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Article

Analysis-Aware Modelling of Spacial Curve for Isogeometric Analysis of Timoshenko Beam

Yang Xia*, Luting Deng and Jian Zhao

School of Automotive Engineering, State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian, 116024, China
*Corresponding Author: Yang Xia. Email: yangxia@dlut.edu.cn
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Abstract: Geometric fitting based on discrete points to establish curve structures is an important problem in numerical modeling. The purpose of this paper is to investigate the geometric fitting method for curved beam structure from points, and to get high-quality parametric model for isogeometric analysis. A Timoshenko beam element is established for an initially curved spacial beam with arbitrary curvature. The approximation and interpolation methods to get parametric models of curves from given points are examined, and three strategies of parameterization, meaning the equally spaced method, the chord length method and the centripetal method are considered. The influences of the different geometric approximation algorithms on the precision of isogeometric analysis are examined. The static analysis and the modal analysis with the established parametric models are carried out. Three examples with different complexities, the quarter arc curved beam, the Tschirnhausen beam and the Archimedes spiral beam are examined. The results show that for the geometric approximation the interpolation method performs good and maintains high precision. The fitting algorithms are able to provide parametric models for isogeometric analysis of spacial beam with Timoshenko model. The equally spaced method and centripetal method perform better than the chord length method for the algorithm to carry out the parameterization for the sampling points.

Keywords: Analysis-aware modelling; curve fitting; Timoshenko beam; spatial curve; isogeometric analysis

1 Introduction

Isogeometric analysis (IGA) is proposed by Hughes et al. to bridge the gap between computer aided design and analysis, and has been proved to be effective in structural analysis, fluid mechanics, biomechanical analysis and microelectronics simulation [1,2]. Compared with the traditional finite element analysis (FEA) method, the IGA uses the computer aided design (CAD) model for simulation, avoiding the complicated meshing process and improving the calculation efficiency. However, similar to the requirement of quality of the mesh in the FEA, in the process of IGA there are also problems with the quality of the CAD model in geometric representation. The CAD model used in IGA is determined by

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the B-spline or NURBS basis functions and the control points. The parameterization of the basis functions is implicitly defined in the geometry, which directly affects the accuracy of simulation.

To characterize the impact of the model quality on analysis, Cohen et al. proposed the analysis-aware modelling concept within IGA to establish an analogous concept of mesh quality [3]. The compositions in the analysis-aware modelling that affect the results of simulation include the domain parameterization, knot spacing, the order and the distribution of control points. Researchers investigated deeply into this subject and obtained fruitful findings. Jaxon et al. investigated the influence of domain parameterization, knot spacing, mesh regularization, and bijective mapping to improve the performance of isogeometric analysis with various geometric models [4–7]. For two-dimensional (2D) problems, Collin et al. introduced the analysis-suitable G^1 geometry parameterizations and proved its optimal convergence property for bilinear two-patch geometries [8]. Nian et al. studied the parameterization of the plane domain for isogeometric analysis based on the Teichmuller mapping, which improved the accuracy of isogeometric analysis [9]. Gondegaon et al. proposed a method to map the 2D domain to a circle and then a square to build parameterization for tensor-product B-spline [10]. With their method models with complex shapes can be efficiently calculated, but domains with holes are not considered. In 2017, Wu et al. presented a method to represent complex planar domains with holes by Bi-cubic splines and obtain optimized results which are suitable for IGA [11]. For three-dimensional modelling, Martin et al. presented a methodology to parameterize a volumetric model based on discrete volumetric harmonic functions so that the resulting model is suitable for isogeometric analysis [12]. Chan et al. derived a PHT-spline representation of a domain from level set boundary representation [13]. Arioli et al. proposed a scaled boundary parameterizations for IGA which is suitable for domains with boundary description, and their method is able to deal with three-dimensional geometry [14].

Xu et al. carried out productive research works on the analysis-aware modelling for IGA. In 2011, Xu et al. studied the parameterization method in two-dimensional computational domain, and optimized the parameterization by repositioning the internal control points [15]. Based on the study of two-dimensional problems, Xu et al. further introduced the NURBS volume parameterization into IGA to analyze the multiple three-dimensional entities, which is of great importance to the industrial applications of IGA [16–18]. They proved that different parameterizations of a three-dimensional computational domain have different impacts on the simulation results and efficiency in IGA. By optimization of parameterization of the boundary before constructing the inner control points and weights, high-quality NURBS body parameters can be obtained, which is suitable for subsequent isogeometric analysis [19,20]. Xu et al. also proposed a computation reuse method to calculate the parameterization of three-dimensional by the help from simple models with similar semantic features [21].

Beam structures are widely used in engineering, and the simulations of beam structures with IGA are investigated by the researchers. Luu et al. used h-refinement, p-refinement, and k-refinement to refine the Timoshenko curved beam model to improve the calculation accuracy [22]. Ghafari et al. studied isogeometric analysis of composite beam using time-saving dimensional reduction method and got results with better convergence rate and more efficient refinement method than FEM [23]. Maurin et al. utilized the advantage of IGA in higher order inter-element continuity to simulate the pantographic lattice, and the energy terms of the structure which depend on the second-order derivatives of the displacements are efficiently calculated [24]. Rezaiee-Pajand et al. investigated the thermo-mechanical static response of curved circular beam and proved that the temperature distribution leads to both in-plane and out-of-plane deformations in beam [25]. Hosseini et al. explored the formulation of IGA Timoshenko beam with large deformations [26]. Numerical locking phenomena is an important topic in IGA analysis of Timoshenko beam. Liu et al. proposed the selective reduced one-point integration and B-bar projection element based on stiffness ratio to deal with the locking problem and obtain accurate results for slender models [27].

Faroughi et al. developed s displacement-based IGA beam element for laminated composite beams [28]. Ghafari et al. proposed a reduced beam model which is developed using cross-sectional properties from 2D beam sectional analysis and applied the model to isogeometric analysis of shear refined delaminated composite beams [29].

The anaylsis-aware modelling of beam structure is also important to improve the calculation accuracy by IGA. To get the model for beam, the spline curve fitting is an important method to get curve model from points in computer aided design and computational graphics. In CNC machining, the fitting with splines of the poly-line machining tool path can be used for smooth tool path generation and data compression [30]. To carry out the fitting the selection of the number and location of the knots and the calculation of the spline coefficients must be determined. Most existing spline curve fitting algorithms need the parameterization procedure to create a parameter value for each data point. A widely used parameterization method is to make sure that the point on the curve corresponding to the parameter is the nearest point on the curve to the target point, and generally an iteration method is used to calculate the parameterization. The point distance minimization (PDM) is the most widely used method [31], which minimize the square distance to get the parameterization. Imani et al. developed an algorithm to construct 2D profile based on NURBS parametric curve from ordered data point cloud [32]. Wang et al. proposed a curvature-based square distance minimization method for the curve fitting [33]. Optimization methods are used to calculate the number and positions of knots for the curve fitting [34]. Garcia-Capulin et al. propose a hierarchical genetic algorithm which allows the number and location of knots and the B-spline coefficients can be calculated automatically [35]. Pagani et al. proposed the curvature based sampling for curves and surfaces so higher density of sample points is assigned where there are some significant features [36]. While generally the curve fitting is challenging to use optimization method for the parameterization.

