INTELLIGENT AUTOMATION & SOFT COMPUTING, 2017
https://doi.org/10.1080/10798587.2017.1330311

Introduction to U-Number Calculus

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ABSTRACT
Commonsense reasoning plays a pivotal role in the development of intelligent systems for decision-making, system analysis, control and other applications. As Prof. L. Zadeh mentions a kernel of the theory of commonsense is the concept of usuality. Zadeh suggested main principles of the theory of usuality, unfortunately up to present day; a fundamental and systemic approach to reasoning with usual knowledge is not developed.
In this study, we develop a new approach to calculus of usual numbers (U-numbers). We consider a U-number as a Z-number, where the second component is "usually". Validity of the suggested approach is verified by examples.

KEYWORDS
Discrete fuzzy number; convolution; usuality; Z-number; U-number

1. Introduction

The theory of usuality should be considered as a basis of decision analysis, system analysis and control in problems where commonsense knowledge plays an important role. As a rule, this knowledge is imprecise, incomplete, and partially reliable. The concept of usuality is naturally characterized by bimodal information. Formally, it may be handled by possibilistic-probabilistic constraints like

$$X \text{ is } u\ A \quad or \quad Usuality(r = u),$$

Where X is the constrained variable and A is a constraining relation. The usuality constraint presupposes that X is a random variable and the probability that X *isu* A is "usually":

$Prob\{X \text{ is } A\}$; *is usually*, or $Prob\{X \text{ is } A\}$ *is* B

Where A is a usual value of X, e.g. A is "*small*"; B is modality of a generalized constraint, for example, "*almost always*".

In the proposed study, usuality is considered as a special case of a Z-number valued information, where the second component is "usually", and is referred to as *U-number*. Humans mainly use U-numbers in everyday commonsense reasoning. Thus, calculus of U-numbers should be rather approximate than exact.

In Zadeh (1983, 1984a, b) Zadeh has suggested main principles of the theory of usuality. In Zadeh (1985) the author shows that the concept of dispositionality is closely related to the notion of usuality. Theory of usuality is defined as a tool for computational framework for commonsense reasoning.

In Zadeh (1996) the author outlines a theory of usuality, which is on representing the meaning of usuality-qualified propositions. A system of inference for usuality-qualified propositions is developed. Yager (1986) introduces a formal mechanism for representing and manipulating of usual values. This mechanism is based on a combination of the linguistic variables and Shafer evidential structures (Shafer, 1976). In Whalen & Schott (1992) the authors analyze the concept usuality, regularity and dispositional reasoning from the point of view of approximate reasoning. Schwartz (2010) discusses fuzzy quantifiers, fuzzy usuality modifiers and fuzzy likelihood modifiers. He analyzes these notions with unified semantics.

The main conclusion stemming from the review of the works mentioned above is that research in the scope of U-number calculus in existing literature is very scarce. In this study we develop a new approach to calculus of U-numbers.

The rest of the paper is structured as follows: In Section 2 we present some prerequisite material to be used in the sequel. In Section 3 we present some arithmetic and algebraic operations on U-numbers. In Section 4 we consider approximate reasoning with usual information. In Section 5 we provide several examples to illustrate the application of the proposed approach and provide comparative analysis. Section 6 concludes.

2. Preliminaries

Definition 1. Arithmetic operations over random variables (Charles, Grinstead, & Snell, 1997; Springer, 1979; Williamson & Downs, 1990). Let X_1 and X_2 be two independent continuous random variables with pdfs p_1 and p_2. A pdf p_{12} of $X_{12} = X_1 * X_2$, where $*$ is a two-place operation, is referred to as a convolution of p_1 and p_2, and is defined as $p_{12}(x) = \iint\limits_{\Omega} p_1(x_1)p_2(x_2)dx_1dx_2$, $\Omega = \{(x_1, x_2)|x = x_1 * x_2\}$.

