2017), the delayed state derivative terms including $\dot{y}(t-\tau(t))$ and $\dot{y}(t-\tau_m)$ are contained in the derivative of LKF. However, if these terms are replaced by the delayed neural networks, it can be found that the time-varying delay is doubled or the delay interval $[0,\tau_m]$ imperceptibly becomes $[0,2\tau_m]$. This means that the calculation results listed in Kwon, et al. (2013), Zheng, et al. (2010), Kwon, et al. (2013), Rakkiyappan, et al. (2016) and Yang, et al. (2017) are doubled. Simultaneously, derived activation function $g(y(t-2\tau(t)))$ and $g(y(t-\tau(t)-\tau_m))$ are non-existent for the considered neural networks. 2) For the delay-decomposition LKF introduced in Ge, et al. (2014) and Zeng, et al. (2011), and the delay-partitioning LKF introduced in Wang, et al. (2015) and Lakshmanan, et al. (2013), the larger the number of subintervals is, there is less conservatism of stability results. However, the conservative reduction trends to be in-apparent as the increasing of the number of time delay subintervals, which lead to a large computational burden (Zhang, et al. 2014). 3) For Leibniz-Newton formula, many free weighting matrices are often needed to be introduced in the derived stability criteria and leads to significant increases in the computational burden. In Zhang, et al. (2014), the Leibniz-Newton formula was used to estimate the derivative of LKF, which caused a large number of matrix variables. Thus, how to further reduce the conservatism and computational burden of the stability results may be more attractive.

This paper further investigates the stability condition for continuous recurrent neural networks with time-varying delay by constructing a newly augmented LKF and employing the LMI method. This paper aims to derive a new and less delay-dependent stability criterion for recurrent neural networks with time-varying delay, while reducing the computational burden via less useful matrix variables. The contributions and improvements are summarized as follows: 1) Unlike the augmented LKF in Kwon, et al. (2013), Zheng, et al. (2010), Kwon, et al. (2013), Rakkiyappan, et al. (2016), and Yang, et al. (2017), a newly augmented LKF is constructed to improve the results, where the delayed state derivative terms including $\dot{y}(t-\tau(t))$ and $\dot{y}(t-\tau_m)$ do not appear in the derived stability criterion. 2) The delay-decomposition or delay-partitioning ideas introduced in Ge, et al. (2014), Zeng, et al. (2011), Wang, et al. (2015) and Lakshmanan, et al. (2013) are not used. Instead, by considering more information of the neural states and neuron activation functions as augmented elements, a newly augmented LKF is developed. 3) The commonly used techniques, including integral inequality, reciprocally convex combination, and free-weighting matrix method, are applied in the estimation of the derivative of the constructed LKF. Those improvements lead to the obtained stability results having lower computational burden. Finally, two numerical examples are given to verify the effectiveness of the proposed stability criterion and its improvements over the recent existing ones.

The rest of this paper is organized as follows: Section 2 gives the problem formulation. Section 3 presents the new and less conservative delay-dependent stability criterion. In Section 4, two numerical examples are given to verify the effectiveness of the proposed stability criterion. Finally, the conclusion is made in Section 5.

Throughout this paper, the superscript T means the transpose of a matrix, R^n denotes the n-dimensional Euclidean space, $R^{m\times n}$ denotes the set of all $m\times n$ real matrices, $P>0$ (≥ 0) means that P is a real symmetric and positive- definite (semipositive-definite) matrix, $|\cdot|$ denotes the absolute value, Y_{ij} represents the element in row i and column j of matrix Y, $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix, symmetric term in a symmetric matrix is denoted by *.

2 PROBLEM FORMULATION

CONSIDER the following recurrent neural network with time-varying delay:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + J \ ,\quad (1)$$

where $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^T \in R^n$ is the state vector. $f(\cdot)=[f_1(\cdot), f_2(\cdot),\ldots, f_n(\cdot)]^T$ denotes the activation functions. $C=\mathrm{diag}\{c_1,c_2,\ldots,c_n\}$ is a diagonal matrix with $c_i>0$. A and B are the connection matrices. $J=[J_1,J_2,\ldots,J_n]^T$ is an external constant input vector. $\tau(t)$ is time-varying delay and satisfies

$$0 \leq \tau(t) \leq \tau_m\,,\ \ \dot{\tau}(t) \leq \mu\,, \quad (2)$$

where τ_m and μ are constants.

Assume that the activation function satisfies

$$\sigma_i^- \le \frac{f_i(a)-f_i(b)}{a-b} \le \sigma_i^+, \forall a,b \in R, a \ne b, i=1,2,\ldots,n \quad (3)$$

where σ_i^- and σ_i^+ are known real constants, which can be positive, zero, and negative.

