Bell Polynomial Approach for the Solutions of Fredholm Integro-Differential Equations with Variable Coefficients

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Abstract: In this article, we approximate the solution of high order linear Fredholm integro-differential equations with a variable coefficient under the initial-boundary conditions by Bell polynomials. Using collocation points and treating the solution as a linear combination of Bell polynomials, the problem is reduced to linear system of equations whose unknown variables are Bell coefficients. The solution to this algebraic system determines the approximate solution. Error estimation of approximate solution is done. Some examples are provided to illustrate the performance of the method. The numerical results are compared with the collocation method based on Legendre polynomials and the other two methods based on Taylor polynomials. It is observed that the method is better than Legendre collocation method and as accurate as the methods involving Taylor polynomials.

Keywords: Bell polynomials, collocation points, matrix method, Fredholm integro-differential equations.

1 Introduction

Integro-differential equations are one of the practical tools used in different disciplines of sciences ranging from engineering to social sciences. Many mathematical models in those areas involve integro-differential equations. Nuclear physics [Velasquez, Kelkar and Upadhyay (2019)], molecular biology [Alonso, Bermejo, Pájaro et al. (2018)], biological population models and ecology [Lutscher (2019)], control and stability theory [Alabau-Boussouira, Ancona, Porretta et al. (2019)], elasticity theory [Umesh, Rajagopal and Reddy (2019)], electromagnetic [Barrios, Retamal, Solano et al. (2019)], viscoelasticity [Vlasov and Rautian (2019)], hydrodynamics [Rodrigues, Silva, Ramos et al. (2017)], economics [Rivaz, Moghadam and Baniasadi (2019)] are among the well-known exemplary areas.

Our main interest is Fredholm integro differential equations (FIDE) with variable coefficients. Since it is usually difficult to obtain exact solution for FIDEs, the

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development of numerical techniques has been the centre of attention for a large number of researchers. Since we are specially interested in solution of the linear FIDEs in this work, we overview the some works on the solution of this kind of FIDEs and some Volterra types. Hosseini et al. [Hosseini and Shahmorad (2003)] proposed Tau numerical solution method, Maleknejad et al. [Maleknejad and Mahmoidi (2004)] gave a solution by using hybrid Taylor and block-pulse functions, Mohsen et al. [Mohsen and El-Gamel (2007)] used a Sinc-collocation method for the linear FIDEs, He et al. [He and Wu (2007)] proposed variational iteration method, Kurt et al. [Kurt and Sezer (2008)] presented a Taylor polynomial approach, Farnoosh et al. [Farnoosh and Ebrahimi (2008)] gave the Monte Carlo method. Vahidi et al. [Vahidi, Babolian, Cordshooli et al. (2009)] gave the solution via Adomian's decomposition method. Yüzbaşı et al. [Yüzbaşı, Şahin and Sezer (2011)] suggested numerical solutions for the systems of linear FIDEs with Bessel polynomial bases. Akyüz-Daşcıoğlu et al. [Akyüz-Daşcıoğlu and Sezer (2012)] presented a matrix method to solve approximately the most general higher order linear FIDEs with variable coefficients under the mixed conditions in terms of Taylor polynomials, Yalcınbaş et al. [Yalçınbaş and Akkaya (2012)] proposed a solution via Boubaker polynomial bases, Yüksel et al. [Yüksel, Yüzbaşı and Sezer (2012)] used a Chebyshev method, Erdem et al. [Erdem, Yalçınbaş and Sezer (2013)] presented Bernoulli polynomial approach for mixed linear FIDEs, Fathy et al. [Fathy, El-Gamel and El-Azab (2014)] used Legendre-Galerkin method for the linear FIDEs, Mirzaee et al. [Mirzaee and Hoseini (2014)] suggested solution via Fibonacci polynomials for the systems of linear FIDEs with Fibonacci polynomials, Oğuz et al. [Oğuz and Sezer (2015)] proposed Chelyshkov collocation method, Yüzbaşı et al. [Yüzbaşı, Gök and Sezer (2015)] gives Müntz-Legendre matrix method for solving delay FIDEs with constant coefficients, Savasaneril et al. [Savasaneril and Sezer (2016)] used Laguerre polynomial solution to find an approximate solution of linear FIDE with variable coefficients, Elbeleze et al. [Elbeleze, Kılıçman and Taib (2016)] suggested a modified homotopy perturbation method for solving linear second-order FIDE, Yüzbaşı [Yüzbaşı (2017)] suggested Shifted Legendre method with residual error estimation for the solution of delay linear FIDEs, Mollaoğlu et al. [Mollaoğlu and Sezer (2017)] proposed a numerical solution with residual error estimation by using Gegenbauer polynomials, Başar et al. [Başar and Sezer (2018)] gave numerical solution based on Stirling polynomials for solving generalized linear FIDEs with mixed functional arguments, Biçer et al. [Biçer, Öztürk and Gülsu (2018)] used Bernoulli polynomials for the solution of linear FIDE with piecewise intervals, Yüzbaşı [Yüzbaşı (2018)] suggested an exponential method and Yüzbaşı et al. [Yüzbaşı and Ismailov (2018)] gave operational matrix method to solve linear Fredholm-Volterra integro differential equations, Xue et al. [Xue, Niu, Yu et al. (2018)] developed an improved reproducing kernel method for the solution of FIDE type boundary value problems, Shiralashetti et al. [Shiralashetti and Kumbinarasaiah (2019)] presented new operational matrix of differentiation using CAS wavelets and also collocation method by genocchi polynomials, Jalilian et al. [Jalilian and Tahernezhad (2019)] proposed exponential spline method for the solution of FIDEs of second kind, Chen et al. [Chen, He and Zeng (2020)] developed a fast multiscale Galerkin method based on a matrix compression scheme for approximating the second order FIDE with Dirichlet boundary conditions.