Let X_1 and X_2 be two independent discrete random variables with the corresponding outcome spaces $X_1 = \{x_{11} \ldots x_{1i} \ldots x_{1n_1}\}$ and $X_2 = \{x_{21} \ldots x_{2i} \ldots x_{2n_2}\}$ and the corresponding discrete probability distributions p_1 and p_2. The probability distribution of $X_1 * X_2$, $* \in \{+, -, \cdot, /\}$, comes as the convolution $p_{12} = p_1 \circ p_2$ of p_1 and p_2, which is defined as follows:

$$p_{12}(x) = \sum_{x = x_1 * x_2} p_1(x_1)p_2(x_2), x \in \{x_1 * x_2 | x_1 \in X_1, x_2 \in X_2\}$$

Definition 2. Probability measure of a fuzzy number (Pedrycz & Gomide, 2007; Zadeh, 1968). Let X be a continuous random variable with pdf p. Let A be a continuous fuzzy

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number describing a possibilistic restriction on values of X. A probability measure of A denoted $P(A)$ is defined as

$$P(A) = \int_{R} \mu_A(x)p(x)dx$$

For a discrete fuzzy number and a discrete probability distribution, the probability measure is defined as

$$P(A) = \sum_{i=1}^{n} \mu_A(x_i)p(x_i) = \mu_A(x_1)p(x_1) + \mu_A(x_2)p(x_2) + \cdots + \mu_A(x_n)p(x_n)$$

3. A general approach to computation with U-numbers

Let X be a random variable and A be a fuzzy number playing a role of fuzzy constraints on values that the random variable may take; X is A. The definition of a usual value of X may be expressed in terms of the probability distribution of X as follows (Zadeh, 1996). If $p(x_i)$ is the probability of X taking x_i as its value, then

$$Usually(X \text{ is } A) = \mu_{usually}\left(\sum_i p(x_i)\mu_A(x_i)\right) \quad (1)$$

Or

$$U = (X, A, B) = \mu_B\left(\sum_i p(x_i)\mu_A(x_i)\right) \quad (2)$$

A usual number describing, "usually, temperature in this city is medium" is shown Fig. 1.

As it was mentioned above, in Zadeh (1996), the author provided an outline for the theory of usuality, however, this topic requires further investigation. Needed is a more general approach for other usuality quantifiers. In this paper "usuality" will be a composite term characterized by fuzzy quantities as *always, usually, frequently / often, occasionally, seldom, almost never/rarely, never*. The codebook for "usuality" is shown in Fig. 2.

In Zadeh (1996), the author raised the following questions; "How can the usual values of two or more variables be combined? More concretely, if $X_{12} = X_1 + X_2$, and the usual values of X_1 andX_2 are given, what will be the usual value of X_{12}?", "How can we construct an inference system for reasoning with usuality-qualified propositions"?

Computation with U-numbers is related to usuality constraint propagation. Assume that X is a random variable taking values x_1, x_2 ... and p is probability distribution of X. The constraint propagation is as follows:

$$\frac{X \text{ isu } A}{\text{Prob}\{X \text{ is } B\} \text{ is } C},$$

$$X \text{ isu } A \rightarrow \text{Prob}\{X \text{ is } A\} \text{ is usually} \rightarrow \mu_{usually}\left(\int_R \mu_A(x)p(x)dx\right),$$

$\mu_C(y) = \sup_{p(x)}(\mu_{usually}(\int_R \mu_A(x)p(x)dx))$,
Subject to

$y = \int_R \mu_B(x)p(x)dx$.

First, consider computation with U-numbers according to basic two-place arithmetic operations +, -, ·, /.

Let $U_1 = (A_1, B_1)$ and $U_2 = (A_2, B_2)$ be U-numbers (B_1 and B_2 are fuzzy terms of the usuality codebook) describing values of random variables X_1 and X_2. Assume that it is needed to compute the result $U_{12} = (A_{12}, B_{12})$ of a two-place operation$* \in \{+, -, \cdot, /\}$: $U_{12} = U_1 * U_2$.