Suppose there exists an equilibrium point $x*$ for the neural network (1), one can shift the equilibrium point of (1) to the origin by changing variables

$$y(t) = x(t) - x*,$$

$$g(y(t)) = f(y(t)+x*) - f(x*), \quad (4)$$

Then, (1) is rewritten as

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t-\tau(t))), \quad (5)$$

where $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$,
$g(y(t)) = [g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t))]^T$.

In addition, it is easily obtained from (3) that

$$\sigma_i^- \le \frac{g_i(a)-g_i(b)}{a-b} \le \sigma_i^+, \forall a,b \in R, a \ne b, i=1,2,\ldots,n \quad (6)$$

Let $a \ne 0$ and $b = 0$, then

$$\sigma_i^- \le \frac{g_i(a)}{a} \le \sigma_i^+, \ \forall a \ne 0, \ i=1,2,\ldots,n, \quad (7)$$

and

$$[g_i(a) - \sigma_i^- a][g_i(a) - \sigma_i^+ a] \le 0, \ \ i=1,2,\ldots,n \quad (8)$$

This paper aims to derive a new and less conservative delay-dependent stability criterion guaranteeing that the delayed neural network (5) is globally asymptotically stable, while reducing the computational burden. To obtain the main results, the following lemmas are introduced.

Lemma 1 (He, et al. 2016). For any constant positive-definite matrix $Q \in R^{n\times n}$ and $\beta \le s \le \alpha$, the following inequalities hold:

$$(\alpha-\beta)\int_\beta^\alpha x^T(s)Qx(s)ds \ge \left(\int_\beta^\alpha x(s)ds\right)^T Q\left(\int_\beta^\alpha x(s)ds\right), \quad (9)$$

$$-\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha \dot{x}^T(u)Q\dot{x}(u)duds$$
$$\le -\left(\int_\beta^\alpha\int_s^\alpha \dot{x}(u)duds\right)^T Q\left(\int_\beta^\alpha\int_s^\alpha \dot{x}(u)duds\right), \quad (10)$$

Lemma 2 (Wang, et al. 2015). For any vectors h_1, h_2 with appropriate dimensions, scalars $a > 0$, $b > 0$ and $a + b = 1$, if there exist matrix $M \in R^{n\times n} > 0$ and any matrix S with appropriate dimensions, and the following inequalities hold :

$$\begin{bmatrix} M & S \\ * & M \end{bmatrix} \ge 0,$$

$$-\frac{1}{a}h_1^T Mh_1 - \frac{1}{b}h_2^T Mh_2 \le -\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}^T \begin{bmatrix} M & S \\ * & M \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad (11)$$

3 STABLILITY CRITERION

THIS section discusses the stability of (5) and derives a delay-dependent stability criterion by employing newly augmented LKF and above mentioned lemmas.

Theorem 1. For given $\tau_m > 0$ and μ, and diagonal matrices $L_1 = \text{diag}\{\sigma_1^-, \sigma_2^-, \ldots, \sigma_n^-\}$, $L_2 = \text{diag}\{\sigma_1^+, \sigma_2^+, \ldots, \sigma_n^+\}$, the neural network (5) with (6) and a time-varying delay satisfying condition (2) is globally asymptotically stable if there exist real matrices

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ * & M_{22} & M_{23} \\ * & * & M_{33} \end{bmatrix} > 0, \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0,$$

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix} > 0, \ K_1 > 0,$$

and positive diagonal matrices
$T_i \ (i=1,2,3)$, $H_1 = \text{diag}\{h_{11}, h_{12}, \ldots, h_{1n}\}$, $H_2 = \text{diag}\{h_{21}, h_{22}, \ldots, h_{2n}\}$,

and any matrice $N_i \in R^{n\times n} (i=1,2,\ldots,11)$,with appropriate dimensions, then the following LMIs hold:

$$\Sigma < 0, \quad (12)$$

$$\begin{bmatrix} G & S \\ * & G \end{bmatrix} \ge 0, \quad (13)$$

Calculating the time derivatives of $V_i(t)$, $i=1,2,\ldots,5$ a along the state trajectories of (5) yields

$$\dot{V}_1(t) = 2\begin{bmatrix} y(t) \\ \int_{t-\tau(t)}^t y(s)ds + \int_{t-\tau_m}^{t-\tau(t)} y(s)ds \\ \int_{t-\tau(t)}^t g(y(s))ds + \int_{t-\tau_m}^{t-\tau(t)} g(y(s))ds \end{bmatrix}^T$$

$$\times\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ * & M_{22} & M_{23} \\ * & * & M_{33} \end{bmatrix}$$

$$\times\begin{bmatrix} \dot{y}(t) \\ y(t)-y(t-\tau_m) \\ g(y(t))-g(y(t-\tau_m)) \end{bmatrix}, \qquad (15)$$

$$\dot{V}_2(t) \le \begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix}$$

$$-(1-\mu)\begin{bmatrix} y(t-\tau(t)) \\ g(y(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}$$

$$\times\begin{bmatrix} y(t-\tau(t)) \\ g(y(t-\tau(t))) \end{bmatrix}, \qquad (16)$$

$$\dot{V}_3(t) = 2[g(y(t)) - L_1 y(t)]^T H_1 \dot{y}(t)$$

$$+2[L_2 y(t) - g(y(t))]^T H_2 \dot{y}(t), \qquad (17)$$

$$\dot{V}_4(t) = \tau_m^2 \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}$$

$$-\tau_m \int_{t-\tau_m}^{t} \begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix}$$

$$\times\begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix} ds \qquad (18)$$

