

Estimation of the Stress-Strength Reliability for Exponentiated Pareto Distribution Using Median and Ranked Set Sampling Methods

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Abstract: In reliability analysis, the stress-strength model is often used to describe the life of a component which has a random strength (X) and is subjected to a random stress (Y). In this paper, we considered the problem of estimating the reliability $R=P [Y<X]$ when the distributions of both stress and strength are independent and follow exponentiated Pareto distribution. The maximum likelihood estimator of the stress strength reliability is calculated under simple random sample, ranked set sampling and median ranked set sampling methods. Four different reliability estimators under median ranked set sampling are derived. Two estimators are obtained when both strength and stress have an odd or an even set size. The two other estimators are obtained when the strength has an odd size and the stress has an even set size and vice versa. The performances of the suggested estimators are compared with their competitors under simple random sample via a simulation study. The simulation study revealed that the stress strength reliability estimates based on ranked set sampling and median ranked set sampling are more efficient than their competitors via simple random sample. In general, the stress strength reliability estimates based on median ranked set sampling are smaller than the corresponding estimates under ranked set sampling and simple random sample methods.

Keywords: Stress-Strength model, ranked set sampling, median ranked set sampling, maximum likelihood estimation, mean square error.

1 Introduction

The ranked set sampling (RSS) was first suggested by McIntyre [McIntyre (1952)] to estimate the mean of pasture and forage yields as an alternative method to the commonly

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Received: 08 April 2020; Accepted: 24 April 2020.

used simple random sample (SRS) method based on the same number of measurement units. The RSS can be described as follows:

- Step 1:** Randomly select m^2 units from the population of interest and randomly allocated them into m sets, each of size m .
- Step 2:** The m units within each set are ranked by visual inspection or by any cheap method concerning the variable of interest.
- Step 3:** The smallest ranked unit is measured from the first set of m units; the second smallest ranked unit is measured from the second set of m units. The process is continued until the largest ranked unit is measured from the last set.
- Step 4:** The entire process can be repeated several times; say r times, to produce an RSS sample of size $n=rm$.

Muttalak [Muttalak (1997)] proposed median ranked set sampling (MRSS) as a modification of the RSS to estimate the population means. The MRSS technique can be summarized as follows: Select m random samples each of size m units from the study population. Rank the units within each set concerning a study variable. If the sample size m is odd, from each sample select for measurement the $((m+1)/2)$ th smallest ranked unit, i.e., the median of the sample. If the sample size m is even, select for the measurement from the first $(m/2)$ samples the $(m/2)$ th smallest ranked unit and from the second $(m/2)$ samples the $((m/2)+1)$ th smallest ranked unit. The cycle can be repeated r times if needed to get a sample of size $n=rm$ units from the MRSS data.

For more details about RSS, Hassan [Hassan (2012)] considered the goodness of fit tests for the exponentiated Pareto distribution using an extreme RSS method. Hassan [Hassan (2013)] obtained the maximum likelihood and Bayesian estimators based on RSS. Haq et al. [Haq, Brown, Moltchanova et al. (2014)] introduced ordered double RSS. Haq et al. [Haq, Brown, Moltchanova et al. (2015)] introduced the varied L RSS scheme as a modification of the RSS. Al-Omari [Al-Omari (2015)] considered the estimation of the distribution function based on L ranked set sampling. Al-Omari [Al-Omari (2016)] proposed a new measure of sample entropy of continuous random variables in RSS. Zamanzade et al. [Zamanzade and Al-Omari (2016)] proposed a new generalization of the RSS for estimating the population means and variance. Al-Omari et al. [Al-Omari and Zamanzade (2017)] proposed the goodness of-fit-tests for the Laplace distribution based on RSS. Syam et al. [Syam, Al-Omari and Ibrahim (2017)] investigated the mean estimation using stratified double MRSS. Bouza et al. [Bouza, Al-Omari, Santiago et al. (2017)] considered the ratio type estimation using the knowledge of the auxiliary variable for ranking and estimating. Al-Omari et al. [Al-Omari and Al-Nasser (2018)] introduced new estimators of the population ratio using multistage MRSS. Al-Nasser et al. [Al-Nasser and Al-Omari (2018)] suggested a new sampling method called MiniMax RSS for estimating the population parameters. Al-Omari et al. [Al-Omari and Haq (2019a)] proposed some entropy estimators of a continuous random variable using RSS. Zamanzade [Zamanzade (2019)] developed a non-parametric cumulative distribution function estimator based on pair RSS. Al-Omari et al. [Al-Omari and Haq (2019b)] suggested a new sampling method for estimating the population mean. Zamanzade et al.

[Zamanzade and Wang (2020)] investigated nonparametric estimation using partially ordered sets. Also, see Haq et al. [Haq and Al-Omari (2015); Al-Nasser and Al-Omari (2015); Santiago, Bouza, Sautto et al. (2016)].

Making statistical inferences concerning the reliability $R=P [Y<X]$ have received considerable attention in the context of the reliability. In this sense, X represents the strength of a system that is subjected to stress Y . It is clear that the system fails when the stress exceeds strength. Therefore, the stress-strength (SS) model is a measure of system reliability. In general, statistical inference about the SS model is considered based on SRS data. However, in recent years, statistical inferences about the SS model based on the RSS method have been considered by several researches. Sengupta et al. [Sengupta and Mukhati (2008)] discussed estimation of the SS reliability for exponential populations. Muttlak et al. [Muttlak, Abu-Dayyeh, Saleh et al. (2010)] proposed three estimators of the SS reliability when X and Y are independent one-parameter exponential populations. Hassan et al. [Hassan, Assar and Yahya (2014, 2015)] discussed the estimation of the SS reliability when Y and X are two independent Burr type XII distribution under several modifications of RSS. Estimation of the SS model for Weibull distribution has been discussed by Akgül et al. [Akgül and Şenoğlu (2017)]. Akgül et al. [Akgül, Acıtaş and Şenoğluc (2018)] discussed the estimation of the SS reliability when X and Y independent Lindley populations.

The Pareto distribution is well-known in the literature due to its capability in modeling the heavy-tailed distributions. Applications of the Pareto distribution appear in several areas including economics, finance, stock price fluctuations, environmental studies, insurance risk, etc. Recently, it has also been recognized as a useful model for the analysis of lifetime data. Gupta et al. [Gupta, Gupta and Gupta (1998)] proposed a simple generalization of the well-known standard Pareto distribution of the second kind, named as the exponentiated Pareto (EP) distribution. A two-parameter EP distribution can have decreasing and upside-down bathtub shaped failure rates. The probability density function (pdf) of the EP distribution is given by

$$f(x; \lambda, \theta) = \lambda \theta (1+x)^{-(\lambda+1)} \left(1 - (1+x)^{-\lambda}\right)^{\theta-1}; \quad x, \lambda, \theta > 0, \tag{1}$$

where θ and λ are two shape parameters. The cumulative distribution function of the EP distribution is defined as

$$F(x; \lambda, \theta) = \left(1 - (1+x)^{-\lambda}\right)^\theta. \tag{2}$$

The EP distribution reduces to the standard Pareto distribution of the second kind when $\theta=1$. The studies about the EP distribution have been provided by some authors. Parameter estimation of the EP is considered by Shawky et al. [Shawky and Abu-Zinadah (2009)]. Estimation of the reliability in a multi-component SS model following EP distribution, for identical and non-identical components, were discussed by Hassan et al. [Hassan and Basheikh (2012a, 2012b)]. Chena et al. [Chena and Cheng (2017)] considered the estimation of the SS model for EP populations. Estimation in a partially accelerated life test for the EP distribution based on progressive censoring has been provided by Hassan et al. [Hassan, Abd-Alla and El-Elaa (2017)]. Parameter estimators of the EP distribution in presence of outliers have been considered by Nooghabi

[Nooghabi (2017)]. The ML, least squares, weighted least squares and Bayesian estimators of an extended EP distribution were discussed by Hassan et al. [Hassan, Nassr and Hemeda (2020)].

In this study, we obtained the estimators of the SS reliability based on SRS, RSS and MRSS schemes. It is assumed that the strength $X \sim EP(\lambda, \theta)$ and the stress $Y \sim EP(\lambda, \beta)$ are both independent. The main reason for using the EP distribution is that its flexibility for modeling several lifetime data and its extensive consideration in engineering, life testing and reliability studies. A simulation study is conducted to compare the behavior of the suggested different estimates.