Consider the case of discretized version of components of usual numbers. The first stage is the computation of two-place operations $*$ of fuzzy numbers A_1 and A_2 on the basis of fuzzy arithmetic. For example, for sum $U_{12} = U_1 + U_2$ we have to calculate $A_{12} = A_1 + A_2$:

$$\mu_{A_1+A_2}(x) = \sup_{x_1}(\min\{\mu_{A_1}(x_1), \mu_{A_2}(x - x_1)\}).$$

The second stage involves step-by-step construction of B_{12} and is related to propagation of probabilistic restrictions. We realize that in U-numbers $U_1 = (A_1, B_1)$ and $U_2 = (A_2, B_2)$, the "true" probability distributions p_1 and p_2 are not exactly known. In contrast, the information available is represented by the fuzzy restrictions:

$$\sum_{k=1}^{n_1} \mu_{A_1}(x_{1k})p_1(x_{1k}) \text{ is } B_1, \sum_{k=1}^{n_2} \mu_{A_2}(x_{2k})p_2(x_{2k}) \text{ is } B_2,$$

Which are represented in terms of the membership functions as

$$\mu_{p_1}(p_1) = \mu_{B_1}\left(\sum_{k=1}^{n_1} \mu_{A_1}(x_{1k})p_1(x_{1k})\right), \mu_{p_2}(p_2) = \mu_{B_2}\left(\sum_{k=1}^{n_2} \mu_{A_2}(x_{2k})p_2(x_{2k})\right).$$

Given these fuzzy restrictions, extract probability distributions p_j, $j = 1, 2$ by solving the following goal linear programming problem:

$$c_1 v_1^l + c_2 v_2^l + \cdots + c_n v_n^l \rightarrow b_{jl} \quad (3)$$

Subject to

$$\left.\begin{array}{l} v_1^l + v_2^l + \cdots + v_n^l = 1 \\ v_1^l, v_2^l ... v_n^l \geq 0 \end{array}\right\} \quad (4)$$

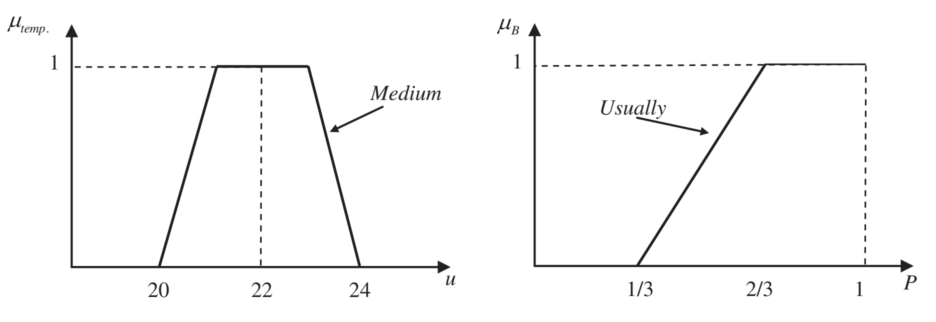


Figure 1. An Example of a U-number.

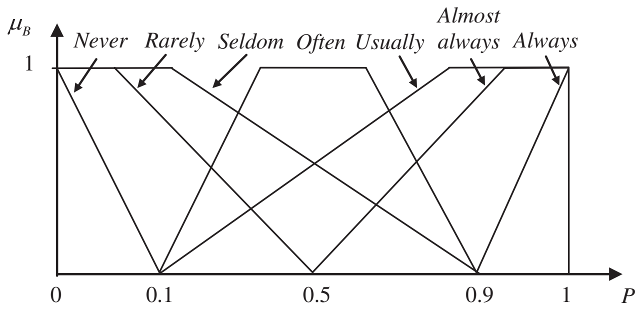


Figure 2. The Codebook of the Fuzzy Quantifiers of Usuality.