Hosseini et al. investigated deeply into the anaylsis-aware modelling of beam structure. Hosseini et al. investigated the effect of parameterization on the results of isogeometric analysis with Euler-Bernoulli beam formulation and proved that the chord length and centripetal methods lead to a less least square error [37]. They used the pseudo-arc length reparameterization method to parameterize the existing parametric model to improve the accuracy of isogeometric simulation [38]. In 2018, Hosseini et al. developed a semi-analytical sensitivity analysis method within IGA framework to solve pre-bent shape design problems in free-form curved beams [39]. Hashemian et al. proposed to apply the fitting and fairing simultaneously in a multi-objective optimization process to reconstruct curves and surfaces [40]. Recently Hashemian et al. investigated the nonlinear bifurcation analysis of statically loaded free-form curved beams and investigated the different knot placement techniques and their influence on the accuracy of simulation [41–43]. Despite their extensive research on the analysis-aware modelling of beam structure, the influence of geometric modelling to an important beam model, which is the Timoshenko beam, is not considered, and the influence of different fitting algorithms are not investigated neither.

The objective of this paper is to focus on the modeling process of curves from a series of sampling points, and to obtain analysis-suitable parametric models for the isogeometric analysis of Timoshenko beam structures. The numerical locking problem is not considered here to allow us concentrate on the analysis-suitable modelling of beam structure, and in the numerical tests slender beams are avoided. To carry out the curve fitting, the parameterization is obtained by the chord length method, the centripetal method and the equal method [44]. Then with the obtained parameters, the curve fitting problem is reduced to a minimization of the quadratic model and the control points can be solved. The influence of the different geometric approximation algorithms on the precision of isogeometric analysis is examined. Compared with previous works, the present paper makes contributions in the following items:

- The analysis-aware modelling for structural static and dynamic simulations of spatial Timoshenko beam structure is proposed.
- The approximation and interpolation methods of curve fitting are examined to investigate their influence on the IGA.
- The effects of three parameterization strategies on the IGA formulation of spatial Timoshenko beam are studied and compared.

The main contents of this paper are outlined as follows. Sections 2 and 3 provide a brief introduction to the B-spline curve and the interpolation and approximation methods. Section 4 introduces the isogeometric analysis algorithm of the spatial Timoshenko beam structure. The numerical tests and discussions are given in Section 5. Finally, in Section 6 the conclusions are given.

2 Parametric Representation of Curve

2.1 B-spline Curves

B-spline is widely used in the representation of free-form geometries. It is well consistent with the commercial CAD softwares and employed widely in engineering applications. In this paper we apply B-spline curves in IGA. The B-spline basis function is defined by the Cox-de Boor recursion formula. For example, the B-spline basis function $N_{i,0}$ with degree $p = 0$ is defined as

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi \in [\xi_i, \xi_{i+1}) \\ 0 & otherwise \end{cases} \tag{1}$$

The ξ_i is the component of the knot vector Ξ, which is a series of non-decending real numbers. An open knot vector is defined with the multiplicity at each end is $p + 1$. For example, a typical open knot vector can be defined as

$$\Xi = \left[\underbrace{0,0,\cdots,0}_{p+1}, \xi_{p+1},\xi_{p+2},\cdots,\xi_n, \underbrace{1,1,\cdots,1}_{p+1}\right] \tag{2}$$

B-spline basis function with higher degree can be defined as,

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \tag{3}$$

A B-spline curve of degree p is expressed with the defined basis function and control point.

$$\mathbf{r}(\xi) = \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{N_{i,p}}(\xi)\mathrm{P}_{\mathbf{i}} \tag{4}$$

$\mathrm{P}_{\mathbf{i}}$ is the control point. With the defined basis functions and $n + 1$ control points, the B-spline curve can be determined.

2.2 Curve Fitting with B-spline Curves for Isogeometric Analysis

Curve fitting with B-splines is a traditional and important problem in engineering applications. Here we describe the problem of curve fitting as follows: given a series of sample points from a given beam structure $\mathbf{S} = (x_j, y_j)$, $j = 0,\ldots, m$, to get the parametric form of the curve

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i \quad u \in [0,1] \tag{5}$$

where $\mathbf{C}(u)$ is the unknown parametric B-spline curve, $N_{i,p}(u)$ represents the basis function defined by the knot vector, p is the degree of the basis function and $\mathbf{P}_i$ represents the control point.

The focus of this paper is put on the problem to obtain a geometric model of curve which is suitable for simulation from a set of given data points. To solve the curve fitting problem, the first issue is the choice of the sample points **S** from the given curve. The given curve can be described by analytical function, or chosen from engineering application which do not have explicit mathematical description. The second issue is the choice of the knot vector Ξ and to estimate the position of control points **P**. Then the required B-spline curve can be described by the determined knot vector and the control points. In this paper we will investigate different methods for curve fitting and consider the requirements of the IGA on the parametrization of model.

3 Algorithms for B-spline Fitting

The B-spline curve fitting involves the following problems. First, the parameters corresponding to the given sample points need to be selected. Then the knot vector should be chosen, and finally the control points should be calculated. In this section we consider both interpolation and approximation methods.

3.1 Sample Points and the Corresponding Parameters

We consider that the sampling points are pre-defined and denoted as $\mathbf{Q}_k, k = 0, \ldots, m$. m represents the total count of sampling points minus 1. To carry out the curve fitting, each parameter must be set corresponding to each sampling point. Three methods for determining the parameter values are applied, namely the equidistance method, the chord length and the centripetal method. The parameters are denoted as $\bar{u}$. In the equally space method, the parameters are equally spaced among the parameter domain [0,1].

$$\begin{aligned} &\bar{u}_0 = 0 \quad \bar{u}_m = 1 \\ &\bar{u}_k = \frac{k}{m} \quad k = 1, \cdots, m-1 \end{aligned} \tag{6}$$

In the chord length method the parameters are calculated as

$$\begin{aligned} &\bar{u}_0 = 0 && \bar{u}_m = 1 \\ &\bar{u}_k = \bar{u}_{k-1} + \frac{|\mathbf{Q}_k - \mathbf{Q}_{k-1}|}{d_1} && k = 1, \cdots, m-1 \end{aligned} \tag{7}$$

The d_1 represents the total length of chords connected by the sampling points.

$$d_1 = \sum_{k=1}^{n} |\mathbf{Q}_k - \mathbf{Q}_{k-1}| \tag{8}$$

In the centripetal method parameters are calculated as,

$$\begin{aligned} &\bar{u}_0 = 0 && \bar{u}_m = 1 \\ &\bar{u}_k = \bar{u}_{k-1} + \frac{\sqrt{|\mathbf{Q}_k - \mathbf{Q}_{k-1}|}}{d_2} && k = 1, \cdots, m-1 \end{aligned} \tag{9}$$

where

$$d_2 = \sum_{k=1}^{n} \sqrt{|\mathbf{Q}_k - \mathbf{Q}_{k-1}|} \tag{10}$$

3.2 Knot Vector Generation

The knot vector has important influence on the results of fitting. The following algorithms are applied for curve interpolation to point data. The knots are defined as,

$$\begin{aligned} &u_0 = \cdots = u_p = 0 \quad u_{n-p} = \cdots = u_n = 1 \\ &u_{j+p} = \frac{1}{p}\sum_{i=j}^{j+p-1} \bar{u}_i \qquad j = 1, \cdots, n-p \end{aligned} \tag{11}$$

n represents the number of knots minus 1.

In curve approximation method, the deBoor algorithm is applied to calculate the knot vectors [44]. First let

$$d = \frac{m+1}{n-p+1} \tag{12}$$

Then the internal knots are defined by

$$\begin{aligned} i &= \mathrm{int}(jd) & \alpha &= jd - i \\ u_{p+j} &= (1-\alpha)t_{i-1} + \alpha t_i & j &= 1, \cdots, n-p \end{aligned} \tag{13}$$

3.3 Fitting

The fitting algorithms are divided into the interpolation method and the approximation method. The B-spline expression of the curve is formulated as

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i \quad u \in [0, 1] \tag{14}$$

In interpolation method, to calculate the control points, the curve should interpolate all the sampling points, and $m = n - p + 1$. With the given sampling points $\mathbf{Q}$ and the corresponding parameters $\bar{\mathbf{u}}$, the interpolation condition can be described by the following equations,

$$\mathbf{Q}_k = \mathbf{C}(\bar{u}_k) = \sum_{i=0}^{n} N_{i,p}(\bar{u}_k)\mathbf{P}_i \tag{15}$$

The equations can be solved to get the control points.