The rest of this article is organized as follows. Section 2 provides the explicit expression of the SS reliability as well as the ML estimator of the reliability R which is obtained under SRS. Section 3 gives the ML estimator of the SS reliability based on RSS. Section 4 comes up with four different reliability estimators based on the MRSS method. The numerical study is provided in Section 5. Eventually, the paper is concluded in Section 6.

2 Reliability estimator based on SRS

This section provides an explicit expression of the SS reliability. The ML estimator of the reliability is derived via SRS.

2.1 Stress strength reliability

Let X is the strength of a system and Y is the stress acting on it. Let X and Y are the two independent strength-stress random variables observed from $EP(\lambda, \theta)$ and $EP(\lambda, \beta)$ respectively. The SS reliability parameter is evaluated under the assumption that the models have the same shape parameter λ but with different shape parameters θ and β that is; $EP(\lambda, \theta)$ and $EP(\lambda, \beta)$. Therefore, the SS reliability parameter R is given by

$$R = P[Y < X] = \int_{x=0}^{\infty} \int_{y=0}^x f(x; \lambda, \theta) f(y; \lambda, \beta) dx dy \quad (3)$$

$$= \int_{x=0}^{\infty} \lambda \theta (1+x)^{-(\lambda+1)} (1-(1+x)^{-\lambda})^{\theta+\beta-1} dx = \frac{\theta}{\theta + \beta}.$$

If θ and β are known, then the reliability parameter is simply calculated using Eq. (3). It can be observed that the SS reliability does not depend on λ . However, if θ and β are unknown and λ is known, the estimators of θ and β depend on λ , and hence so does the estimator of the SS reliability.

2.2 Reliability estimator based on SRS

Suppose that X_1, X_2, \dots, X_n is a random sample from $EP(\lambda, \theta)$ and Y_1, Y_2, \dots, Y_m is a sample from $EP(\lambda, \beta)$. The ML estimator of the reliability given that the sample is obtained. To compute the ML estimator of the reliability, it is required to obtain the ML estimators of θ and β . The joint log-likelihood function for the observed samples is

$$L = n \ln \theta + m \ln \beta + (m + n) \ln \lambda - (\lambda + 1) \left[\sum_{i=1}^n \ln(1 + x_i) + \sum_{j=1}^m \ln(1 + y_j) \right] + (\theta - 1) \sum_{i=1}^n \ln \left(1 - (1 + x_i)^{-\lambda} \right) + (\beta - 1) \sum_{j=1}^m \ln \left(1 - (1 + y_j)^{-\lambda} \right). \tag{4}$$

Differentiating Eq. (4) with respect to unknown parameters, then we obtain the following equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln \left(1 - (1 + x_i)^{-\lambda} \right), \tag{5}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{m}{\beta} + \sum_{j=1}^m \ln \left(1 - (1 + y_j)^{-\lambda} \right), \tag{6}$$

and

$$\frac{\partial \ln L}{\partial \ln \lambda} = \frac{(m + n)}{\lambda} - \left[\sum_{i=1}^n \ln(1 + x_i) + \sum_{j=1}^m \ln(1 + y_j) \right] + \sum_{i=1}^n \frac{(\theta - 1) \ln(1 + x_i)}{\left[(1 + x_i)^\lambda - 1 \right]} + \sum_{j=1}^m \frac{(\beta - 1) \ln(1 + y_j)}{\left[(1 + y_j)^\lambda - 1 \right]}. \tag{7}$$

Setting Eqs. (5)-(7) with zeros we obtain the ML estimators of θ, λ and β . From Eqs. (5) and (6), we obtain the ML estimator of θ and β as a function of λ as follows:

$$\hat{\theta}(\lambda) = \frac{-n}{\sum_{i=1}^n \ln \left(1 - (1 + x_i)^{-\lambda} \right)}, \tag{8}$$

and

$$\hat{\beta}(\lambda) = \frac{-m}{\sum_{j=1}^m \ln \left(1 - (1 + y_j)^{-\lambda} \right)}. \tag{9}$$

If λ is known, the ML estimators of θ and β can be obtained from Eqs. (8) and (9). In case of all the parameters are unknown, then λ can first be estimated by solving the following nonlinear equation:

$$b(\lambda) = \frac{n + m}{\left[\sum_{i=1}^n \ln(1 + x_i) + \sum_{j=1}^m \ln(1 + y_j) \right] - \sum_{i=1}^n \frac{(\hat{\theta}(\lambda) - 1) \ln(1 + x_i)}{\left((1 + x_i)^\lambda - 1 \right)} - \sum_{j=1}^m \frac{(\hat{\beta}(\lambda) - 1) \ln(1 + y_j)}{\left((1 + y_j)^\lambda - 1 \right)}}. \tag{10}$$

Since $\hat{\lambda}$ is a fixed point solution of the nonlinear Eq. (10), hence, it can be obtained by using a simple iterative procedure. Therefore, the ML estimator of the SS reliability, denoted by \hat{R} , can be obtained as follows

$$\hat{R} = \frac{\hat{\theta}}{\hat{\theta} + \hat{\beta}} \quad (11)$$

3 Reliability estimator based on RSS

To obtain the ML estimator of SS reliability; suppose $\{X_{i(i)c}, i=1, 2, \dots, n; c=1, 2, \dots, r_x\}$ be an RSS observed from $X \sim EP(\lambda, \theta)$, with sample size nr_x , where n is the set size and r_x is the number of cycles. Let $\{Y_{j(j)d}, j=1, 2, \dots, m; d=1, 2, \dots, r_y\}$ be an RSS observed from $Y \sim EP(\lambda, \beta)$ with sample size mr_y , where m is the set size and r_y is the number of cycles. Therefore, the likelihood functions L_1 of the observed data will be as follows:

$$L_1 = \prod_{c=1}^{r_x} \prod_{i=1}^n f_i(x_{i(i)c}) \prod_{d=1}^{r_y} \prod_{j=1}^m f_j(y_{j(j)d}), \quad (12)$$

where

$$f_i(x_{i(i)c}) = \frac{n! \lambda \theta}{(i-1)!(n-i)!} \left[1 - \left(1 - (1 + x_{i(i)c})^{-\lambda} \right)^\theta \right]^{n-i} (1 + x_{i(i)c})^{-(\lambda+1)} \left(1 - (1 + x_{i(i)c})^{-\lambda} \right)^{\theta i - 1}, \quad (13)$$

and

$$f_j(y_{j(j)d}) = \frac{m! \lambda \beta}{(j-1)!(m-j)!} \left[1 - \left[1 - (1 + y_{j(j)d})^{-\lambda} \right]^\beta \right]^{m-j} (1 + y_{j(j)d})^{-(\lambda+1)} \left[1 - (1 + y_{j(j)d})^{-\lambda} \right]^{\beta j - 1}, \quad (14)$$

are the pdfs of random variables $X_{i(i)c}$ and $Y_{j(j)d}$, respectively, where $x_{i(i)c} > 0$ and $y_{j(j)d} > 0$. The log-likelihood function of L_1 will be as follows:

$$\begin{aligned} \ln L_1 = & \ln H_1 + r_x n \ln(\lambda \theta) + \sum_{c=1}^{r_x} \sum_{i=1}^n (n-i) \ln \left[1 - \left(1 - z_{i(i)c}^{-\lambda} \right)^\theta \right] - (\lambda+1) \sum_{c=1}^{r_x} \sum_{i=1}^n \ln(z_{i(i)c}) \\ & + \sum_{c=1}^{r_x} \sum_{i=1}^n (\theta i - 1) \ln(1 - z_{i(i)c}^{-\lambda}) + r_y m \ln(\lambda \beta) + \sum_{d=1}^{r_y} \sum_{j=1}^m (m-j) \ln \left[1 - \left(1 - w_{j(j)d}^{-\lambda} \right)^\beta \right] \\ & - (\lambda+1) \sum_{d=1}^{r_y} \sum_{j=1}^m \ln(w_{j(j)d}) + \sum_{d=1}^{r_y} \sum_{j=1}^m (\beta j - 1) \ln(1 - w_{j(j)d}^{-\lambda}), \end{aligned} \quad (15)$$

where H_1 is a constant, $z_{i(i)c} = (1 + x_{i(i)c})$, and $w_{j(j)d} = (1 + y_{j(j)d})$. The ML estimators of θ, β and λ can be obtained by maximizing $\ln L_1$ directly with respect to θ, β and λ . The first partial derivatives of the log-likelihood function with respect to θ, β and λ are given by:

$$\frac{\partial \ln L_1}{\partial \theta} = \frac{r_x n}{\theta} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(n-i) \ln(1 - z_{i(i)c}^{-\lambda})}{(1 - z_{i(i)c}^{-\lambda})^{-\theta} - 1} + \sum_{c=1}^{r_x} \sum_{i=1}^n i \ln(1 - z_{i(i)c}^{-\lambda}), \quad (16)$$

$$\frac{\partial \ln L_1}{\partial \beta} = \frac{r_y m}{\beta} - \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(m-j) \ln(1 - w_{j(j)d}^{-\lambda})}{(1 - w_{j(j)d}^{-\lambda})^{-\beta} - 1} + \sum_{d=1}^{r_y} \sum_{j=1}^m j \ln(1 - w_{j(j)d}^{-\lambda}), \quad (17)$$

and

$$\begin{aligned} \frac{\partial \ln L_1}{\partial \lambda} = & \frac{r_x n + r_y m}{\lambda} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{\theta(n-i)(1-z_{i(i)c}^{-\lambda})^{\theta-1} z_{i(i)c}^{-\lambda} \ln z_{i(i)c}}{1-(1-z_{i(i)c}^{-\lambda})^\theta} - \sum_{c=1}^{r_x} \sum_{i=1}^n \ln z_{i(i)c} \\ & + \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(\theta i - 1) \ln z_{i(i)c}}{z_{i(i)c}^\lambda - 1} + \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{\beta(m-j)(1-w_{j(j)d}^{-\lambda})^{\beta-1} w_{j(j)d}^{-\lambda} \ln w_{j(j)d}}{1-(1-w_{j(j)d}^{-\lambda})^\beta} \\ & - \sum_{d=1}^{r_y} \sum_{j=1}^m \ln w_{j(j)d} + \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(\beta j - 1) \ln w_{j(j)d}}{w_{j(j)d}^\lambda - 1}. \end{aligned} \tag{18}$$

Clearly, it is not easy to obtain a closed-form solution to non-linear Eqs. (16)-(18), so we apply an iterative technique to solve these equations numerically. Hence, we obtain the ML estimators of θ, β and λ , then by using invariance property the ML estimator of the reliability is obtained by using Eq. (3).

4 Reliability estimator based on MRSS

The MRSS has been proposed by Muttlak [Muttalak (1997)] as a sampling technique to estimate the population mean. The MRSS procedure depends on two cases, the first one for odd set size while the second case for even set size. In this section, an estimator of the SS reliability is obtained when both stress and strength have odd or even set sizes. Additionally, the ML estimator of SS reliability is obtained when strength X has an odd set size and stress Y has an even set size and vice versa.

4.1 Reliability estimator with odd set size

In this sub-section, the ML estimator of the SS reliability is provided when both X and Y are drawn using MRSS with an odd set size. Let $X_{i(g)c}; i=1, \dots, n, c=1, \dots, r_x; g=[(n+1)/2]$ be an MRSS observed from $X \sim EP(\lambda, \theta)$, with sample size nr_x , where n is the set size and r_x is the number of cycles. Let $Y_{j(k)d}; j=1, \dots, m, d=1, \dots, r_y; k=[(m+1)/2]$ be an MRSS observed sample from $Y \sim EP(\lambda, \beta)$ with sample size mr_y , where m is the set size and r_y is the number of cycles. Therefore, the likelihood function L_2 of the observed data will be as follows:

$$L_2 = \prod_{c=1}^{r_x} \prod_{i=1}^n f_g(x_{i(g)c}) \prod_{d=1}^{r_y} \prod_{j=1}^m f_k(y_{j(k)d}), \tag{19}$$

where

$$f_g(x_{i(g)c}) = \frac{n! \lambda \theta}{[(g-1)!]^2} \left[1 - \left(1 - (1 + x_{i(g)c})^{-\lambda} \right)^\theta \right]^{g-1} (1 + x_{i(g)c})^{-(\lambda+1)} \left[1 - (1 + x_{i(g)c})^{-\lambda} \right]^{\theta g-1}, \tag{20}$$

and

$$f_k(y_{j(k)d}) = \frac{m! \lambda \beta}{[(k-1)!]^2} \left[1 - \left(1 - (1 + y_{j(k)d})^{-\lambda} \right)^\beta \right]^{k-1} (1 + y_{j(k)d})^{-(\lambda+1)} \left(1 - (1 + y_{j(k)d})^{-\lambda} \right)^{\beta k-1}, \tag{21}$$

are the pdfs of $X_{i(g)c}$ and $Y_{j(k)d}$, respectively, where $x_{i(g)c} > 0$, and $y_{j(k)d} > 0$. The log-likelihood function of L_2 will be as follows:

$$\begin{aligned} \ln L_2 = & \ln H_2 + r_x n \ln(\lambda\theta) + \sum_{c=1}^{r_x} \sum_{i=1}^n (g-1) \ln \left[1 - \left(1 - Q_{i(g)c}^{-\lambda} \right)^\theta \right] - (\lambda+1) \sum_{c=1}^{r_x} \sum_{i=1}^n \ln Q_{i(g)c} \\ & + \sum_{c=1}^{r_x} \sum_{i=1}^n (\theta g - 1) \ln \left(1 - Q_{i(g)c}^{-\lambda} \right) + r_y m \ln(\lambda\beta) + \sum_{d=1}^{r_y} \sum_{j=1}^m (k-1) \ln \left[1 - \left(1 - T_{j(k)d}^{-\lambda} \right)^\beta \right] \\ & - (\lambda+1) \sum_{d=1}^{r_y} \sum_{j=1}^m \ln T_{j(k)d} + \sum_{d=1}^{r_y} \sum_{j=1}^m (\beta k - 1) \ln \left(1 - T_{j(k)d}^{-\lambda} \right), \end{aligned} \quad (22)$$

where H_2 is a constant, $Q_{i(g)c} = 1 + x_{i(g)c}$, and $T_{j(k)d} = 1 + y_{j(k)d}$. The ML estimators of θ, β and λ can be obtained by maximizing $\ln L_2$ directly with respect to θ, β and λ . The first partial derivatives of $\ln L_2$ with respect to θ, β and λ are given by:

$$\frac{\partial \ln L_2}{\partial \theta} = \frac{r_x n}{\theta} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(g-1) \ln \left(1 - Q_{i(g)c}^{-\lambda} \right)}{\left[\left(1 - Q_{i(g)c}^{-\lambda} \right)^{-\theta} - 1 \right]} - \sum_{c=1}^{r_x} \sum_{i=1}^n g \ln \left(1 - Q_{i(g)c}^{-\lambda} \right), \quad (23)$$

$$\frac{\partial \ln L_2}{\partial \beta} = \frac{r_y m}{\beta} - \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(k-1) \ln \left(1 - T_{j(k)d}^{-\lambda} \right)}{\left[\left[1 - T_{j(k)d}^{-\lambda} \right]^{-\beta} - 1 \right]} + \sum_{d=1}^{r_y} \sum_{j=1}^m k \ln \left(1 - T_{j(k)d}^{-\lambda} \right), \quad (24)$$

and

$$\begin{aligned} \frac{\partial \ln L_2}{\partial \lambda} = & \frac{r_x n + r_y m}{\beta} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(g-1) \theta \left(1 - Q_{i(g)c}^{-\lambda} \right)^{\theta-1} Q_{i(g)c}^{-\lambda} \ln(Q_{i(g)c})}{\left[1 - \left(1 - Q_{i(g)c}^{-\lambda} \right)^\theta \right]} - \sum_{c=1}^{r_x} \sum_{i=1}^n \ln(Q_{i(g)c}) \\ & + \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(\theta g - 1) \ln(Q_{i(g)c})}{\left(Q_{i(g)c}^\lambda - 1 \right)} - \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(k-1) \beta \left(1 - T_{j(k)d}^{-\lambda} \right)^{\beta-1} T_{j(k)d}^{-\lambda} \ln(T_{j(k)d})}{\left[1 - \left(1 - T_{j(k)d}^{-\lambda} \right)^\beta \right]} \\ & - \sum_{d=1}^{r_y} \sum_{j=1}^m \ln(T_{j(k)d}) + \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(\beta k - 1) \ln(T_{j(k)d})}{\left(T_{j(k)d}^\lambda - 1 \right)}. \end{aligned} \quad (25)$$

Since Eqs. (23)-(25) have no closed-form solution, so an iterative technique is employed to solve these equations numerically. Hence, the ML estimators of the unknown parameters based on the MRSS in case of an odd set size are obtained. Therefore, the ML estimator of the SS reliability is obtained by substituting the ML estimators of population parameters in Eq. (3).