Where $c_k = \mu_{A_j}(x_{jk})$ and $v_k = p_j(x_{jk})$, $k = 1 \cdots .. n_j$, $k = 1 .. n_j$. As a result, $p_{jl}(x_{jk})$, $k = 1 \ldots n_j$ is found and, therefore, distribution p_{jl} is obtained. Thus, to construct the distributions p_{jl}, we need to solve m simple problems (3)-(4).

Distributions $p_{jl}(x_{jk})$, $k = 1 \ldots n_j$ naturally induce probabilistic uncertainty over the result $X_{12} = X_1 * X_2$. This is a critical point of computation of U-numbers, at which the issue of dependence between X_1 and X_2 should be considered. For simplicity, here we consider the case of independence between X_1 and X_2. This implies that given a pair p_{1l_1}, p_{2l_2}, the convolution $p_{12s} = p_{1l_1} \circ p_{2l_2}$, $s = 1 \ldots m^2$ is to be computed as on the basis of Definition 1.

For the case of dependence between X_1 and X_2, p_{12s} should be computed as a joint probability distribution by taking into account dependence between random variables (Williamson, 1989; Williamson & Downs, 1990; Wise & Henrion, 1986).

Given p_{12s}, the value of probability measure of A_{12}, $P(A_{12}) = \sum_{k=1}^{n} \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k})$ can be computed. However, the "true" p_{12s} is not exactly known as the "true" p_{1l_1}, p_{2l_2} are described by fuzzy restrictions. These fuzzy restrictions induce the fuzzy set of convolutions p_{12s}, $s = 1 \ldots m^2$ with the membership function defined as

$$\mu_{p_{12}}(p_{12s}) = \max_{p_{12s} = p_{1l_1} \circ p_{2l_2}} [\mu_{p_1}(p_{1l_1}) \wedge \mu_{p_2}(p_{2l_2})] \tag{5}$$

Subject to

$$\mu_{p_j}(p_{jl_j}) = \mu_{B_j}\left(\sum_{k=1}^{n_j} \mu_{A_j}(x_{jk}) p_{jl_j}(x_{jk}) \right), \ j = 1, 2 \tag{6}$$

Where $\wedge$ is *min* operation. The *min* operation is the simplest t-norm operation. In general, any t-norm operation can be used, but the use of a t-norm operation is context-dependent. For example, the use of the product t-norm (or another t-norm, which provides a smaller value than the *min* does) would produce a more strict constraint $\mu_{p_{12}}$. In turn, this will result in a more strict constraint $\mu_{B_{12}}$ (see below). In other words, this may reduce an entropy of resulted B_{12} and help to better interpret the result (one will have a more concentrated B_{12}). Indeed, as fuzzy restrictions over p_{1l_1}, p_{2l_2} are of a subjective basis, it may be helpful to reduce an effect of small membership degrees μ_{p_1} and μ_{p_2} on $\mu_{p_{12}}$. This can be done by using the product t-norm. Thus, the use of some t-norm operations may improve interpretability of results without sufficient loss of information.

As a result, fuzziness of information on p_{12s} described by Z_{12} induces fuzziness of the value of the probability measure $P(A_{12})$ in a form of a fuzzy number B_{12}. The membership function of B_{12} is defined as

$$\mu_{B_{12}}(b_{12s}) = \max(\mu_{p_{12}}(p_{12s})) \tag{7}$$

Subject to

$$b_{12s} = \sum_k \mu_{A_{12}}(x_k) p_{12s}(x_k) \tag{8}$$

As a result, $U_{12} = U_1 * U_2$ is obtained as $U_{12} = (A_{12}, B_{12})$.

Let us now consider one-place algebraic operations as a square and a square root of U-numbers.