Since $\begin{bmatrix} G & S \\ * & G \end{bmatrix} \ge 0$, by applying Lemma 2, one can obtains

$$\dot{V}_4(t) \le \tau_m^2 \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}$$

$$-\begin{pmatrix} \int_{t-\tau(t)}^{t} \begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix} ds \\ \int_{t-\tau_m}^{t-\tau(t)} \begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix} ds \end{pmatrix}^T \begin{bmatrix} G & S \\ * & G \end{bmatrix}$$

$$\times\begin{pmatrix} \int_{t-\tau(t)}^{t} \begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix} ds \\ \int_{t-\tau_m}^{t-\tau(t)} \begin{bmatrix} y(s) \\ g(y(s)) \\ \dot{y}(s) \end{bmatrix} ds \end{pmatrix}$$

$$= \tau_m^2 \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \\ \dot{y}(t) \end{bmatrix}$$

$$-\begin{bmatrix} \int_{t-\tau(t)}^{t} y(s)ds \\ \int_{t-\tau(t)}^{t} g(y(s))ds \\ y(t)-y(t-\tau(t)) \\ \int_{t-\tau_m}^{t-\tau(t)} y(s)ds \\ \int_{t-\tau_m}^{t-\tau(t)} g(y(s))ds \\ y(t-\tau(t))-y(t-\tau_m) \end{bmatrix}^T$$

$$\times\begin{bmatrix} G_{11} & G_{12} & G_{13} & S_{11} & S_{12} & S_{13} \\ * & G_{22} & G_{23} & S_{21} & S_{22} & S_{23} \\ * & * & G_{33} & S_{31} & S_{32} & S_{33} \\ * & * & * & G_{11} & G_{12} & G_{13} \\ * & * & * & * & G_{22} & G_{23} \\ * & * & * & * & * & G_{33} \end{bmatrix}$$

$$\times\begin{bmatrix} \int_{t-\tau(t)}^{t} y(s)ds \\ \int_{t-\tau(t)}^{t} g(y(s))ds \\ y(t)-y(t-\tau(t)) \\ \int_{t-\tau_m}^{t-\tau(t)} y(s)ds \\ \int_{t-\tau_m}^{t-\tau(t)} g(y(s))ds \\ y(t-\tau(t))-y(t-\tau_m) \end{bmatrix}. \qquad (19)$$

By using Lemma 1, one can obtain

$$\dot{V}_5(t) = (\tau_m^2/2)^2 \dot{y}^T(t) K_1 \dot{y}(t)$$

$$-(\tau_m^2/2)\int_{t-\tau_m}^{t}\int_{s}^{t} \dot{y}^T(u) K_1 \dot{y}(u) du ds$$

$$\le (\tau_m^2/2)^2 \dot{y}^T(t) K_1 \dot{y}(t)$$

$$-\left(\int_{t-\tau_m}^{t}\int_{s}^{t} \dot{y}(u) du ds\right)^T K_1 \left(\int_{t-\tau_m}^{t}\int_{s}^{t} \dot{y}(u) du ds\right)$$

$$= (\tau_m^2/2)^2 \dot{y}^T(t) K_1 \dot{y}(t)$$

$$-\left(\tau_m y(t) - \int_{t-\tau(t)}^{t} y(s)ds - \int_{t-\tau_m}^{t-\tau(t)} y(s)ds\right)^T K_1$$

$$\times\left(\tau_m y(t)-\int_{t-\tau(t)}^{t} y(s)ds-\int_{t-\tau_m}^{t-\tau(t)} y(s)ds\right). \qquad (20)$$