The approximation of the curve is more complicated than the interpolation. In the approach method, the least square algorithm is used to minimize the error function

$$f = \sum_{k=0}^{m} |\mathbf{Q}_k - \mathbf{C}(\bar{u}_k)|^2 \tag{16}$$

In the method of curve approximation, the curve generally do not go through the sample points. To minimize the error function, the following equations must hold [44].

$$\left(\mathbf{N}^{\mathrm{T}}\mathbf{N}\right)\mathbf{P} = \mathbf{R} \tag{17}$$

In the equation,

$$\mathbf{N} = \begin{bmatrix} N_{1,p}(\bar{u}_1) & \cdots & N_{n-1,p}(\bar{u}_1) \\ \vdots & \ddots & \vdots \\ N_{1,p}(\bar{u}_{m-1}) & \cdots & N_{n-1,p}(\bar{u}_{m-1}) \end{bmatrix} \tag{18}$$

$$\mathbf{P} = [\mathbf{P}_1 \quad \cdots \quad \mathbf{P}_{n-1}]^{\mathrm{T}} \tag{19}$$

$$\mathbf{R} = \begin{bmatrix} N_{1,p}(\bar{u}_1)\mathbf{R}_1 + \cdots + N_{1,p}(\bar{u}_{m-1})\mathbf{R}_{m-1} \\ \vdots \\ N_{n-1,p}(\bar{u}_1)\mathbf{R}_1 + \cdots + N_{n-1,p}(\bar{u}_{m-1})\mathbf{R}_{m-1} \end{bmatrix} \tag{20}$$

$$\mathbf{R}_k = \mathbf{Q}_k - N_{0,p}(\bar{u}_k)\mathbf{Q}_0 - N_{n,p}(\bar{u}_k)\mathbf{Q}_m \quad k = 1, \cdots, m-1 \tag{21}$$

Here **P** represents the coordinates of control points and can be calculated.

4 Isogeometric Analysis of Timoshenko Beam

In this section we briefly introduce the isogeometric analysis algorithm for static and modal analysis of Timoshenko beam structures. To describe the geometry of beam structure, two sets of coordinate systems, which are the local and global coordinate systems are established. The local coordinate system is defined according to the Frenet curvilinear system as,

$$\begin{bmatrix} \mathbf{t}(u) \\ \mathbf{n}(u) \\ \mathbf{b}(u) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{C}}(u)/|\dot{\mathbf{C}}(u)| \\ \mathbf{b}(u) \times \mathbf{t}(u) \\ (\dot{\mathbf{C}}(u) \times \ddot{\mathbf{C}}(u))/|\dot{\mathbf{C}}(u) \times \ddot{\mathbf{C}}(u)| \end{bmatrix} \tag{22}$$

According to the Timoshenko beam theory, the displacement within the beam structure can be described by the transit displacements and rotations of the points in the central line, which can be formulated as

$$\begin{aligned} \bar{u}_t &= u_t + y\theta_b + z\theta_n \\ \bar{u}_n &= u_n - z\theta_t \\ \bar{u}_b &= u_b - y\theta_t \end{aligned} \tag{23}$$

The displacements are formulated in local coordinate system, and are denoted as $\tilde{\mathbf{u}} = \{ u_t \quad u_n \quad u_b \}^{\mathrm{T}}$ and $\tilde{\mathbf{r}} = \{ \theta_t \quad \theta_n \quad \theta_b \}^{\mathrm{T}}$. When transformed into the global coordinate system, the displacements are denoted as $\mathbf{u} = \{ u_x \quad u_y \quad u_z \}^{\mathrm{T}}$ and $\mathbf{r} = \{ \theta_x \quad \theta_y \quad \theta_z \}^{\mathrm{T}}$, with transformation through rotation $\tilde{\mathbf{u}} = \mathbf{R}\mathbf{u}$, $\tilde{\mathbf{r}} = \mathbf{R}\mathbf{r}$. The rotation matrix is defined as,

$$\mathbf{R} = \begin{bmatrix} \mathbf{t}(\xi) \\ \mathbf{n}(\xi) \\ \mathbf{b}(\xi) \end{bmatrix} = \begin{bmatrix} t_x & t_y & t_z \\ n_x & n_y & n_z \\ b_x & b_y & b_z \end{bmatrix} \tag{24}$$

The strain matrix is calculated through the displacements of the control points in global coordinate system [45],

$$\Theta = \begin{bmatrix} \varepsilon_t \\ \gamma_n \\ \gamma_b \\ \eta_t \\ \chi_n \\ \chi_b \end{bmatrix} = \begin{bmatrix} t_x \dfrac{\mathrm{d}u_x}{\mathrm{d}s} + t_y \dfrac{\mathrm{d}u_y}{\mathrm{d}s} + t_z \dfrac{\mathrm{d}u_z}{\mathrm{d}s} \\ n_x \dfrac{\mathrm{d}u_x}{\mathrm{d}s} + n_y \dfrac{\mathrm{d}u_y}{\mathrm{d}s} + n_z \dfrac{\mathrm{d}u_z}{\mathrm{d}s} - b_x\theta_x - b_y\theta_y - b_z\theta_z \\ b_x \dfrac{\mathrm{d}u_x}{\mathrm{d}s} + b_y \dfrac{\mathrm{d}u_y}{\mathrm{d}s} + b_z \dfrac{\mathrm{d}u_z}{\mathrm{d}s} + n_x\theta_x + n_y\theta_y + n_z\theta_z \\ t_x \dfrac{\mathrm{d}\theta_x}{\mathrm{d}s} + t_y \dfrac{\mathrm{d}\theta_y}{\mathrm{d}s} + t_z \dfrac{\mathrm{d}\theta_z}{\mathrm{d}s} \\ n_x \dfrac{\mathrm{d}\theta_x}{\mathrm{d}s} + n_y \dfrac{\mathrm{d}\theta_y}{\mathrm{d}s} + n_z \dfrac{\mathrm{d}\theta_z}{\mathrm{d}s} \\ b_x \dfrac{\mathrm{d}\theta_x}{\mathrm{d}s} + b_y \dfrac{\mathrm{d}\theta_y}{\mathrm{d}s} + b_z \dfrac{\mathrm{d}\theta_z}{\mathrm{d}s} \end{bmatrix} \tag{25}$$

Each control point has six degrees of freedom,

$$\begin{aligned} \mathbf{U} &= \left\{ u \quad v \quad w \quad \theta_x \quad \theta_y \quad \theta_z \right\}^{\mathrm{T}} \\ &= [N_0 \quad \mathbf{I}_6 \quad \cdots \quad N_n\mathbf{I}_6]\{\mathbf{U_1} \cdots \mathbf{U}_n\}^{\mathrm{T}} \end{aligned} \tag{26}$$

The strain matrix can be calculated as

$$\Theta = \mathbf{BU} \tag{27}$$

Finally, the element stiffness matrix is formulated as,

$$\mathbf{K}_e = \int_e \mathbf{B}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{B}\mathrm{d}s \tag{28}$$

The Jacob value is calculated as,

$$J = \frac{\mathrm{ds}}{\mathrm{d}u} = \sqrt{\left(\frac{\mathrm{dx}}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{dy}}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{dz}}{\mathrm{d}u}\right)^2} \tag{29}$$

The **D** represents the generalized elastic matrix of Timoshenko beam model. The mass matrix is formulated as [46]

$$\mathbf{M}_e = \int_e \mathbf{N}^{\mathrm{T}} \cdot \Gamma \cdot N\mathrm{d}s \tag{30}$$