4.2 Reliability estimator with even set size

Here, the reliability estimator is investigated when both X and Y are drawn based on MRSS with even set size. In case of the even set sizes, the q th smallest unit is chosen from the first q ordered sets, while the $(q+1)$ th smallest unit is chosen from each of the

remaining q sets. Let the set $\{X_{i(q)c}, i=1, \dots, q; c=1, \dots, r_x\} \cup \{X_{i(q+1)c}, i=q+1, \dots, n; c=1, \dots, r_x\}$ be an MRSS drawn from EP(λ, θ) with even set sizes where $q = n/2$. By a similar way, let the set $\{Y_{j(v)d}, j=1, \dots, v; d=1, \dots, r_y\} \cup \{Y_{j(v+1)d}, j=v+1, \dots, m; d=1, \dots, r_y\}$ be an MRSS drawn from the EP(λ, β) with even set sizes where $v = m/2$. Therefore, the likelihood functions L_3 of the observed data will be as follows:

$$L_3 = \prod_{c=1}^{r_x} \prod_{i=1}^q f_q(x_{i(q)c}) \prod_{c=1}^{r_x} \prod_{i=q+1}^n f_{q+1}(x_{i(q+1)c}) \prod_{d=1}^{r_y} \prod_{j=1}^v f_v(y_{j(v)d}) \prod_{d=1}^{r_y} \prod_{j=v+1}^m f_{v+1}(y_{j(v+1)d}), \quad (26)$$

where

$$f_q(x_{i(q)c}) = \frac{n! \lambda \theta}{(q-1)! q!} \left[1 - \left(1 - (1 + x_{i(q)c})^{-\lambda} \right)^\theta \right]^q (1 + x_{i(q)c})^{-(\lambda+1)} \left(1 - (1 + x_{i(q)c})^{-\lambda} \right)^{\theta q - 1}, \quad (27)$$

$$f_{q+1}(x_{i(q+1)c}) = \frac{n! \lambda \theta}{(q-1)! q!} \left[1 - \left(1 - (1 + x_{i(q+1)c})^{-\lambda} \right)^\theta \right]^{q-1} \times (1 + x_{i(q+1)c})^{-(\lambda+1)} \left(1 - (1 + x_{i(q+1)c})^{-\lambda} \right)^{\theta q + \theta - 1}, \quad (28)$$

$$f_v(y_{j(v)d}) = \frac{m! \lambda \beta}{(v-1)! v!} \left[1 - \left(1 - (1 + y_{j(v)d})^{-\lambda} \right)^\beta \right]^v (1 + y_{j(v)d})^{-(\lambda+1)} \left(1 - (1 + y_{j(v)d})^{-\lambda} \right)^{\beta v - 1}, \quad (29)$$

and

$$f_{v+1}(y_{j(v+1)d}) = \frac{m! \lambda \beta}{(v-1)! v!} \left[1 - \left(1 - (1 + y_{j(v+1)d})^{-\lambda} \right)^\beta \right]^{v-1} (1 + y_{j(v+1)d})^{-(\lambda+1)} \times \left(1 - (1 + y_{j(v+1)d})^{-\lambda} \right)^{\beta v + \beta - 1}. \quad (30)$$

be the pdfs of $X_{i(q)c}$, $X_{i(q+1)c}$, $Y_{j(v)d}$ and $Y_{j(v+1)d}$, respectively, where $x_{i(q)c} > 0, x_{i(q+1)c} > 0, y_{j(v)d} > 0$, and $y_{j(v+1)d} > 0$. The log-likelihood function of L_3 will be as follows:

$$\ln L_3 = \ln H_3 + nr_x \ln(\lambda \theta) + \sum_{c=1}^{r_x} (A_1 + A_2) - (\lambda + 1) \sum_{c=1}^{r_x} \left[\sum_{i=1}^q \ln M_{i(q)c} + \sum_{i=q+1}^n \ln M_{i(q+1)c} \right] + mr_y \ln(\lambda \beta) + \sum_{d=1}^{r_y} (B_1 + B_2) - (\lambda + 1) \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \ln N_{j(v)d} + \sum_{j=v+1}^m \ln N_{j(v+1)d} \right], \quad (31)$$

where H_3 is a constant, $M_{i(q)c} = 1 + x_{i(q)c}$, and $N_{j(v)d} = 1 + y_{j(v)d}$,

$$A_1 = \sum_{i=1}^q q \ln \left[1 - \left(1 - M_{i(q)c}^{-\lambda} \right)^\theta \right] + \sum_{i=q+1}^n (q-1) \ln \left[1 - \left(1 - M_{i(q+1)c}^{-\lambda} \right)^\theta \right], \quad (32)$$

$$A_2 = \sum_{i=1}^q (\theta q - 1) \ln \left(1 - M_{i(q)c}^{-\lambda} \right) + \sum_{i=q+1}^n (\theta q + \theta - 1) \ln \left(1 - M_{i(q+1)c}^{-\lambda} \right), \quad (33)$$

$$B_1 = \sum_{j=1}^v \nu \ln \left[1 - \left(1 - N_{j(\nu)d}^{-\lambda} \right)^\beta \right] + \sum_{j=\nu+1}^m (\nu-1) \ln \left[1 - \left(1 - N_{j(\nu+1)d}^{-\lambda} \right)^\beta \right], \quad (34)$$

and

$$B_2 = \left[\sum_{j=1}^v (\beta\nu-1) \ln \left(1 - N_{j(\nu)d}^{-\lambda} \right) + \sum_{j=\nu+1}^m (\beta\nu+\beta-1) \ln \left(1 - N_{j(\nu+1)d}^{-\lambda} \right) \right]. \quad (35)$$

The ML estimators of θ, β and λ can be obtained by maximizing $\ln L_3$ directly with respect to θ, β and λ . The first partial derivatives of log-likelihood function with respect to θ, β and λ are given by:

$$\begin{aligned} \frac{\partial \ln L_3}{\partial \theta} &= \frac{nr_x}{\theta} - \sum_{c=1}^{r_x} \left[\sum_{i=1}^q \frac{q \ln \left(1 - M_{i(q)c}^{-\lambda} \right)}{\left[\left(1 - M_{i(q)c}^{-\lambda} \right)^{-\theta} - 1 \right]} + \sum_{i=q+1}^n \frac{(q-1) \ln \left(1 - M_{i(q+1)c}^{-\lambda} \right)}{\left[\left(1 - M_{i(q+1)c}^{-\lambda} \right)^{-\theta} - 1 \right]} \right] \\ &+ \sum_{c=1}^{r_y} \left[\sum_{i=1}^q q \ln \left(1 - M_{i(q)c}^{-\lambda} \right) + \sum_{i=q+1}^n (q+1) \ln \left(1 - M_{i(q+1)c}^{-\lambda} \right) \right], \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial \ln L_3}{\partial \beta} &= \frac{mr_y}{\beta} - \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \frac{\nu \ln \left(1 - N_{j(\nu)d}^{-\lambda} \right)}{\left[\left(1 - N_{j(\nu)d}^{-\lambda} \right)^{-\beta} - 1 \right]} + \sum_{j=\nu+1}^m \frac{(\nu-1) \ln \left(1 - N_{j(\nu+1)d}^{-\lambda} \right)}{\left[\left(1 - N_{j(\nu+1)d}^{-\lambda} \right)^{-\beta} - 1 \right]} \right] \\ &+ \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \nu \ln \left(1 - N_{j(\nu)d}^{-\lambda} \right) + \sum_{j=\nu+1}^m (\nu+1) \ln \left(1 - N_{j(\nu+1)d}^{-\lambda} \right) \right], \end{aligned} \quad (37)$$

and

$$\begin{aligned} \frac{\partial \ln L_3}{\partial \lambda} &= \frac{nr_x + mr_y}{\lambda} + \sum_{c=1}^{r_x} (A'_1 + A'_2) - \sum_{c=1}^{r_x} \left[\sum_{i=1}^q \ln M_{i(q)c} + \sum_{i=q+1}^n \ln M_{i(q+1)c} \right] \\ &+ \sum_{d=1}^{r_y} (B'_1 + B'_2) - \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \ln N_{j(\nu)d} + \sum_{j=\nu+1}^m \ln N_{j(\nu+1)d} \right], \end{aligned} \quad (38)$$