In our study, we search for the solutions of the high-order linear Fredholm integro-differential equation with variable coefficients in terms of Bell polynomials. The basic idea is to approximate the solution function via Bell polynomials. On the examples, the efficiency and accuracy is given and also method is compared with three solution methods for integro differential equations given previously in literature: Legendre matrix collocation method given by Yalçinbaş et al. [Yalçinbaş, Sezer and Sorkun (2009)] and two methods based on Taylor polynomials by Yalçinbaş et al. [Yalçınbaş and Sezer (2000); Akyüz-Daşcıoğlu and Sezer (2012)].

The high order linear Fredholm integro differential equation is given as follows:

Definition 1.1. Let m be positive integer and $a_{jk}, b_{jk}, \ \lambda, \lambda_j$ be real numbers for j, k=0, 1,.. $m-1$. Suppose that $g(x), \ P_k(x)$ is continuous on $[a, b]$ and the kernel function $K(x, t)$ is continuous on $[a, b]\times[a, b]$ and

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda \int_a^b K(x, t)y(t)dt, \quad a \leq x, t \leq b \tag{1}$$

$$\sum_{k=0}^{m-1} \left(a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b) \right) = \lambda_j; \ \ j = 0, 1, 2, \ldots, m-1 \tag{2}$$

(1) is called the high-order linear Fredholm integro-differential equation with variable coefficients under the mixed conditions (2).

The Bell polynomials were given by Bell in 1934 [Bell (1934)]. These polynomials can be expressed in some ways. It can be written as a series expansion of a generating exponential function, also it can be given by the second kind of Stirling numbers. Bell polynomials are used in number theory, analysis, combinatorial analysis and statistics. Also, Mirzaee [Mirzaee (2017)] used Bell polynomials to solve integral equations.