In which

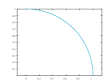
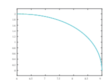
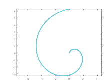
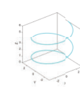
$$\Gamma = \begin{bmatrix} \rho A\mathbf{I}_3 & \\ & \Psi \end{bmatrix}, \quad \Psi = \begin{bmatrix} \rho I_t & & \\ & \rho I_n & \\ & & \rho I_b \end{bmatrix} \tag{31}$$

5 Numerical Experiments

The effect of parameterization of B-spline curves from sample points on the result of isogeometric analysis of Timoshenko beam is explored. In the numerical tests three examples are discussed, which are a quadrant arc, a Tschirnhausen cantilever beam and an Archimedes spiral curved beam. With these curves, the sample points are collected from the parametric definitions. Then the centripetal method, chord length method and the equally spaced method are used to define the knots of the B-spline function. Finally interpolation and approximation methods are applied to calculate the control points. With the obtained B-spline curves, the deflection under certain forces and natural frequencies are tested with isogeometric analysis. The influence of the different methods of parametrization on the results of the static and dynamic performances is investigated. The parametric form of functions of the test models are listed in Tab. 1. The sample points are calculated by the mathematical functions shown in the table with parameters uniformly distributed with in the parametric range. Let P and Q be two sets of points of the input shape and the calculated approximation spline curve. The discreted Hausdorff distance from P to Q is denoted by

$$h(P,Q) = \max_{p \in P}\{\min_{q \in Q}\|p - q\|\} \tag{32}$$

Table 1: The models applied for numerical tests. The range column gives the parameter ranges of the models, and the L column gives the lengths of the models

Model	Function	Range	L	Shape
Quadrant arc	$\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}$	$0 \le \theta \le \frac{\pi}{2}$	1.57	
Tschirnhausen beam	$\begin{cases} x = -3\left(\xi^2 - 3\right) \\ y = -\xi\left(\xi^2 - 3\right) \end{cases}$	$0 \le \xi \le 1$	4	
Archimedes beam	$\begin{cases} x = a\sin a \\ y = a\cos a \end{cases}$	$0 \le a \le 2\pi$	21.255	
Helix	$\begin{cases} x = 3\cos(720t) \\ y = 3\sin(720t) \\ z = 6t \end{cases}$	$0 \le t \le 1$	38.1723	

The Hausdorff distance between P and Q is defined as

$$H(P,Q) = \max\{h(P,Q), h(Q,P)\} \tag{33}$$

To consider the absolute size of the shapes, the geometrical error introduced by the approximation is described by the relative Hausdorff distance. In present paper H_r is used to represent the geometric error in fitting.

$$H_r = \frac{H(P,Q)}{L(P)} \tag{34}$$

5.1 Quadrant Arc: Comparison between Approximation and Interpolation Method

Fig. 1 shows the configuration of the quadrant arc which is defined with formula in Tab. 1. Ten sample points are also shown in the figure which are uniformly distributed. With the pre-defined ten sample points, the chord length, equally space and centripetal method obtain the same knots for the present test. While for approximation and interpolation methods, the calculated knots are different. Fig. 2 shows the knots calculated from the test with ten input sample points. The distribution of control points is calculated and is shown in Fig. 3.

The performance of approximation and interpolation algorithms in geometric fitting is investigated. The following phenomenons are observed:

- First, we set the number of sampling points to be the same with the number of the control points of the curve. The results show that the interpolation method gives stable results with small geometric error. While for the approximation method, the geometric fitting error is acceptable when the number of sampling points is small, and increases rapidly when the number increases. Therefore, suggestion is given that with approximation method the number of sample points should be larger than that of control points. The detailed results are shown in Tab. 2 in Appendix section.

- Second, we set the number of sample points as 50, and increase gradually the number of control points in approximation method. The results show that the fitting error is limited when the control points are less than 40, and then the fitting error grows rapidly.

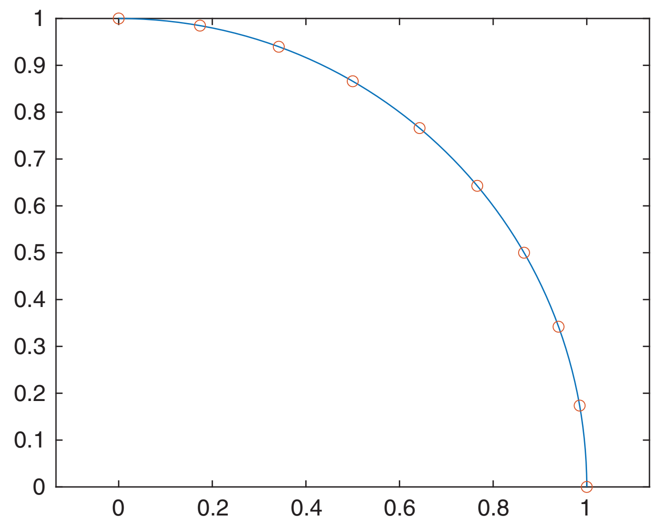


Figure 1: The geometry and input data points of quadrant arc

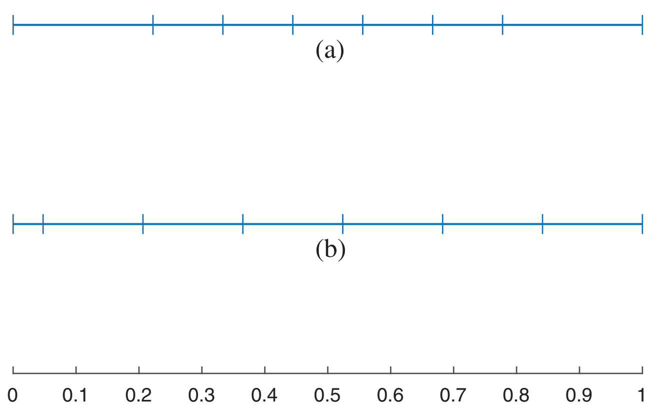


Figure 2: The distribution of knots calculated with the pre-defined ten sample points. (a) The interpolation method, (b) the approximation method

The influence of the geometric model on the isogeometric analysis of the quadratic arc is studied. The material properties are set as Young's modulus $E = 1.0E6$, Poisson's ratio $\mu = 0.3$, density $\rho = 7900$. The section of the beam is set as round circle with radius as $r = 0.05$. The downward vertical force with magnitude 1 is applied on the right end of the arc and the left end is pinched. The simulation result with commercial FEA software ABAQUS is used as reference and B31 beam element is applied. An "overkill" refined mesh with 1000 elements is used for the simulation. The calculated reference solution for the deformation of the free end in x direction is -0.160. The reference result by FEA is also validated, and IGA simulation on NURBS described quadrant arc model is carried out which obtains the same result as FEA.

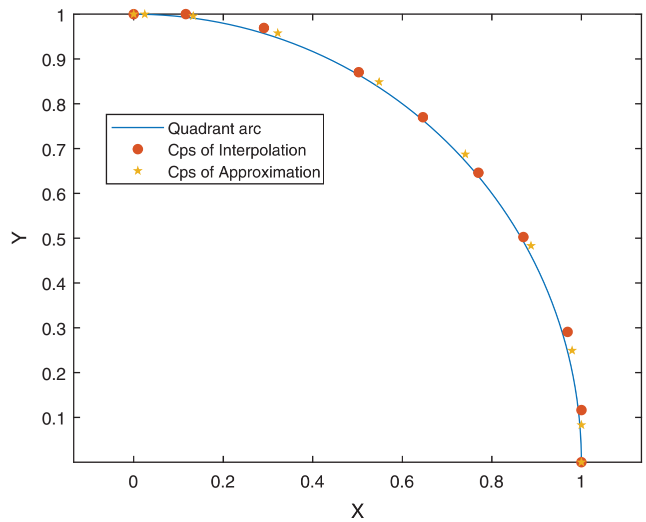


Figure 3: The distribution of control points. Ten control points are shown obtained by the approximation and interpolation method with ten input points

The result in Fig. 4 shows the relative error of deflection of the free end in x direction of the arc calculated with the fitted model using IGA. In the test the same number of sample and control points are used for both approximation and interpolation method. The result proves that good precision is obtained with the fitting model using isogeometric analysis, and that the proposed fitting methods from points are effective. The results also show that with approximation method the number of sample points should be larger than that of control points. For models calculated with interpolation method, the number of sample and control points are always the same and stable performances in geometric fitting and also IGA simulation are observed.