where

$$A'_1 = - \left[\sum_{c=1}^q \frac{\theta q \left(1 - M_{i(q)c}^{-\lambda} \right)^{\theta-1} M_{i(q)c}^{-\lambda} \ln M_{i(q)c}}{1 - \left(1 - M_{i(q)c}^{-\lambda} \right)^\theta} + \sum_{c=q+1}^n \frac{(q-1) \theta \left(1 - M_{i(q+1)c}^{-\lambda} \right)^{\theta-1} M_{i(q+1)c}^{-\lambda} \ln M_{i(q+1)c}}{1 - \left(1 - M_{i(q+1)c}^{-\lambda} \right)^\theta} \right], \quad (39)$$

$$A'_2 = \sum_{i=1}^q \frac{(\theta q - 1) M_{i(q)c}^{-\lambda} \ln M_{i(q)c}}{1 - M_{i(q)c}^{-\lambda}} + \sum_{i=q+1}^n \frac{(\theta q + \theta - 1) M_{i(q+1)c}^{-\lambda} \ln M_{i(q+1)c}}{1 - M_{i(q+1)c}^{-\lambda}}, \quad (40)$$

$$B'_1 = - \left[\sum_{j=1}^{\nu} \frac{\nu\beta \left(1 - N_{j(\nu)d}^{-\lambda}\right)^{\beta-1} N_{j(\nu)d}^{-\lambda} \ln N_{j(\nu)d}}{1 - \left(1 - N_{j(\nu)d}^{-\lambda}\right)^{\beta}} + \sum_{j=\nu+1}^m \frac{(\nu-1)\beta \left(1 - N_{j(\nu+1)d}^{-\lambda}\right)^{\beta-1} \times N_{j(\nu+1)d}^{-\lambda} \ln N_{j(\nu+1)d}}{1 - \left(1 - N_{j(\nu+1)d}^{-\lambda}\right)^{\beta}} \right]. \quad (41)$$

and

$$B'_2 = \left[\sum_{j=1}^{\nu} \frac{(\beta\nu - 1)N_{j(\nu)d}^{-\lambda} \ln N_{j(\nu)d}}{1 - N_{j(\nu)d}^{-\lambda}} + \sum_{j=\nu+1}^m \frac{(\beta\nu + \nu - 1)N_{j(\nu+1)d}^{-\lambda} \ln N_{j(\nu+1)d}}{1 - N_{j(\nu+1)d}^{-\lambda}} \right]. \quad (42)$$

Eqs. (36)-(38) are solved numerically via iterative technique to obtain the ML estimators of population parameters. Further, the ML estimators of population parameters will be employed in Eq. (3) to get the reliability estimator.

4.3 Reliability estimator with odd strength and even stress set sizes

Here, the reliability estimator is obtained when the strength of X has MRSS with an odd set size, while the stress of Y has MRSS with an even set size. Let $X_{i(g)c}; i=1, \dots, n, c=1, \dots, r_x; g = [(n+1)/2]$ be an MRSS observed from $X \sim EP(\lambda, \theta)$ with sample size nr_x , where n is the odd set size and r_x is the number of cycles. Let the set $\{Y_{j(\nu)d}; j=1, \dots, \nu; d=1, \dots, r_y\} \cup \{Y_{j(\nu+1)d}; j=\nu+1, \dots, m; d=1, \dots, r_y\}$ be an MRSS drawn from $EP(\lambda, \beta)$ with even set sizes where $\nu = m/2$. Therefore, the likelihood functions L_4 of the observed data will be as follows:

$$L_4 = \prod_{c=1}^{r_x} \prod_{i=1}^n f_g(x_{i(g)c}) \prod_{d=1}^{r_y} \prod_{j=1}^{\nu} f_{\nu}(y_{j(\nu)d}) \prod_{d=1}^{r_y} \prod_{j=\nu+1}^m f_{\nu+1}(y_{j(\nu+1)d}). \quad (43)$$

Therefore, the likelihood functions L_4 of the observed data will be as follows

$$\begin{aligned} \ln L_4 = & \ln H_4 + r_x n \ln(\lambda\theta) + \sum_{c=1}^{r_x} \sum_{i=1}^n (g-1) \ln \left[1 - \left(1 - Q_{i(g)c}^{-\lambda}\right)^{\theta} \right] - (\lambda+1) \sum_{c=1}^{r_x} \sum_{i=1}^n \ln(Q_{i(g)c}) \\ & + mr_y \ln(\lambda\beta) + \sum_{c=1}^{r_x} \sum_{i=1}^n (\theta g - 1) \ln \left(1 - Q_{i(g)c}^{-\lambda}\right) + \sum_{d=1}^{r_y} (B_1 + B_2) \\ & - (\lambda+1) \sum_{d=1}^{r_y} \left[\sum_{j=1}^{\nu} \ln N_{j(\nu)d} + \sum_{j=\nu+1}^m \ln N_{j(\nu+1)d} \right], \end{aligned} \quad (44)$$

where H_4 is a constant, $Q_{i(g)c} = 1 + x_{i(g)c}$, and $N_{j(\nu)d} = 1 + y_{j(\nu)d}$. The ML estimators of θ, β and λ can be obtained by maximizing $\ln L_4$ directly with respect to the parameters θ, β and λ . The first partial derivatives of log-likelihood function with respect to θ, β and λ are given by:

$$\frac{\partial \ln L_4}{\partial \theta} = \frac{r_x n}{\theta} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(g-1) \ln(1 - Q_{i(g)c}^{-\lambda})}{\left[(1 - Q_{i(g)c}^{-\lambda})^{-\theta} - 1 \right]} - \sum_{c=1}^{r_x} \sum_{i=1}^n g \ln(1 - Q_{i(g)c}^{-\lambda}), \quad (45)$$

$$\begin{aligned} \frac{\partial \ln L_4}{\partial \beta} = & \frac{mr_y}{\beta} - \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \frac{v \ln(1 - N_{j(v)d}^{-\lambda})}{\left[(1 - N_{j(v)d}^{-\lambda})^{-\beta} - 1 \right]} + \sum_{j=v+1}^m \frac{(v-1) \ln(1 - N_{j(v+1)d}^{-\lambda})}{\left[(1 - N_{j(v+1)d}^{-\lambda})^{-\beta} - 1 \right]} \right] \\ & + \sum_{d=1}^{r_y} \left[\sum_{j=1}^v v \ln(1 - N_{j(v)d}^{-\lambda}) + \sum_{j=v+1}^m (v+1) \ln(1 - N_{j(v+1)d}^{-\lambda}) \right], \end{aligned} \quad (46)$$

and

$$\begin{aligned} \frac{\partial \ln L_4}{\partial \lambda} = & \frac{r_x n + r_y m}{\lambda} - \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(g-1) \theta (1 - Q_{i(g)c}^{-\lambda})^{\theta-1} Q_{i(g)c}^{-\lambda} \ln(Q_{i(g)c})}{\left[1 - (1 - Q_{i(g)c}^{-\lambda})^\theta \right]} \\ & - \sum_{c=1}^{r_x} \sum_{i=1}^n \ln(Q_{i(g)c}) + \sum_{c=1}^{r_x} \sum_{i=1}^n \frac{(\theta g - 1) \ln(Q_{i(g)c})}{Q_{i(g)c}^{-\lambda} - 1} \\ & + \sum_{d=1}^{r_y} (B'_1 + B'_2) - \sum_{d=1}^{r_y} \left[\sum_{j=1}^v \ln N_{j(v)d} + \sum_{j=v+1}^m \ln N_{j(v+1)d} \right]. \end{aligned} \quad (47)$$

As it seems the likelihood Eqs. (45)-(47) have no closed-form solutions. Therefore, the numerical technique will be utilized to get the solution. The ML estimators of population parameters will be employed in Eq. (3) to get the reliability estimator.