Construction of $U = U_1^2$ is as follows: $A = A_1^2$ is determined as

$$A_1^2 = \bigcup_{\alpha \in [0,1]} \alpha [A_1^2]^\alpha, \tag{9}$$

$$[A_1^2]^\alpha = \{x_1^2 | x_1 \in A_1^\alpha\}. \tag{10}$$

The probability distribution p is determined given p_1 as (Aliev, Huseynov, Aliyev, & Alizadeh, 2015),

$$p(x) = \frac{1}{2\sqrt{x}} \left[p_1(\sqrt{x}) + p_1(-\sqrt{x}) \right], \ x \geq 0. \tag{11}$$

Next by noting that a "true" p_1 is not known, one has to consider fuzzy constraint μ_{p_1} to be constructed by solving a certain LP problem (3)-(4).The fuzzy set of probability distributions $p_{1,l}$ with membership function μ_{p_1} naturally induces the fuzzy set of probability distributions p_l with the membership function $\mu_p(p_l)$ defined as

$\mu_p(p_l) = \mu_{p_1}(p_{1l})$, l = 1\cdotsm

Where p is determined from p_1 based on (11).

The probability measure $P(A)$ given p is produced on basis of Definition 2. Finally, given a fuzzy restriction on p described by μ_p, we extend $P(A)$ to a fuzzy set B by solving a problem analogous to (7)-(8). As a result, U^2 is obtained on the basis of the extension principle for computation with U-numbers as $U^2 = (A, B)$. Let us mention that for $X_1 \geq 0$, it is not needed to compute B, because it is the same as B_1 (Aliev, Alizadeh, & Huseynov, 2015; Aliev et al., 2015). Computation of $U = U_1^n$, where n is any natural number, is carried out in an analogous fashion.

Let us consider computation of $U = \sqrt{U_1}$ based on the extension principle for computation with U-numbers. $A = \sqrt{A_1}$ is determined as follows:

$$\sqrt{A} = \bigcup_{\alpha \in [0,1]} \alpha [\sqrt{A_1}]^\alpha, \tag{12}$$

$$[\sqrt{A_1}]^\alpha = \{\sqrt{x_1} | x_1 \in A_1^\alpha\}. \tag{13}$$

The probability distribution p is determined given as (Aliev et al., 2015).

$$p(x) = 2x p_1(x^2). \tag{14}$$

Then we compute μ_p by solving problems (3)-(4) and recall that

$$\mu_p(p_l) = \mu_{p_1}(p_{1l}),$$

Where p is determined from p_1 on the basis of (14). Next we compute probability measure $P(A)$. Finally, given the membership function μ_p, we construct a fuzzy set B by solving a problem analogous to (7)-(8). Let us mention that for the square root of a U-number, it is not needed to carry out computation of B, because it is the same as $B_1 B_1$ (Aliev et al., 2015).

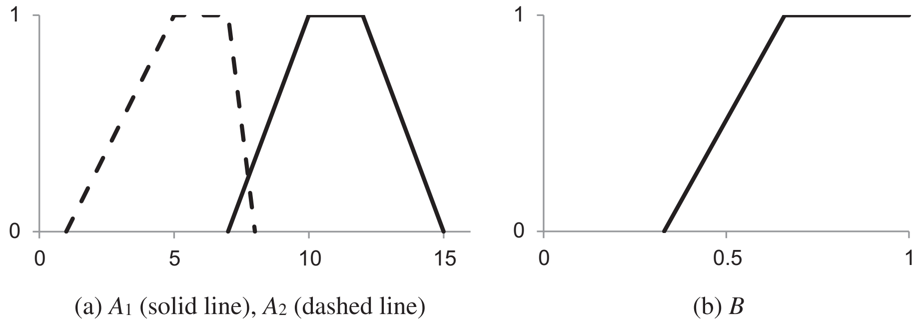


Figure 3. U-numbers $U_1=(A_1, B_1)$ and $U_2=(A_2, B_2)$.