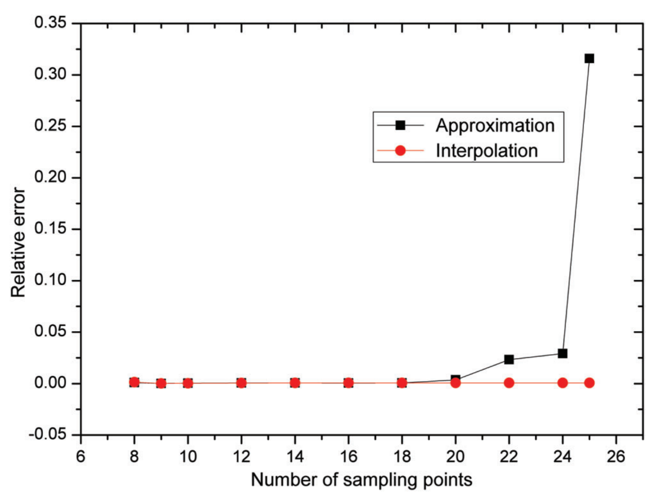


Figure 4: The relative error of deflection of the free end in x direction of the arc calculated with the established model by the fitting algorithms

5.2 Tschirnhausen Beam: The Influence of Parameters of Sample Points

Fig. 5 shows the configuration of Tschirnhausen beam. Ten sample points are also shown in the figure which are uniformly distributed in the parameter domain according to formula in Tab. 1. As the curvature varies along the beam, the samples are un-uniformly distributed on the curve. The parameters corresponding to the sample points are also different when calculated by the equally space method, chord length method and the centripetal method.

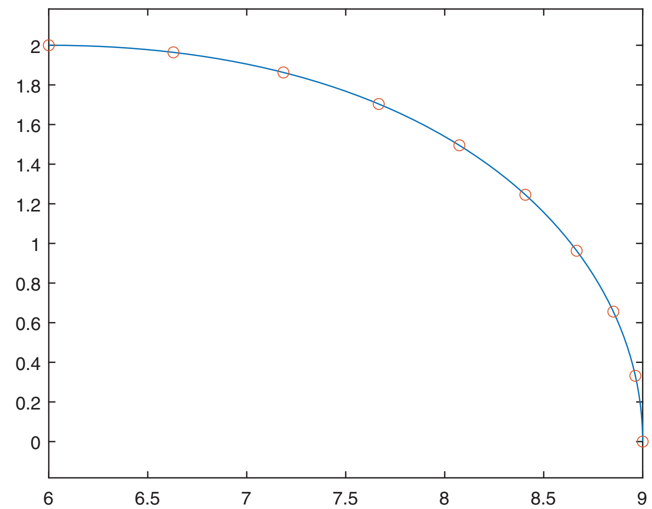


Figure 5: The distribution of sample points in Tschirnhausen beam

The knot vector calculated with ten input sample points are shown in Fig. 6. The knot vector calculated with interpolation method are shown in the first row, and that with approximation method in the second row. In the first column, the knots are calculated with centripetal method. Chord length method is used for second column, and equally space method for the third column. The distribution of control points is shown in Fig. 7.

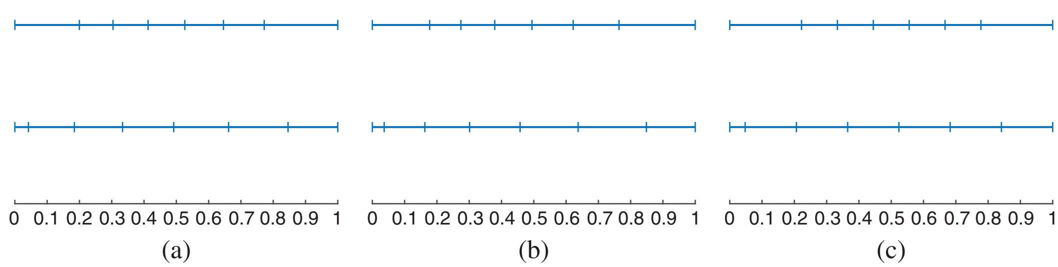
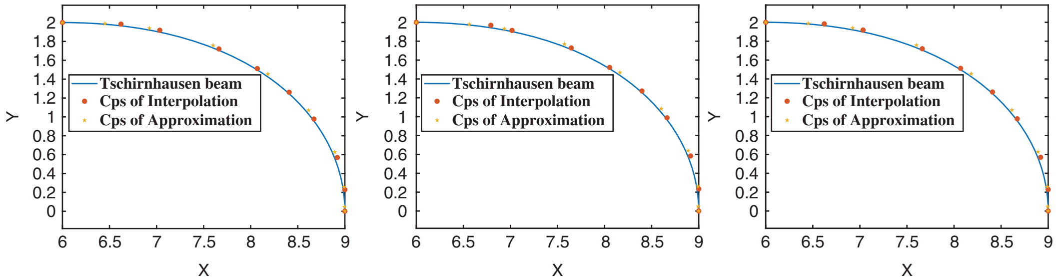


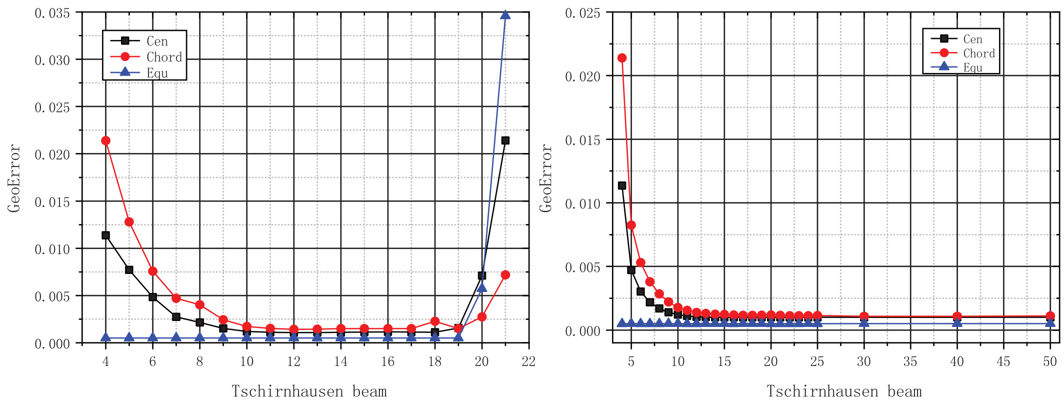
Figure 6: The distribution of knot vector in Tschirnhausen beam. Results with interpolation method are shown in the first row, and that with approximation method are shown in the second row. (a) Centripetal method. (b) Chord length method. (c) Equally space method

The performance in geometric fitting is first examined. First the number of control points is set the same as the number of parameters, and the performance of interpolation and approximation methods are tested, as shown in Fig. 8. It is clear than the precision is improved when the number of sampling points increases. Among the three method to calculate the parameters for the sample points, the equally spaced method obtains the best result. It is also noticed that the interpolation method is stable, while the approximation method gets larger geometric error when the number of sampling points is beyond a threshold value. This phenomena is similar with that in the first numerical test for quadratic arch.