According to (6)-(8), it implies that, for any diagonal matrices $T_i = diag\{t_{i1}, t_{i2}, \ldots, t_{in}\} > 0$, $i = 1, 2, \ldots, 5$, the following inequality holds:

$$\begin{aligned}
0 \le & -2[g(y(t)) - L_1 y(t)]^T T_1 [g(y(t)) - L_2 y(t)] \\
& - 2[g(y(t-\tau(t))) - L_1 y(t-\tau(t))]^T \\
& \times T_2 [g(y(t-\tau(t))) - L_2 y(t-\tau(t))] \\
& - 2[g(y(t-\tau_m)) - L_1 y(t-\tau_m)]^T \\
& \times T_3 [g(y(t-\tau_m)) - L_2 y(t-\tau_m)] \\
& -2[g(y(t)) - g(y(t-\tau(t))) \\
& - L_1(y(t) - y(t-\tau(t)))]^T \\
& \times T_4 [g(y(t)) - g(y(t-\tau(t))) \\
& - L_2(y(t) - y(t-\tau(t)))] \\
& -2[g(y(t-\tau(t))) - g(y(t-\tau_m)) \\
& - L_1(y(t-\tau(t)) - y(t-\tau_m))]^T \\
& \times T_5 [g(y(t-\tau(t))) - g(y(t-\tau_m)) \\
& \quad - L_2(y(t-\tau(t)) - y(t-\tau_m))]. \qquad (21)
\end{aligned}$$

By combination of the concerned neural network (5) and free-weighting matrix method, the following equation is true for any matrices N_i $(i = 1, 2 \cdots, 11)$

$$\begin{aligned}
0 = & 2[y^T(t)N_1 + y^T(t-\tau(t))N_2 + y^T(t-\tau_m)N_3 \\
& + g^T(y(t))N_4 + g^T(y(t-\tau(t)))N_5 \\
& + g^T(y(t-\tau_m))N_6 + (\int_{t-\tau(t)}^{t} y(s)ds)^T N_7 \\
& + (\int_{t-\tau_m}^{t-\tau(t)} y(s)ds)^T N_8 + (\int_{t-\tau(t)}^{t} g(y(s))ds)^T N_9 \\
& + (\int_{t-\tau_m}^{t-\tau(t)} g(y(s))ds)^T N_{10} + \dot{y}^T(t)N_{11}] \\
& \times[-Cy(t) + Ag(y(t)) + Bg(y(t-\tau(t))) - \dot{y}(t)]. \\
& \qquad (22)
\end{aligned}$$

Finally, by combining (15)-(22), one can derive that

$$\dot{V}(t) \le \xi^T(t)\Sigma\xi(t). \qquad (23)$$

If $\Sigma < 0$, then $\dot{V}(t) < 0$ for $\forall \xi(t) \ne 0$, the concerned system (5) is globally asymptotically stable. This completes the proof.

Remark 1: Recently, the extended reciprocally convex combination approach to reduce the conservatism of the stability criteria for recurrent neural networks with time-varying delays is used in Wang, et al. (2015). Motivated by this idea, the approach is applied in this paper, which is shown in (19) and has potential to yield less conservative condition.

Remark 2: From the perspective of a computational burden, the stability criterion obtained in this paper has lower computational burden than the criteria obtained by the delay-decomposition LKF in Ge, et al. (2014) and Zeng, et al. (2011) and delay-partitioning LKF in Wang, et al. (2015) and Lakshmanan, et al. (2013), for the reason that the number of decision matrix variables is greatly reduced. Similarly, compared with the recent constructions of augmented LKF in Kwon, et al. (2013), Zheng, et al. (2010), Kwon, et al. (2013), Rakkiyappan, et al. (2016), and Yang, et al. (2017), the main difference is that the delayed state derivative terms are not considered in the constructed LKF in this paper, which greatly reduces the dimensions and computational burden of the stability criterion. Moreover, in Kwon, et al. (2013), Zheng, et al. (2010), Kwon, et al. (2013), and Rakkiyappan, et al. (2016), the free-weighting matrix method was not employed when they established the delay-dependent stability criteria. Therefore, the computational burden of the delay-dependent stability criterion obtained in this paper is lower.

Remark 3: In Theorem 1, the free-weighting matrix method used in (22) plays an important role in reducing the conservatism of stability criterion via incorporating the system model (5), in which many free weighting matrices are involved. Therefore, the obtained criterion will have better results than the existing ones.

Remark 4: The stability criteria derived in this paper give the matrix variables to be determined and the LMI-based constraint conditions for ensuring the stability of delayed neural networks; one can use the **feasp** function in MATLAB/LMI toolbox to solve those variables from the corresponding criterion.

4 NUMERICAL EXAMPLES

IN this section, two numerical examples are considered to verify the effectiveness of the obtained stability criterion. The main objective is to derive an acceptable maximum upper bound (AMUB) on time delays such that delayed neural networks are globally asymptotically stable. Meanwhile, the larger the AMUB is, there is less conservatism of the corresponding stability criterion.