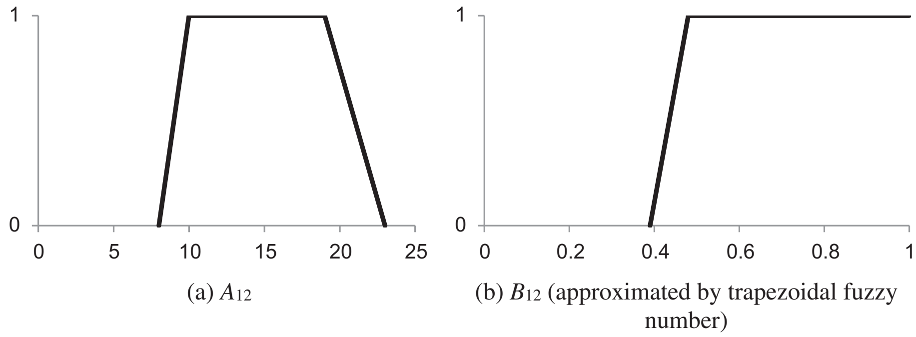


Figure 4. U-number $U_{12}=(A_{12}, B_{12})$.

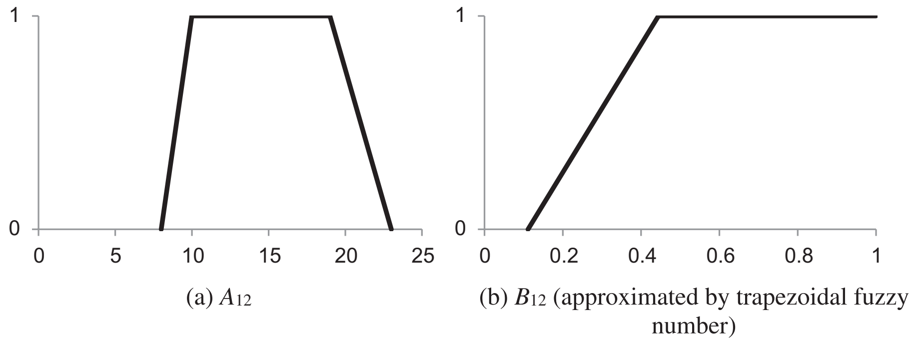


Figure 5. U-number U_{12}.

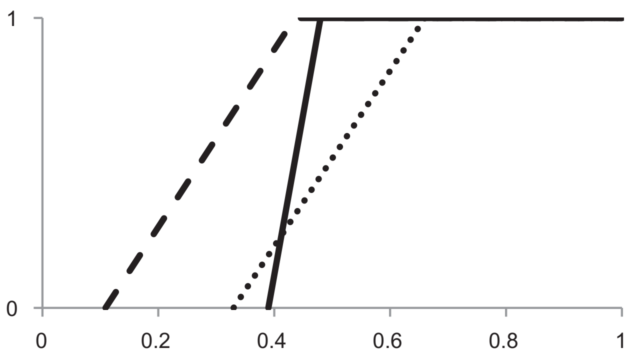


Figure 6. The Given Value B (Dotted Line), and the Computed Value B_{12}: B_{12} Computed by using the Proposed Approach (Dashed Line) and B_{12} Computed by Using the Yager 's Approach (Solid Line).

4. Approximate reasoning with usual information

The approximate reasoning can be considered as a formal model of commonsense knowledge-based reasoning with imprecise and uncertain information (Aliev, Alizadeh, & Guirimov, 2010; Aliev, Mamedova, & Aliev, 1993; Aliev, Pedrycz, & Huseynov, 2012). Approximate reasoning is based on fuzzy logic (Aliev, 1994; Aliev & Tserkovny, 2011; Jamshidi, 1997) and provides many successful applications in various fields (Aliev, 2013; Aliev & Memmedova, 2015).