(a) (b) (c)

Figure 7: The distribution of control points in Tschirnhausen beam. (a) Centripetal method. (b) Chord length method. (c) Equally space method



(a) (b)

Figure 8: The geometric error of fitting algorithms with different sampling and control points. (a) Approximation method. (b) Interpolation method

The influence of the number of control points in approximation method is tested. The number of sample point is fixed as 50. The equally spaced method gives stable results, and all the three parametric methods for sampling create large geometric error when the number of control points get close with the number of sampling point, as shown in Fig. 9.

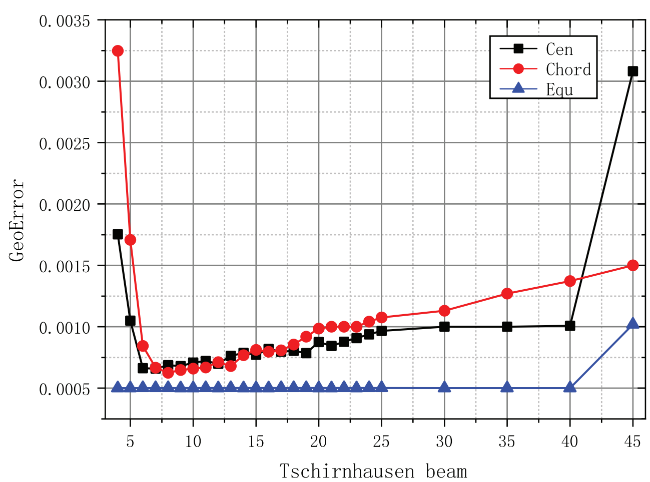
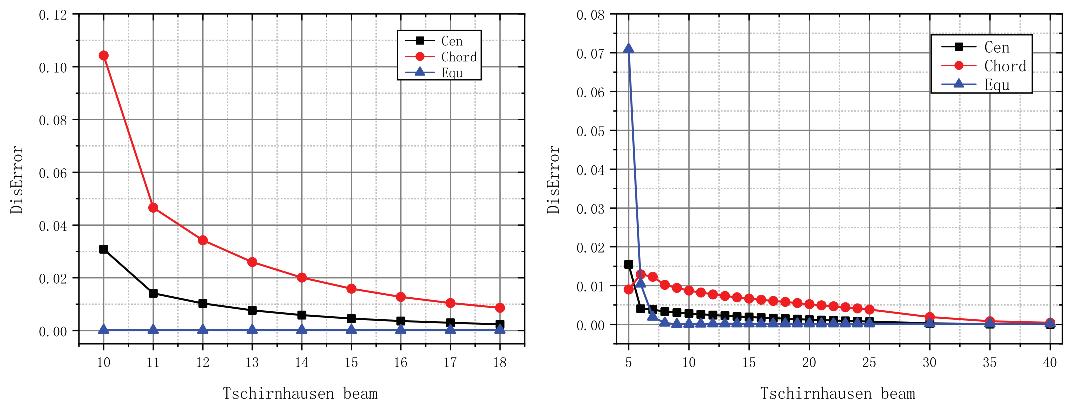


Figure 9: The geometric error of approximation method. The number of control points is changing, and the number of sampling points is fixed as 50

The influence of the geometric model on the isogeometric analysis of the Tschirnhausen beam is studied. The material properties are set the same as that of quadratic arc beam. The section of the beam is set as round circle with radius as $r = 0.2$. The downward vertical force with magnitude 1 is applied on the right end of the beam and the left end is pinched. The reference solution with Abaqus is used, and an "overkill" refined mesh with 3000 elements is used for the simulation. The deformation of the free end in x direction is calculated as -0.7208. The simulation results are shown in Fig. 10. Within the three parametric methods for sampling points, the equally space method gives the best results. With models obtained with approximation method as shown in Fig. 9, the simulation results are examined for models with geometric error less than 0.002. As shown in Fig. 11, the models performs well in the simulation, and the equally space method gives the most precise results.



(a) (b)

Figure 10: The relative error of displacement calculated with models by approximation and interpolation methods. (a) Approximation method. (b) Interpolation method

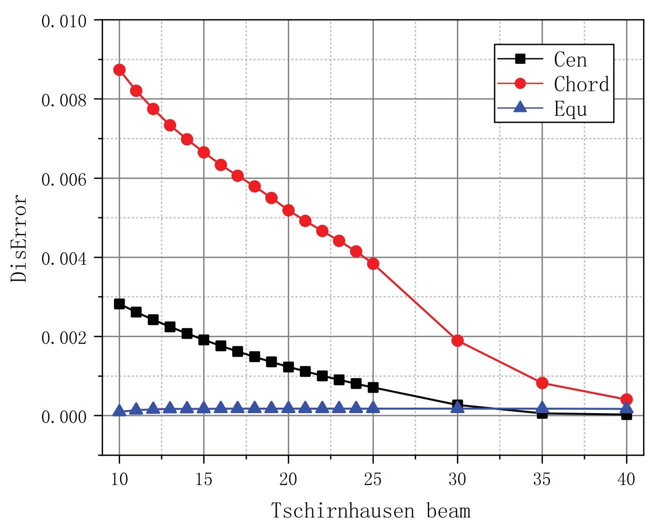


Figure 11: The error of displacement calculated in IGA with models from approximation method. To get the geometric models, 50 sampling points are used

Through the test of Tschirnhausen beam, the results show that the fitting algorithms work well to obtain models for IGA. And, it is proved again that in approximation method the number of control points should be less that of the sampling points. The precision of both geometric approximation and simulation increases with the number of sampling points, and the effect in improving the precision decreases once the number reaches a threshold. With the methods for parametrization, the equally space method performs better than the centripetal and Chord length method.

5.3 Archimedes Beam

In the third example the Archimedes beam model is applied. Compared with the first two examples, the Archimedes model's curvature varies more sharply. Fig. 12 shows the configuration along with ten sample points which are uniformly distributed within the parametric field. As shown in Figs. 13 and 14, the knots and distribution of control points are calculated with the ten sample points as input data. With interpolation method the geometric error decreases with the increase of number of sampling points, as shown in Fig. 15. With approximation method, the number of sampling point is chosen as 50, and we test the influence of number of sampling point on the geometric error. The results are shown in Fig. 16.