Example 1: Consider the following 4-neuron delayed neural network (5) with the parameters (Zhang, et al. 2014)

$$C = \text{diag}\{1.2769, 0.6231, 0.9230, 0.4480\}$$

$$A=\begin{bmatrix}-0.0373 & 0.4852 & -0.3351 & 0.2336\\ -1.6033 & 0.5988 & -0.3224 & 1.2352\\ 0.3394 & -0.0860 & -0.3824 & -0.5785\\ -0.1311 & 0.3253 & -0.9534 & -0.5015\end{bmatrix}$$

$$B=\begin{bmatrix}0.8674 & -1.2405 & -0.5325 & 0.0220\\ 0.0474 & -0.9164 & 0.0360 & 0.9816\\ 1.8495 & 2.6117 & -0.3788 & 0.8428\\ -2.0413 & 0.5179 & 1.1734 & -0.2775\end{bmatrix}$$

$\sigma_1^+ = 0.1137$, $\sigma_2^+ = 0.1279$, $\sigma_3^+ = 0.7994$, $\sigma_4^+ = 0.2368$.

First case: $\sigma_i^- = 0,\ i = 1, 2, \ldots, 4$. The comparison results on the AMUB of time-varying delay via the different methods presented in recent works (Zhang & Han, 2011; Ge, et al. 2014; Wang, et al. 2015; Kwon, et al. 2013; Kwon, et al. 2013; Zeng, et al. 2015; Liu, et al. 2015; Zhang, et al. 2014; Zhang, et al. 2016; Zhang, et al. 2017; Yang, et al. 2017) are listed in Table 1. From Table 1, it can be seen that Theorem 1 provides larger AMUBs of time-varying delay than the existing results in the literatures, especially when $\mu \geq 0.5$, which sufficiently shows the advantage of the developed method in this paper.

Table 1. AMUBs τ_m for Various μ for Example 1 (First Case).

Methods\ μ	0.1	0.5	0.9
Zhang & Han 2011	3.5204	2.7167	2.2141
Ge, et al. 2014	3.8428	2.7081	2.2485
Zhang, et al. 2014	3.8739	2.7821	2.3279
Wang, et al. 2015	3.4886	2.6056	2.2522
Kwon, et al. 2013	3.7857	3.0546	2.6703
Zeng, et al. 2015	4.1903	3.0779	2.8268
Liu, et al. 2015	4.2143	3.1059	2.7494
Kwon, et al. 2013	3.8102	3.1518	2.8402
Zhang, et al. 2016	4.2993	3.1577	2.8371
Zhang, et al. 2017	4.2778	3.2152	2.9361
Yang, et al. 2017	4.4530	3.4929	3.0726
Theorem 1	4.6962	3.6220	2.7335

Table 2. AMUBs τ_m for Various μ for Example 1 (Second Case).

Methods\ μ	0.1	0.5	0.9
Zhang, et al. 2014	2.1321	1.3752	0.3602
Zhang, et al. 2014	2.1326	1.3759	0.3654
Zhang, et al. 2014	2.2019	1.4307	0.3767
Zhang, et al. 2014	2.2013	1.4307	0.3767
Liu, et al. 2015	3.0064	2.1112	1.6383
Theorem 1	3.1248	2.3432	1.6203

Second case:, $\sigma_1^- = -0.4$, $\sigma_2^- = 0.1$, $\sigma_3^- = 0$, $\sigma_4^- = -0.3$,. The AMUBs of time-varying delay for various μ obtained by Theorem 1 and the methods presented in Zhang, et al. (2014) and Liu, et al. (2015) are listed in Table 2. From Table 2, it can be found that the proposed method in Theorem 1 significantly enhances the feasible region of stability criterion compared to those results in Zhang, et al. (2014) and Liu, et al. (2015), especially when $\mu \geq 0.5$. When $\mu = 0.9$, the AMUBs are 0.3602, 0.3654 and 0.3767 in Zhang, et al. (2014). Applying Theorem 1, the AMUB is 1.6203, which is much better than those results in Zhang, He, Jiang, Wu, and Wu (2014).

When $\mu = 0.9$, the AMUB of time-varying delay is obtained by Theorem 1, $\tau_m = 2.7335$, and the initial values are randomly chosen as $[2, 5, -5, 2]^T$, the simulation result is shown in Figure 1. Obviously, the concerned neural network is globally asymptotically stable.

When $\mu = 0.5$, the AMUB of time-varying delay is obtained by Theorem 1, $\tau_m = 3.6220$, and the initial values are randomly chosen as $[0.2, 0.5, -0.5, 0.2]^T$, the simulation result is shown in Figure 2. Obviously, the concerned neural network is globally asymptotically stable.

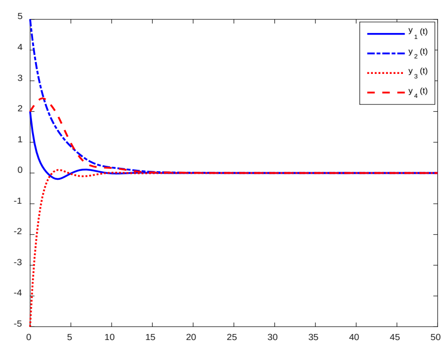

Figure 1. State Trajectories of $y_1(t), y_2(t), y_3(t), y_4(t)$ for Example 1 when $\mu = 0.9$, $\tau_m = 2.7335$.