4.4 Reliability estimator with even strength and odd stress set sizes

Here, the reliability estimator is obtained when the strength of X is selected using MRSS with even set size, while the stress of Y is selected based on MRSS with an odd set size. Let the set $\{X_{i(q)c}, i=1, \dots, q; c=1, \dots, r_x\} \cup \{X_{i(q+1)c}, i=q+1, \dots, n; c=1, \dots, r_x\}$ be an MRSS drawn from EP(λ, θ) with even set sizes where $q = n/2$. Let $Y_{j(k)d}; j=1, \dots, m, d=1, \dots, r_y; k = [(m+1)/2]$ be MRSS observed from $Y \sim \text{EP}(\lambda, \beta)$ with sample size mr_y , where m is the set size and r_y is the number of cycles. Therefore, the likelihood functions L_5 of observed data, where X and Y independent, is given by:

$$L_5 = \prod_{c=1}^{r_x} \prod_{i=1}^q f_q(x_{i(q)c}) \prod_{c=1}^{r_x} \prod_{i=q+1}^n f_{q+1}(x_{i(q+1)c}) \prod_{d=1}^{r_y} \prod_{j=1}^m f_k(y_{j(k)d}),$$

Therefore, the likelihood functions L_5 of the observed data will be as follows:

$$\begin{aligned} \ln L_5 = & \ln H_5 + nr_x \ln(\lambda \theta) + \sum_{i=1}^{r_x} (A_1 + A_2) - (\lambda + 1) \sum_{i=1}^{r_x} \left[\sum_{c=1}^q \ln M_{i(q)c} + \sum_{c=q+1}^n \ln M_{i(q+1)c} \right] \\ & + r_y m \ln(\lambda \beta) + \sum_{d=1}^{r_y} \sum_{j=1}^m (k-1) \ln \left[1 - (1 - T_{j(k)d}^{-\lambda})^\beta \right] - (\lambda + 1) \sum_{d=1}^{r_y} \sum_{j=1}^m \ln(T_{j(k)d}) \\ & + \sum_{d=1}^{r_y} \sum_{j=1}^m (\beta k - 1) \ln(1 - T_{j(k)d}^{-\lambda}), \end{aligned}$$

where H_5 is a constant, $M_{i(q)c} = 1 + x_{i(q)c}$ and $T_{j(k)d} = 1 + y_{j(k)d}$. The ML estimators of θ, β and λ can be obtained by maximizing $\ln L_5$ directly with respect to θ, β and λ . The first partial derivatives of log-likelihood function with respect to θ, β and λ are given by:

$$\frac{\partial \ln L_5}{\partial \theta} = \frac{nr_x}{\theta} - \sum_{c=1}^{r_x} \left[\sum_{i=1}^q \frac{q \ln(1 - M_{i(q)c}^{-\lambda})}{\left[(1 - M_{i(q)c}^{-\lambda})^{-\theta} - 1 \right]} + \sum_{i=q+1}^n \frac{(q-1) \ln(1 - M_{i(q+1)c}^{-\lambda})}{\left[(1 - M_{i(q+1)c}^{-\lambda})^{-\theta} - 1 \right]} \right] + \sum_{c=1}^{r_x} \left[\sum_{i=1}^q q \ln(1 - M_{i(q)c}^{-\lambda}) + \sum_{i=q+1}^n (q+1) \ln(1 - M_{i(q+1)c}^{-\lambda}) \right], \tag{48}$$

$$\frac{\partial \ln L_5}{\partial \beta} = \frac{r_y m}{\beta} - \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(k-1) \ln(1 - T_{j(k)d}^{-\lambda})}{\left[(1 - T_{j(k)d}^{-\lambda})^{-\beta} - 1 \right]} + \sum_{d=1}^{r_y} \sum_{j=1}^m k \ln(1 - T_{j(k)d}^{-\lambda}), \tag{49}$$

and

$$\frac{\partial \ln L_5}{\partial \lambda} = \frac{nr_x + mr_y}{\lambda} - \sum_{i=1}^{r_x} (A'_1 + A'_2) - \sum_{c=1}^{r_x} \left[\sum_{i=1}^q \ln M_{i(q)c} + \sum_{i=q+1}^n \ln M_{i(q+1)c} \right] - \sum_{d=1}^{r_y} \sum_{j=1}^m \ln(T_{j(k)d}) - \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(k-1)\beta \ln(1 - T_{j(k)d}^{-\lambda}) T_{j(k)d}^{-\lambda} \ln(T_{j(k)d})}{\left[1 - (1 - T_{j(k)d}^{-\lambda})^\beta \right]} + \sum_{d=1}^{r_y} \sum_{j=1}^m \frac{(\beta k - 1) \ln(T_{j(k)d})}{T_{j(k)d}^\lambda - 1}. \tag{50}$$

It is clear that the Eqs. (48)-(50) do not have an analytical solution, so the ML estimators of θ, β and λ can also be obtained by numerical methods. Hence, the ML estimators of θ and β are employed in Eq. (3) to get reliability estimator.

5 Simulation study

In this section, an extensive simulation study is provided to compare the performance of the reliability estimates based on RSS and MRSS with their SRS counterparts. In this comparison, the absolute bias, the mean square error and the relative efficiency criteria are utilized. In the simulation setup, the set sizes and the number of cycles are selected as $(m_x, m_y) = (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5), (6, 6), (7, 7)$ and $C = r_x = r_y = 5$, respectively. Thus, the sample sizes are obtained as $n = m_x r$ and $m = m_y r$ for RSS and MRSS samples. Also, we use $(n, m) = (10, 10), (10, 15), (15, 10), (15, 15), (15, 20), (20, 15), (20, 20), (25, 25), (30, 30), (35, 35)$ as a sample sizes for SRS samples. The values of parameters are selected as $(\theta, \beta) = (3, 2), (5, 2), (5, 1), (8, 0.8)$ and $\lambda = 1$, where the true values of system reliability $R = 0.600, 0.714, 0.833$ and 0.933 . We generate 1000 random samples from EP(λ, θ) and EP(λ, β) distributions. The estimated reliability is examined via absolute bias (AB), mean square errors (MSEs), efficiencies and difference of efficiencies (D) criteria. The efficiencies of RSS and MRSS with respect to SRS are defined as:

$$Efficiency(R_{RSS}, R_{SRS}) = \frac{MSE(R_{SRS})}{MSE(R_{RSS})}, \quad Efficiency(R_{MRSS}, R_{SRS}) = \frac{MSE(R_{SRS})}{MSE(R_{MRSS})}$$

and $D = Efficiency(RSS) - Efficiency(MRSS)$.

While the absolute bias is defined as $AB(R_\delta) = |E(R_\delta) - R|$, $\delta = SRS, RSS, MRSS$.

Tabs. 1 to 4 summarize the ABs and MSEs of the reliability estimate based on SRS, RSS and MRSS. They also give the efficiencies of the reliability estimate based on RSS and MRSS with respect to SRS for various values of the sample sizes and distribution parameters.

From Tabs. 1 to 5 and Figs. 1 to 6 we can conclude the following:

- Tabs. 1 to 4 show that the reliability estimates under RSS and MRSS schemes are more efficient than the corresponding under SRS in approximately most of the situations.
- At true value $R=0.6$, Fig. 1 shows that the MSE of the reliability estimate based on SRS takes the largest values as compared with the corresponding one based on RSS and MRSS. While the MSE of the reliability estimate based on MRSS picks the smallest values in most cases considered in this study.
- The MSE of the reliability estimate based on MRSS, at true value $R=0.714$, gets the smallest values while the MSE of the reliability estimate based on SRS has the largest values.

Table 1: MSEs and ABs of R estimates based on SRS, RSS and MRSS and their efficiencies with respect to SRS for even set size for 5 cycles

C	(m_x, m_y)	SRS		RSS		MRSS		Efficiency	
		AB	MSE	AB	MSE	AB	MSE	RSS	MRSS
R=0.6, $(\theta, \beta)=(3, 2)$									
5	(2,2)	0.3630	0.1674	0.3953	0.1562	0.1974	0.0395	1.0717	4.2380
5	(4,4)	0.3821	0.1635	0.1591	0.0256	0.1545	0.0239	6.3867	6.8410
5	(6,6)	0.4000	0.1600	0.1867	0.0350	0.1393	0.0194	4.5714	8.2474
R=0.714, $(\theta, \beta)=(5, 2)$									
5	(2,2)	0.2827	0.0829	0.2809	0.0789	0.2590	0.0671	1.0507	1.2355
5	(4,4)	0.2841	0.0821	0.2825	0.0798	0.2643	0.0699	1.0288	1.1745
5	(6,6)	0.2830	0.0802	0.2822	0.0796	0.1535	0.0236	1.0075	3.3983
R=0.833, $(\theta, \beta)=(5, 1)$									
5	(2,2)	0.1667	0.0278	0.1620	0.0263	0.1314	0.0173	1.0570	1.6069
5	(4,4)	0.1653	0.0273	0.1631	0.0266	0.1341	0.0180	1.0263	1.5167
5	(6,6)	0.1667	0.0278	0.1473	0.0218	0.1535	0.0236	1.2752	1.1780
R=0.9, $(\theta, \beta)=(8, 0.8)$									
5	(2,2)	0.0809	0.0165	0.0500	0.0026	0.0597	0.0036	6.3462	4.5833
5	(4,4)	0.0958	0.0093	0.0470	0.0024	0.0908	0.0083	3.8750	1.1205
5	(6,6)	0.0918	0.0086	0.0450	0.0021	0.0909	0.0083	4.0952	1.0361