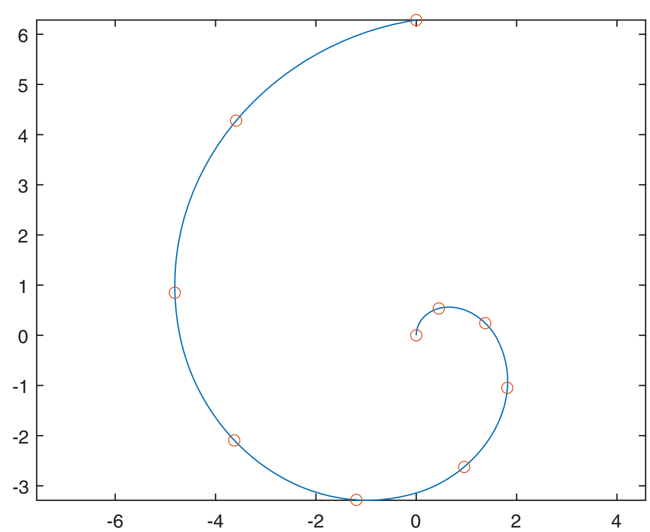


Figure 12: The distribution of sample points in the Archimedes beam

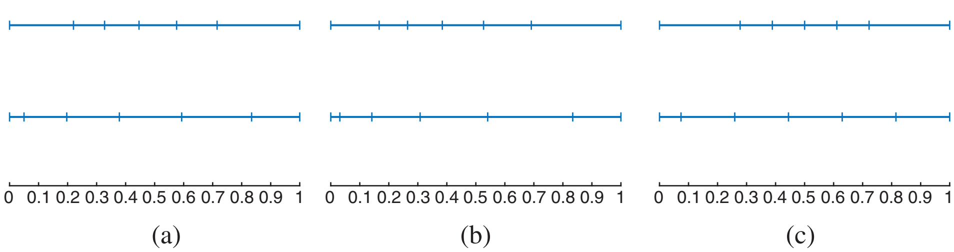
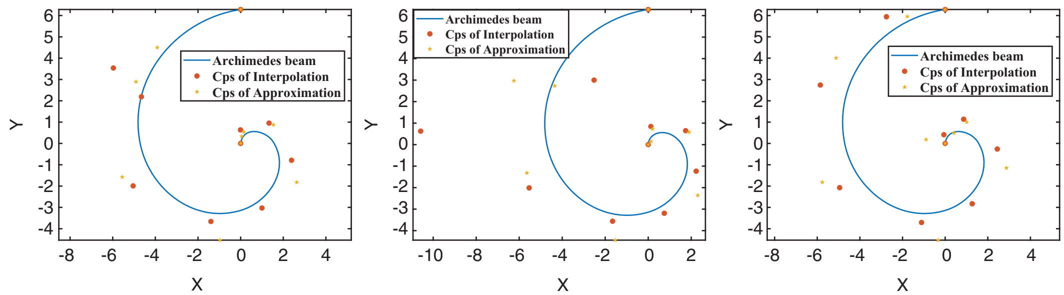


Figure 13: The distribution of knot vector in Archimedes beam. (a) Centripetal method. (b) Chord length method. (c) Equally space method

The influence of the geometric model on the isogeometric analysis of the Archimedes beam is studied. The material properties are set the same as that of quadratic arc beam. The section of the beam is set as round circle with radius as $r = 1$. The downward vertical force with magnitude 1 is applied on the right end of the



(a) (b) (c)

Figure 14: The distribution of control points in the Archimedes beam. (a) Centripetal method. (b) Chord length method. (c) Equally space method

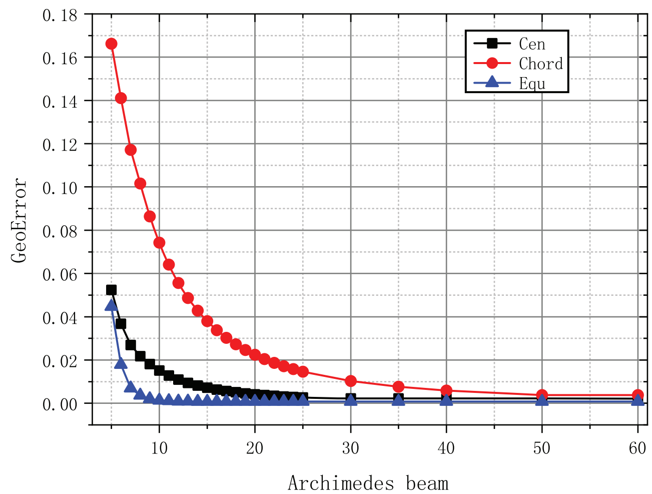


Figure 15: The geometric error of model obtained with interpolation method. The x axis shows the number of sampling points

beam and the left end is pinched. The reference solution with Abaqus is used, and an "overkill" refined mesh with 10000 elements is used. The deformation of the free end in x direction is calculated as -0.1156, and the natural frequency at first rank is $1.0996E-2$. With interpolation method, the performances of obtained models in IGA simulation are shown in Fig. 17. In the calculation, models with geometric error less than $5E-3$ are used. The results shown that the equally space method and centripetal method out-perform the chord length method.

The models calculated by approximation method shown in Fig. 16 are tested in IGA. The results are shown in Fig. 18. The models are calculated with 50 sampling points. When the number of control points reaches 10, the precisions of the simulations for displacement and model analysis are both high. When the number of control point gets close to the number of sampling point (44 in this test), the errors in simulations will rise along with the geometric error.

5.4 Helix

In this example we consider the Helix model which is more complicated than previous examples. The parametric definition of Helix is listed in Tab. 1. The material properties are set as Young's modulus $E = 10E6$, Poisson's ratio $\mu = 0.3$, density $\rho = 7900$. The section of the beam is set as round circle with radius as $r = 0.5$. In Fig. 19 the configuration along with 15 sample points are displayed. In this example the

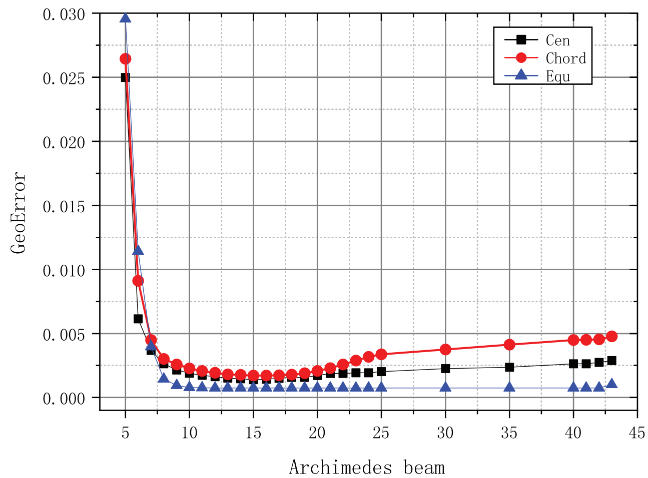
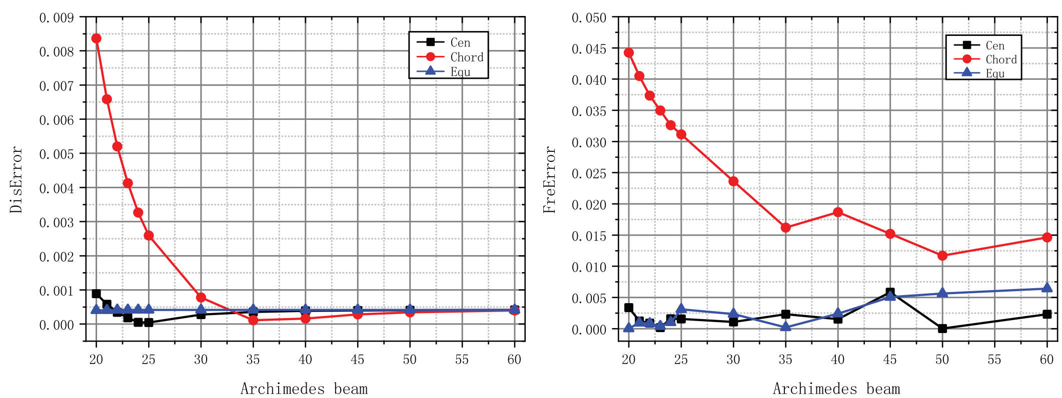
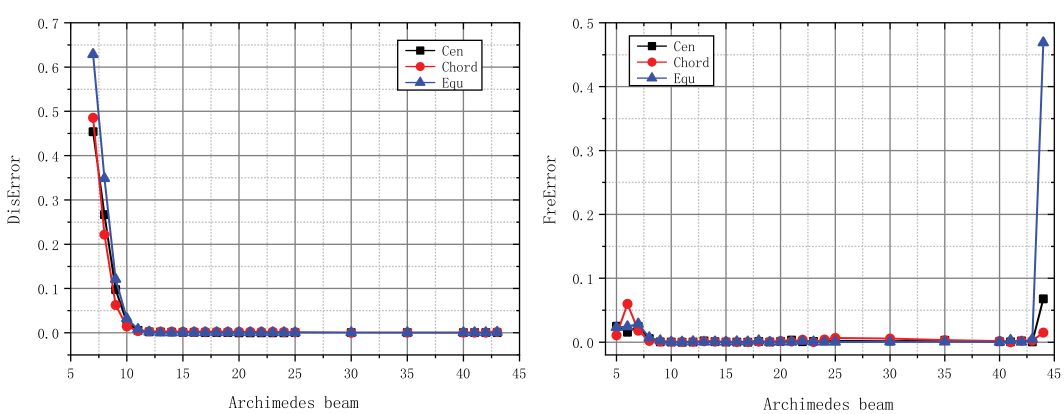