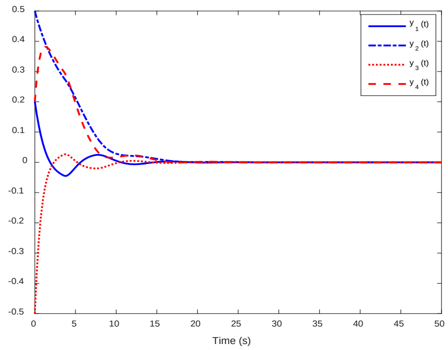

Figure 2. **State Trajectories of** $y_1(t), y_2(t), y_3(t), y_4(t)$ **for Example 1 when** $\mu = 0.5$, $\tau_m = 3.6220$.

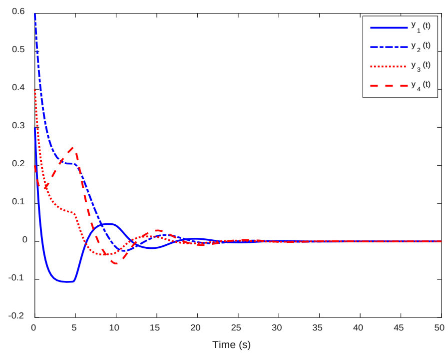

Figure 3. **State Trajectories of** $y_1(t), y_2(t), y_3(t), y_4(t)$ **for Example 1 when** $\mu = 0.1$, $\tau_m = 4.6962$.

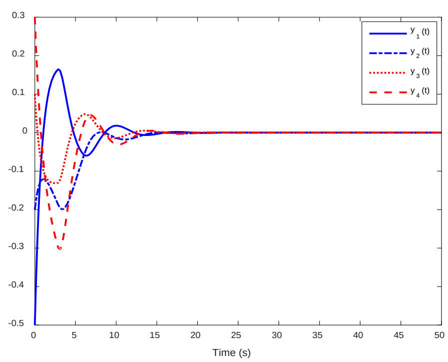

Figure 4. **State Trajectories of** $y_1(t), y_2(t), y_3(t), y_4(t)$ **for Example 1 when** $\mu = 0.8324$, $\tau_m = 2.8853$.

Similarly, when $\mu = 0.1$, the AMUB of time-varying delay is obtained by Theorem 1, $\tau_m = 4.6962$, and the initial values are randomly taken as $[0.3, 0.6, 0.4, 0.2]^T$, the simulation result is shown in Figure 3. Obviously, the concerned neural network is globally asymptotically stable.

Inspired by the application given in Wang, Liu, Shan and Zhang (2015), the application is utilized to further verify the effectiveness of the stability criteria obtained in this paper. When the time delay is $\tau(t) = 1.7637 + 0.3536\sin(0.4260t)$ $+ 0.7725\cos(0.5\sqrt{3}t)$, $\tau_m = 2.8853$, $\mu = 0.8324$, and the initial values are $[-0.5, -0.2, 0.1, 0.3]^T$, the simulation result is shown in Figure 4. Obviously, the concerned neural network is globally asymptotically stable. Furthermore, compared with the result reported in Wang, Liu, Shan and Zhang (2015) ($\tau_m = 2.2477$, $\mu = 0.8324$), the acceptable upper bound of time-varying delay in this paper is better.

Example 2. Consider the following 2-neuron delayed neural network (5) with the parameters (Zhang, et al. 2014)

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\sigma_1^+ = 0.4,\ \sigma_2^+ = 0.8,\ \sigma_i^- = 0,\ i = 1, 2.$$

When $\mu = 0.9$, the AUMBs τ_m of $\tau(t)$ obtained by Theorem 1 and different methods presented in Zhang, et al. (2013), Zhang, et al. (2014), Ge, et al. (2014), He, et al. (2016), Wang, et al. (2017), Kwon, et al. (2013) and Rakkiyappan, et al. (2016) are listed in Table 3. Seen from Table 3, it is obviously found that the result calculated based on the criterion in this paper is less conservative than the existing ones.

When $\mu = 0.9$, $\tau_m = 2.9158$, and the initial values are randomly chosen as $[-0.1,\ 0.1]^T$, the simulation result is shown in Figure 5. Obviously, the concerned neural network is globally asymptotically stable.

All above, the obtained stability criterion in this paper is much effective and less conservative than most of the existing results in the literatures.