The problem of approximate reasoning with usual information is stated as follows:

Given the following U-rules:

If X_1 is $U_{X_1,1} = (A_{X_1,1}, B_{X_1,1})$ and… and X_m is $U_{X_m,1} = (A_{X_m,1}, B_{X_m,1})$ then Y is $U_Y = (A_{Y,1}, B_{Y,1})$

If X_1 is $U_{X_1,2} = (A_{X_1,2}, B_{X_1,2})$ and… and X_m is $U_{X_m,2} = (A_{X_m,2}, B_{X_m,2})$ then Y is $U_Y = (A_{Y,2}, B_{Y,2})$

…

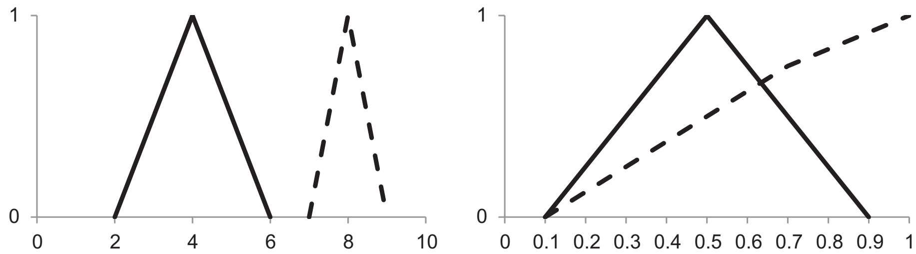
If X_1 is $U_{X_1,n} = (A_{X_1,n}, B_{X_1,n})$ and… and X_m is $U_{X_m,n} = (A_{X_m,n}, B_{X_m,n})$ then Y is $U_Y = (A_{Y,n}, B_{Y,n})$

And a current observation

X_1 is$U_{X_1} = (A'_{X_1}, B'_{X_1})$ and… and X_m is $U'_{X_m} = (A'_{X_m}, B'_{X_m})$,

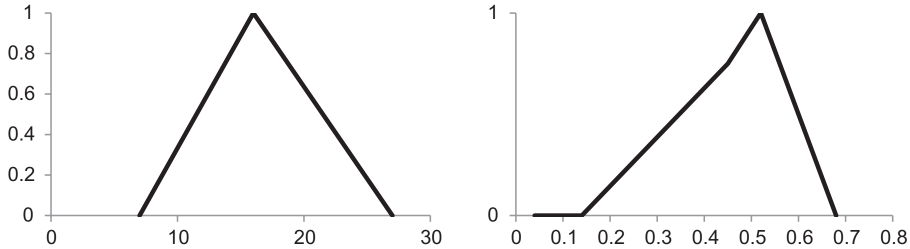
Find the U-value of Y.

The idea underlying the suggested interpolation approach is that the resulting output is computed as a convex combination of consequent parts. The coefficients of linear interpolation are determined on the basis of the similarity between a current input and antecedent parts (Kóczy & Hirota, 1991). This implies that the resulting output U'_Y is computed as



(a) A_1 (solid line), A_2 (dashed line) (b) B_1 (solid line), B_2 (dashed line)

Figure 7. U-numbers U_1=(A_1, B_1) and U_2=(A_2, B_2).



(a) A_{12} (approximated to triangular fuzzy number) (b) B_{12}

Figure 8. U-number U_{12}=(A_{12}, B_{12}).

$$U'_Y = \sum_{j=1}^{n} w_j U_{Y,j} = \sum_{j=1}^{n} w_j (A_{Y,j}, B_{Y,j}), \tag{15}$$

Where $U_{Y,j}$ is the U-valued consequent of the j-th rule, $w_j = \frac{\rho_j}{\sum_{k=1}^{n} \rho_k}$, $j = 1 \ldots n$; $k = 1 \ldots n$ are coefficients of linear interpolation, n is the number of U-rules, ρ_j is defined as follows:

$$\rho_j = \min_{i=1 \ldots m} S(U'_{X_i}, U_{X_i,j}), \tag{16}$$

Where S is the similarity between current i-th U-valued input and the i-th U-valued antecedent of the j-th rule. Thus, ρ_j computes the similarity between a current input vector and a vector of antecedents of j-th rule.