Table 2: MSEs and ABs of R estimates based on SRS, RSS and MRSS and their efficiencies with respect to SRS for odd set size for 5 cycles

C	(m_x, m_y)	SRS		RSS		MRSS		Efficiency	
		AB	MSE	AB	MSE	AB	MSE	RSS	MRSS
$R=0.6, (\theta, \beta)=(3, 2)$									
5	(3,3)	0.354	0.1692	0.3878	0.1504	0.3685	0.1658	1.125	1.021
5	(5,5)	0.386	0.1628	0.1591	0.0256	0.2910	0.085	6.359	1.915
5	(7,7)	0.400	0.1600	0.1582	0.0251	0.2100	0.044	6.375	3.636
$R=0.714, (\theta, \beta)=(5, 2)$									
5	(3,3)	0.2847	0.0821	0.2788	0.0777	0.2071	0.0434	1.057	1.892
5	(5,5)	0.2857	0.0816	0.2823	0.0797	0.1150	0.0130	1.024	6.277
5	(7,7)	0.2787	0.0779	0.2796	0.0782	0.1140	0.0130	0.996	5.992
$R=0.833, (\theta, \beta)=(5, 1)$									
5	(3,3)	0.1654	0.0274	0.1630	0.0266	0.1248	0.0159	1.030	1.723
5	(5,5)	0.1653	0.0273	0.1499	0.0226	0.0420	0.0022	1.208	12.409
5	(7,7)	0.1658	0.0275	0.1456	0.0213	0.0320	0.0014	1.291	19.643
$R=0.9, (\theta, \beta)=(8, 0.8)$									
5	(3,3)	0.0809	0.0164	0.0480	0.0023	0.0082	0.0011	7.130	14.909
5	(5,5)	0.0933	0.0088	0.0460	0.0022	0.0055	0.0008	4.000	11.000
5	(7,7)	0.0915	0.0084	0.0450	0.0020	0.0033	0.0005	4.200	16.800

Table 3: MSEs and ABs of R estimates based on SRS, RSS and MRSS and their efficiencies with respect to SRS when X has odd set size and Y has even set sizes for 5 cycles

C	(m_x, m_y)	SRS		RSS		MRSS		Efficiency	
		AB	MSE	AB	MSE	AB	MSE	RSS	MRSS
$R=0.6, (\theta, \beta)=(3, 2)$									
5	(3,2)	0.3620	0.1676	0.3950	0.1560	0.1306	0.0180	1.0744	9.3111
5	(3,4)	0.3621	0.1675	0.3990	0.1602	0.2918	0.0852	1.0456	1.9660
$R=0.714, (\theta, \beta)=(5, 2)$									
5	(3,2)	0.2857	0.0816	0.2769	0.0766	0.2232	0.0513	1.0653	1.5906
5	(3,4)	0.2847	0.0821	0.2822	0.0796	0.2813	0.0792	1.0314	1.0366
$R=0.833, (\theta, \beta)=(5, 1)$									
5	(3,2)	0.1657	0.0275	0.1456	0.0218	0.1413	0.0253	1.2615	1.0870
5	(3,4)	0.1657	0.0275	0.1670	0.0280	0.1661	0.0276	0.9821	0.9964
$R=0.9, (\theta, \beta)=(8, 0.8)$									
5	(3,2)	0.0291	0.0120	0.0470	0.0022	0.0895	0.0080	5.4550	1.5000
5	(3,4)	0.0960	0.0093	0.0510	0.0026	0.0907	0.0082	3.5770	1.1340

Table 4: MSEs and ABs of R estimates based on SRS, RSS and MRSS and their efficiencies with respect to SRS when X has even set size and Y has odd set size for 5 cycles

C	(m_x, m_y)	SRS		RSS		MRSS		Efficiency	
		AB	MSE	AB	MSE	AB	MSE	RSS	MRSS
$R=0.6, (\theta, \beta)=(3, 2)$									
5	(2,3)	0.358	0.1684	0.396	0.1608	0.2068	0.0431	1.047	3.907
5	(4,3)	0.375	0.165	0.1527	0.0237	0.2717	0.0847	6.962	1.948
$R=0.714, (\theta, \beta)=(5, 2)$									
5	(2,3)	0.2837	0.0825	0.2857	0.0816	0.26	0.0676	1.011	1.22
5	(4,3)	0.2857	0.0816	0.2825	0.0798	0.2735	0.0748	1.023	1.091
$R=0.833, (\theta, \beta)=(5, 1)$									
5	(2,3)	0.1658	0.0275	0.0346	0.0012	0.1471	0.0217	22.917	1.267
5	(4,3)	0.1658	0.0275	0.1461	0.0216	0.0797	0.0321	1.273	0.857
$R=0.9, (\theta, \beta)=(8, 0.8)$									
5	(2,3)	0.0809	0.0164	0.049	0.0024	0.0076	0.0155	6.833	1.058
5	(4,3)	0.0943	0.009	0.049	0.0024	0.015	0.0003	3.75	3

Table 5: A comparison between RSS and MRSS based on Tab. 1 to Tab. 4

Table 1		D	Table 2		D
RSS	MRSS		RSS	MRSS	
1.0717	4.238	-3.1663	1.125	1.021	0.104
6.3867	6.841	-0.4543	6.359	1.915	4.444
4.5714	8.2474	-3.676	6.375	3.636	2.739
1.0507	1.2355	-0.1848	1.057	1.892	-0.835
1.0288	1.1745	-0.1457	1.024	6.277	-5.253
1.0075	3.3983	-2.3908	0.996	5.992	-4.996
1.057	1.6069	-0.5499	1.03	1.723	-0.693
1.0263	1.5167	-0.4904	1.208	12.409	-11.201
1.2752	1.178	0.0972	1.291	19.643	-18.352
6.3462	4.5833	1.7629	7.13	14.909	-7.779
3.875	1.1205	2.7545	4	11	-7
4.0952	1.0361	3.0591	4.2	16.8	-12.6
Table 3		D	Table 4		D
RSS	MRSS		RSS	MRSS	
1.0744	9.3111	-8.2367	1.047	3.907	-2.86
1.0456	1.966	-0.9204	6.962	1.948	5.014
1.0653	1.5906	-0.5253	1.011	1.22	-0.209
1.0314	1.0366	-0.0052	1.023	1.091	-0.068
1.2615	1.087	0.1745	22.917	1.267	21.65
0.9821	0.9964	-0.0143	1.273	0.857	0.416

5.455	1.5	3.955	6.833	1.058	5.775
3.577	1.134	2.443	3.75	3	0.75

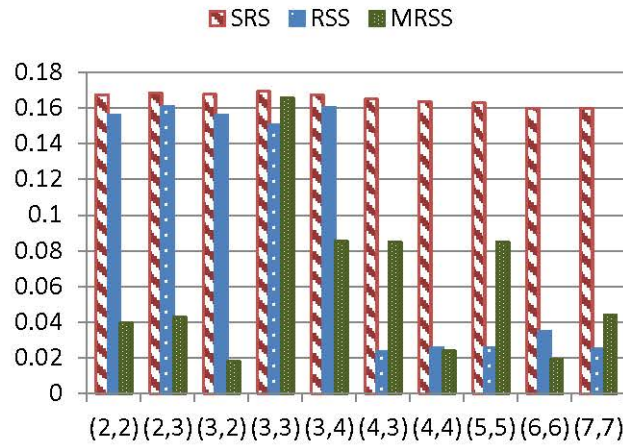


Figure 1: MSE of the reliability estimate at $R=0.6$

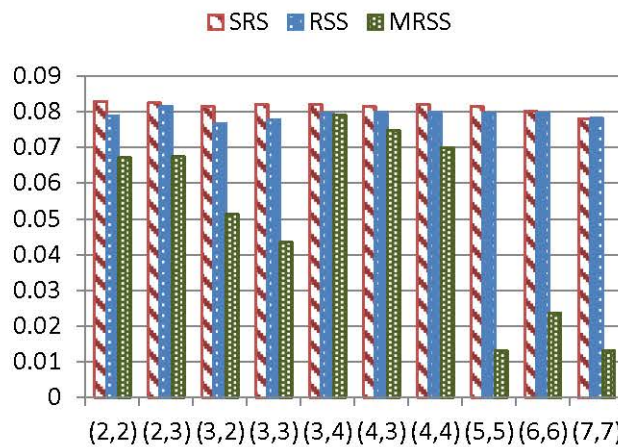


Figure 2: MSE of the reliability estimate at $R=0.714$

- The AB of the reliability estimate under SRS takes the largest values compared with the corresponding for all values of R in most of the cases (see Figs. 3 and 4).