Table 3. **AMUBs** τ_m **for Example 2.**

Methods	$\mu = 0.9$
Zhang, et al. 2013	1.6375
Ge, et al. 2014	1.9562
Zhang, et al. 2014	1.9603
He, et al. 2016	2.2201
Wang, et al. 2017	2.3582
Kwon, et al. 2013	2.8339
Rakkiyappan, et al. 2016	2.8881
Theorem 1	2.9158

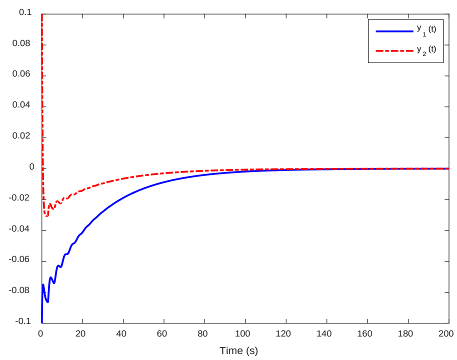

Figure 5. State Trajectories of $y_1(t), y_2(t)$ for Example 2 when $\mu = 0.9$, $\tau_m = 2.9158$.

5 CONCLUSION

THIS paper has studied the delay-dependent stability for continuous recurrent neural networks with time-varying delay and lower computational burden. A new and less conservative delay-dependent stability criterion has been established by constructing a newly augmented LKF and employing LMI method. It has been proven that the obtained stability criterion has a lower computational burden. The new LKF that considers more information of the slope of the activation functions has been further developed. Finally, two numerical examples have been considered to demonstrate the effectiveness of the proposed stability criterion.

6 ACKNOWLEDEMENT

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8 NOTES ON CONTRIBUTORS

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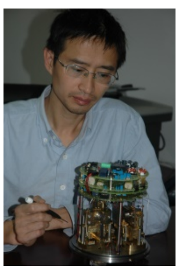

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APPENDIX

S with appropriate dimensions, then the following LMIs hold:

$$\Sigma < 0,$$

$$\begin{bmatrix} G & S \\ * & G \end{bmatrix} \geq 0,$$

Where

$$\Sigma = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \cdots & \Phi_{1,11} \\ * & \Phi_{22} & \Phi_{23} & \cdots & \Phi_{2,11} \\ * & * & \Phi_{33} & \cdots & \Phi_{3,11} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \Phi_{11,11} \end{bmatrix},$$

$$\Phi_{11} = 2M_{12} + P_{11} + \tau_m^2 G_{11} - G_{33} - \tau_m^2 K_1 - 2L_1^T(T_1 + T_4)L_2 - 2N_1C,$$

$$\Phi_{12} = G_{33} - S_{33} + 2L_1^T T_4 L_2 - C^T N_2^T,$$

$$\Phi_{13} = -M_{12} + S_{33} - C^T N_3^T,$$

$$\Phi_{14} = M_{13} + P_{12} + \tau_m^2 G_{12} + (L_1^T + L_2^T)(T_1 + T_4) + N_1 A - C^T N_4^T,$$

$$\Phi_{15} = -(L_1^T + L_2^T)T_4 + N_1 B - C^T N_5^T,$$

$$\Phi_{16} = -M_{13} - C^T N_6^T,$$

$$\Phi_{17} = M_{22} - G_{13}^T + \tau_m K_1 - C^T N_7^T,$$

$$\Phi_{18} = M_{22} - S_{31} + \tau_m K_1 - C^T N_8^T,$$

$$\Phi_{19} = M_{23} - G_{23}^T - C^T N_9^T,$$

$$\Phi_{1,10} = M_{23} - S_{32} - C^T N_{10}^T,$$

$$\Phi_{1,11} = M_{11} - L_1^T H_1 + L_2^T H_2 + \tau_m^2 G_{13} - N_1 - C^T N_{11}^T,$$