5. Examples and comparative analysis

Example 1. Addition of U-numbers. Let U-numbers $U_1 = (A_1, B_1)$ and $U_2 = (A_2, B_2)$ be given, where $B_1 = B_2 = B$ (Figure 3).

The U-number $U_{12}=U_1+U_2$ obtained by using the proposed approach is shown in Figure 4.

If we apply the bandwidth-based computation approach (Zadeh, 2011) as a special case of the proposed theory, the result is defined as $U_{12} \approx (A_1+A_2, B^2)$. This result is shown in Figure 5.

For comparison, the given value of reliability, B, and the results of computation of B_{12} (approximated by trapezoidal fuzzy number) are shown in Figure 6.

This example shows that Yager's approach to computation with usual numbers is a special case of the proposed theory, and is valid if 1st parts of U-numbers are non-fuzzy, particularly, intervals.

Example 2. Multiplication of U-numbers. Let U-numbers $U_1=(A_1, B_1)$ and $U_2=(A_2, B_2)$ be given (Figure 7.)

The result of computation of multiplication $U_{12}=U1{*}U2$ obtained by using the proposed approach (Section 3) is shown in Figure 8.

6. Conclusion

Prof. Zadeh suggested a theory of usuality, which plays a central role in common sense reasoning. However, this topic requires further investigation. It is needed to develop a more general approach for computation and approximate reasoning with usual information. Up to present day, no systematic approach is suggested to solve such problems of the usuality theory as a combination of usual values of two and more variables and reasoning with usuality-based IF-Then rules. In this study, we suggest a new approach to computation with U-numbers for modeling commonsense reasoning on the basis of usual information. The suggested basics of U-number calculus are illustrated by numerical examples. A comparative analysis shows effectiveness of the suggested approach as compared to existing works.

Acknowledgement

I would like to express my gratefulness to Prof. L.A. Zadeh for his outstanding idea to develop a theory of approximate arithmetic operations on U-numbers and for his valuable comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the author.

Notes on contributor

Rafik A. Aliev was born in Aghdam, Azerbaijan, 1942. He received Ph.D. and Doctorate degrees from the Institute of Control Problems, Moscow, Russia, in 1967 and 1975, respectively. His major fields of study are decision theory with imperfect information, arithmetic of Z-numbers, fuzzy logic, soft computing and control theory. He is a professor and the Head of the Department of the joint MBA Program between the Georgia State University (Atlanta, GA, USA) and the Azerbaijan State University of Oil and Industry (Baku, Azerbaijan), and a Visiting Professor with the University of Siegen, (Siegen, Germany) and with Near East University, (Nicosia, North Cyprus). He is also an invited speaker in Georgia State University (Atlanta, GA).

Dr. Aliev is a regular Chairman of the International Conferences on Applications of Fuzzy Systems and Soft Computing, International Conferences on Soft Computing and Computing with Words, and World Conferences on Intelligent Systems for Industrial Automation. He is an Editor of the Journal of Advanced Computational Intelligence and Intelligent Informatics (Japan), an Associate Editor of the Information Sciences journal, a member of the Advisory board of the International Journal of Information Technology and Decision Making, member of Editorial Boards of International Journal of Web-based Communities (The Netherlands), Iranian IEEE Systems journal, Journal of Fuzzy Systems (Iran), International Journal of Advances in Fuzzy Mathematics (Italy), and International Journal "Intelligent Automation and Soft Computing." He was awarded USSR State Prize in field of Science (1983), Lifetime Achievement Award (2014), and International Fuzzy Systems Association fuzzy fellow award (2015).

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