Figure 16: When the number of sampling points is 50, the relationship between the number of control points and the geometric error in approximation method



(a) (b)

Figure 17: The relative error of deflection calculated with models obtained by interpolation method. (a) Displacement in x direction, (b) First rank of natural frequency



(a) (b)

Figure 18: Influence of the number of control points on simulation accuracy in approximation method. (a) Relative error in displacement in x direction, (b) First rank of natural frequency

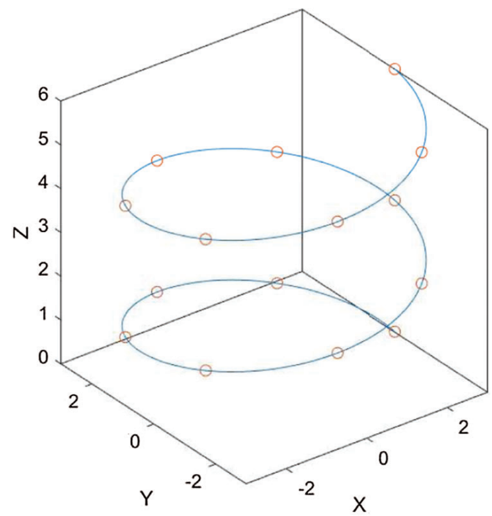


Figure 19: The shape and sample points in Helix curve

centripetal, chord length and equally space parametric methods obtains the same results, therefore we do not investigate the difference between these methods. The performance of geometric fitting and isogeometric analysis by the models obtained with interpolation and approximation algorithms are investigated. For approximation algorithm, 50 sample points are applied when the number of control points changes. For interpolation method the number of sample points is always equal to the number of control points. The geometric error is shown in Fig. 20. With increasing control points, the relative geometric error, which is defined in Eq. (34) decreases rapidly. The static and modal analysis are carried out. For static analysis, the endpoint at $z = 6$ is fixed, and a concentrated force is applied along the negative direction of x axis with magnitude as 10. For modal analysis, both ends are fixed and the first rank of natural frequency is calculated. Results by the FEA methods which are calculated by Abaqus are used again as the reference solutions. 15000 elements are applied, and we have proved that the solution is already convergent and remains the same when more elements are used. The reference solution for displacement in x direction of endpoint at $z = 0$ is 0.1436, and the first rank of natural frequency is 0.009620. The results calculated with geometric model from interpolation and approximation methods are shown in Fig. 21. The relative

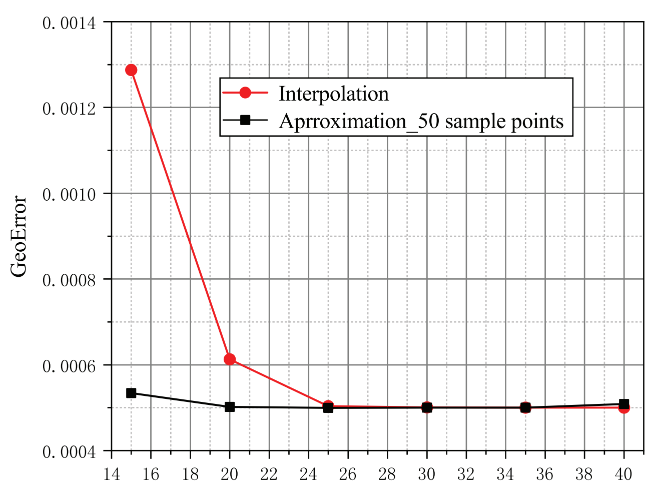
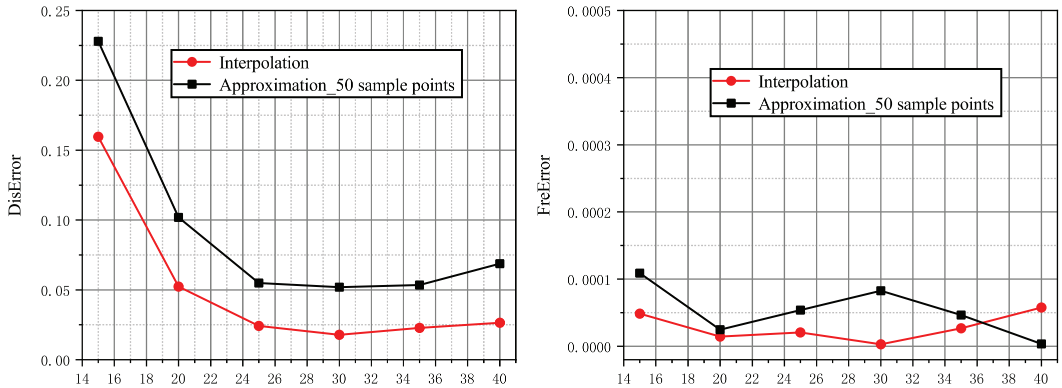


Figure 20: The geometric error of approximation and interpolation algorithms in Helix curve



(a) (b)

Figure 21: The relative error of deflection in x direction of endpoint at $z = 0$ and first rank of natural frequency calculated with Helix models obtained by interpolation and approximation methods. (a) Deflection in x direction, (b) First rank of natural frequency

error in deflection decreases with the increased number of control points, and generally the interpolation method performs better than the approximation method. The results of natural frequency are quite precise, which have the relative error at around 0.005% for both methods.

6 Discussions and Conclusions

In this paper, the analysis-suitable modelling of spatial beam with Timoshenko theory for IGA is investigated. Two basic fitting algorithms, which are the approximation and interpolation methods, are discussed. Three strategies of parameterization for the sampling points, meaning the equally spaced method, the chord length method and the centripetal method are considered. The static and the modal analysis with the established parametric models are carried out. The results show that both the interpolation and approximation method can obtain precise parametric model with low geometric error for IGA. The approximation method should be used when the number of control points is much less than that of the sampling points to assure the precision for geometric fitting. The interpolation method obtains stable and precise geometry model, and the geometric error decreases with the increase of sampling point. Within the parameterization methods for sampling points, the equally spaced method and centripetal method perform better than the chord length method.

In present work the sampling points are obtained by uniform distribution within the parametric space according to the explicit formula of curve in the present paper. In future work the variation of sampling points need to be considered. The point clouds obtained by scanning should also be considered.

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Appendix A. Results of the Numerical Tests

Tab. 2 shows the geometric error of the fitting algorithms.

Table 2: The geometric error of the fitting algorithm for quadratic arc in Numerical example 1. The degree of B-spline basis is set as three

	4	6	8	10	15	20
Interpolation	0.00176	5.0042E-4	5.0040E-4	5.0038E-4	5.0038E-4	5.0038E-4
Approximation	0.00176	7.2119E-4	5.0038E-4	5.0084E-4	5.0038E-4	0.0043
	22	24	25	30	40	50
Interpolation	5.0038E-4	5.0038E-4	5.0038E-4	5.0038E-4	5.0038E-4	5.0038E-4
Approximation	0.2959	9.7218	71.3655	7489.1009	7131.2582	1.0116E6