$$\Phi_{22} = -(1-\mu)P_{11} - 2G_{33} + 2S_{33} - 2L_1^T(T_2 + T_4 + T_5)L_2,$$

$$\Phi_{23} = -S_{33} + G_{33} + 2L_1^T T_5 L_2,$$

$$\Phi_{24} = -(L_1^T + L_2^T)T_4 + N_2 A,$$

$$\Phi_{25} = -(1-\mu)P_{12} + (L_1^T + L_2^T)(T_2 + T_4 + T_5) + N_2 B,$$

$$\Phi_{26} = -(L_1^T + L_2^T)T_5,$$

$$\Phi_{27} = G_{13}^T - S_{13}^T,$$

$$\Phi_{28} = S_{31} - G_{13}^T,$$

$$\Phi_{29} = G_{23}^T - S_{23}^T,$$

$$\Phi_{2,10} = S_{32} - G_{23}^T,$$

$$\Phi_{2,11} = -N_2,$$

$$\Phi_{33} = -G_{33} - 2L_1^T(T_3 + T_5)L_2,$$

$$\Phi_{34} = N_3 A,$$

$$\Phi_{35} = -(L_1^T + L_2^T)T_5 + N_3 B,$$

$$\Phi_{36} = (L_1^T + L_2^T)(T_3 + T_5),$$

$$\Phi_{37} = -M_{22} + S_{13}^T,$$

$$\Phi_{38} = -M_{22} + G_{13}^T,$$

$$\Phi_{39} = -M_{23} + S_{23}^T,$$

$\Phi_{3,10} = -M_{23} + G_{23}^T$,
$\Phi_{3,11} = -N_3$,
$\Phi_{44} = P_{22} + \tau_m^2 G_{22} - 2T_1 - 2T_4 + 2N_4 A$,
$\Phi_{45} = 2T_4 + N_4 B + A^T N_5^T$,
$\Phi_{46} = A^T N_6^T$,
$\Phi_{47} = M_{23}^T + A^T N_7^T$,
$\Phi_{48} = M_{23}^T + A^T N_8^T$,
$\Phi_{49} = M_{33} + A^T N_9^T$,
$\Phi_{4,10} = M_{33} + A^T N_{10}^T$,
$\Phi_{4,11} = H_1 - H_2 + \tau_m^2 G_{23} - N_4 + A^T N_{11}^T$,
$\Phi_{55} = -(1-\mu) P_{22} - 2T_2 - 2T_4 - 2T_5 + 2N_5 B$,
$\Phi_{56} = 2T_5 + B^T N_6^T$,
$\Phi_{57} = B^T N_7^T$,
$\Phi_{58} = B^T N_8^T$,
$\Phi_{59} = B^T N_9^T$,
$\Phi_{5,10} = B^T N_{10}^T$,
$\Phi_{5,11} = -N_5 + B^T N_{11}^T$,
$\Phi_{66} = -2T_3 - 2T_5$,
$\Phi_{67} = -M_{23}^T$,
$\Phi_{68} = -M_{23}^T$,
$\Phi_{69} = -M_{33}$,
$\Phi_{6,10} = -M_{33}$,
$\Phi_{6,11} = -N_6$,
$\Phi_{77} = -G_{11} - K_1$,
$\Phi_{78} = -S_{11} - K_1$,
$\Phi_{79} = -G_{12}$,
$\Phi_{7,10} = -S_{12}$,
$\Phi_{7,11} = M_{12}^T - N_7$,
$\Phi_{88} = -G_{11} - K_1$,
$\Phi_{89} = -S_{21}^T$,
$\Phi_{8,10} = -G_{12}$,
$\Phi_{8,11} = M_{12}^T - N_8$,
$\Phi_{99} = -G_{22}$,
$\Phi_{9,10} = -S_{22}$,
$\Phi_{9,11} = M_{13}^T - N_9$,
$\Phi_{10,10} = -G_{22}$,
$\Phi_{10,11} = M_{13}^T - N_{10}$,
$\Phi_{11,11} = \tau_m^2 G_{33} + (\tau_m^2/2)^2 K_1 - 2N_{11}$.

Proof. Consider the following candidate for the LKF:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \quad (14)$$

Where

$$V_1(t) = \begin{bmatrix} y(t) \\ \int_{t-\tau_m}^{t} y(s)ds \\ \int_{t-\tau_m}^{t} g(y(s))ds \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ * & M_{22} & M_{23} \\ * & * & M_{33} \end{bmatrix} \times \begin{bmatrix} y(t) \\ \int_{t-\tau_m}^{t} y(s)ds \\ \int_{t-\tau_m}^{t} g(y(s))ds \end{bmatrix},$$

$$V_2(t) = \int_{t-\tau(t)}^{t} \begin{bmatrix} y(s) \\ g(y(s)) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \begin{bmatrix} y(s) \\ g(y(s)) \end{bmatrix} ds,$$

$$V_3(t) = 2\sum_{i=1}^{n} \int_0^{y_i(t)} [h_{1i}(g_i(s) - \sigma_i^- s) + h_{2i}(\sigma_i^+ s - g_i(s))]ds,$$

$$V_4(t) = \tau_m \int_{t-\tau_m}^{t} \int_s^t \begin{bmatrix} y(u) \\ g(y(u)) \\ \dot{y}(u) \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ * & G_{22} & G_{23} \\ * & * & G_{33} \end{bmatrix} \times \begin{bmatrix} y(u) \\ g(y(u)) \\ \dot{y}(u) \end{bmatrix} duds,$$

$$V_5(t) = (\tau_m^2/2) \int_{t-\tau_m}^{t} \int_s^t \int_u^t \dot{y}^T(v) K_1 \dot{y}(v) dvduds,$$

$$\xi^T(t) = [y^T(t), y^T(t-\tau(t)), y^T(t-\tau_m), \times g^T(y(t)), g^T(y(t-\tau(t))), \times g^T(y(t-\tau_m)), \int_{t-\tau(t)}^{t} y^T(s)ds, \times \int_{t-\tau_m}^{t-\tau(t)} y^T(s)ds, \int_{t-\tau(t)}^{t} g^T(y(s))ds, \times \int_{t-\tau_m}^{t-\tau(t)} g^T(y(s))ds, \dot{y}^T(t)].$$