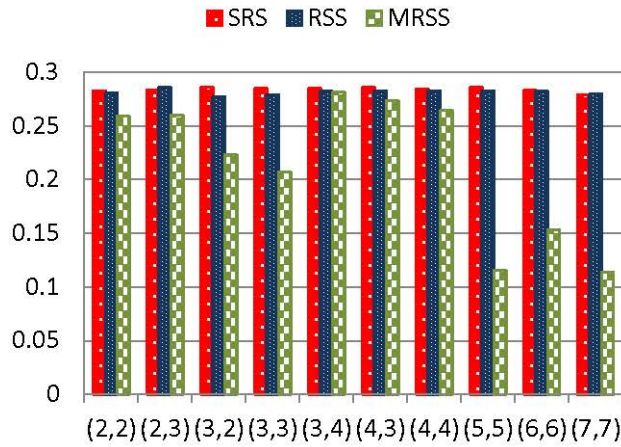


Figure 3: AB of the reliability estimate at $R=0.714$

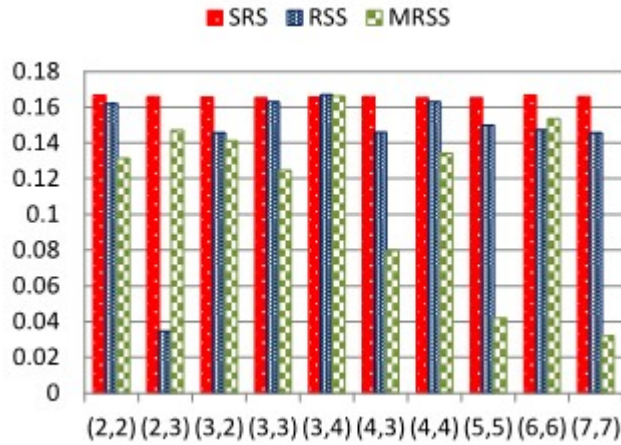


Figure 4: AB of the reliability estimates at $R=0.833$

- At true value $R=0.833$, the MSE of the reliability estimate under MRSS and RSS have smallest values except at $(m_x, m_y)=(3, 4), (4, 3)$ (see Fig. 5). At true value, $R=0.9$, Fig. 6 demonstrates that the reliability estimate under RSS has less MSE than that the corresponding under MRSS except at $(m_x, m_y)=(3, 3), (4, 3), (5, 5)$ and $(7, 7)$.

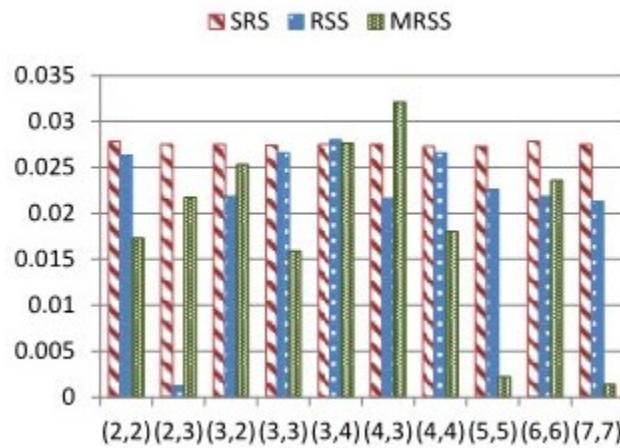


Figure 5: MSE of all reliability estimates at R=0.833

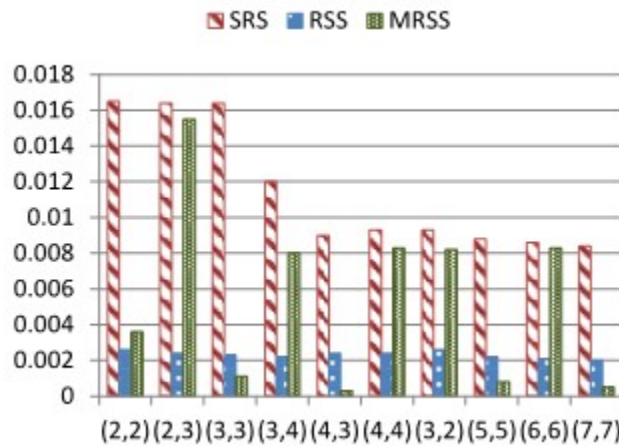


Figure 6: MSE of all reliability estimates at R=0.9

- Fig. 7 shows that the efficiency of the reliability estimate under MRSS is greater than the efficiency of the reliability estimate under RSS expect at $(m_x, m_y)=(3, 3), (5, 5), (7, 7)$, where the true value of $R=0.6$.
- The reliability estimate under MRSS has greater efficiency than the reliability estimate based on RSS in case of the true value of $R=0.714$ (see Fig. 8).

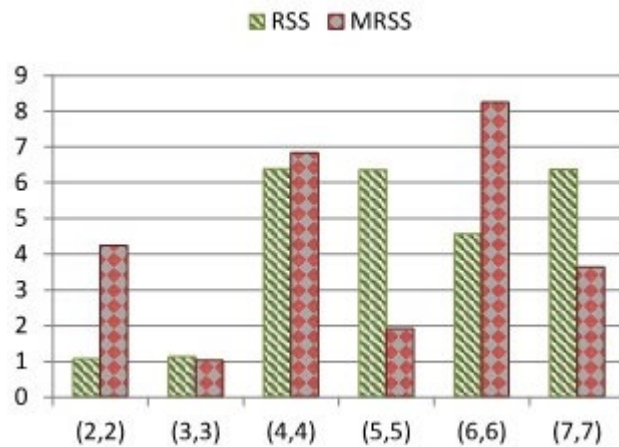


Figure 7: Efficiency of the reliability estimate at $R=0.6$

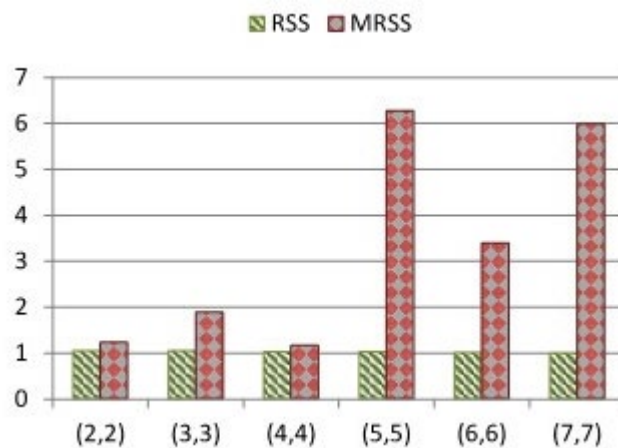


Figure 8: Efficiency of the reliability estimate at $R=0.714$

- For the true value $R=0.833$, the reliability estimate under MRSS has greater efficiency than the reliability estimate under RSS for all values of (m_x, m_y) expect at $(2, 3)$.
- As the set size increases, the efficiency of all the reliability estimates increases in almost all cases.

6 Conclusions

In this article, the estimation of the reliability $R=P[Y<X]$ is studied when the strength X and stress Y are independent variables follow the exponentiated Pareto distribution. The maximum likelihood estimators of R are computed using SRS, RSS and MRSS. Based on MRSS, the reliability estimate is considered in four different cases. The simulation study is performed to evaluate the performance of the different proposed estimates. From the simulation study, it is found that, the MSEs of the reliability estimates based on SRS data are greater than their competitors based on RSS and MRSS data, respectively. The MSEs

of the reliability estimates under MRSS are the smallest in most of the cases as compared with the corresponding estimates using RSS and SRS data. The efficiency of all reliability estimates increases as the set size increases in almost cases.

Also, it is found that the reliability estimates using MRSS is more efficient than the reliability estimates based on RSS. Also, the reliability estimates based on RSS and MRSS are more efficient than the reliability estimates under SRS. In general, the reliability estimates under MRSS are more efficient as compared with the other reliability estimates based on RSS and SRS methods.

Acknowledgement: The authors are sincerely grateful to the anonymous referees for their time and effort in providing very constructive, helpful, and valuable comments and suggestions that have led to a substantial improvement in the quality of the paper.

Funding Statement: The authors extend their appreciation to Deanship of Scientific Research at King Khalid University for funding this work through Research Groups Program under grant number R. G. P. 2/67/41. I. M. A. and A. I. A. who received the grant.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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