Definition 1.2. Bell [Bell (1934)] Let n be a natural number and $S(n, k)$ be the Stirling numbers of second kind, i.e.,

$$S(n, k) = \sum_{j=0}^{k} \frac{(-1)^j}{k!} \binom{k}{j} (k-j)^n.$$

Then

$$B_n(x) = \sum_{k=0}^{n} S(n, k)x^k \tag{3}$$

is called a Bell polynomial of degree n.

We look for an approximate solution of (1) in the following form:

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x) \tag{4}$$

where $a_n, \;\; n = 0, 1, 2, \ldots, N$ are coefficients and $B_n(x)$ are Bell polynomials.

The organization of the paper is as follows: In Section 2 and its subsections, we present matrix representations of Bell polynomials, differential part and integral part of (1). In Section 3, the solution procedure is given. In Section 4, error estimation is done, In Section 5 some numerical examples are given and the comparison of the method with previous methods are given on the examples.

2 Basic matrix relations

Proposition 2.1. Let N be natural number. If the Bell polynomials $B_k(x)$ from k=0 to N are written as the matrix, i.e.,

$\mathbf{B}(\mathrm{x}) = [B_0(x) \;\; B_1(x) \ldots \; B_N(x)]$.

Then

$$\mathbf{B}(\mathrm{x}) = \mathbf{X}(\mathrm{x})\mathbf{S} \tag{5}$$

where

$$\mathbf{X}(\mathrm{x}) = [1 \; x \; x^2 \; \ldots \; x^N], \qquad \mathbf{S} = \begin{bmatrix} S(0,0) & S(1,0) & S(2,0) & \cdots & S(N,0) \\ 0 & S(1,1) & S(2,1) & \cdots & S(N,1) \\ 0 & 0 & S(2,2) & \cdots & S(N,2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S(N,N) \end{bmatrix}.$$

For the sake of clarity, let us denote Eq. (1) in the form

$$D(x) = g(x) + \lambda I(x) \tag{6}$$

where

$$D(x) = \sum_{k=0}^{m} P_k(x) y^{(k)}(x) \text{ and } \quad I(x) = \int_a^b K(x,t) y(t) dt$$

Now we transform the parts $D(x), \; I(x)$ and the conditions (2) into matrix form.

2.1 Matrix relation for the differential part D(x)

Since the approximate solution $y(x)$ is the linear combination of Bell polynomials as shown (3), it can be written as the product of unknown coefficient matrix **A** and **B**(x), i.e.,

$$y(x) = \mathbf{B}(\mathrm{x})\mathbf{A};\ \mathbf{A} = [a_0\ \ a_1\ \ \ldots\ \ a_N]^T \tag{7}$$

By substituting (5) into (7) we obtain

$$y(x) = \mathbf{X}(\mathrm{x})\mathbf{SA} \tag{8}$$

Theorem 2.2. Yalçınbaş et al. [Yalçınbaş and Akkaya (2012)] let $\mathbf{X}(\mathrm{x})$ denoted as in Proposition 2.1. and $\mathbf{X}^{(k)}(\mathrm{x})$ denote the kth derivative of each entry. Then the following equality holds:

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k \tag{9}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{M^0} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus, by means of the equalities (8) and (9), the following equality can be written:

$$y^{(k)}(x) = \mathbf{B}^{(k)}(x)\mathbf{A} = \mathbf{X}^{(k)}(x)\mathbf{SA} = \mathbf{X}(x)\mathbf{M}^k\mathbf{SA} \tag{10}$$

By substituting the expression (10) into Eq. (6), we get the relation

$$D(x) = \sum_{k=0}^{m} P_k(x)\mathbf{X}(x)\mathbf{M}^k\mathbf{SA}.$$

2.2 Matrix representation of Fredholm integral part

Let us find the matrix form of the Fredholm integral part $I(x)$.

Theorem 2.3. Let $K(x,t)$ be an analytic function on $[a,b] \times [a,b]$ where $0 \in [a,b]$. For every $\epsilon > 0$, there exists a natural number N such that

$$\left| K(x,t) - \sum_{p=0}^{N}\sum_{q=0}^{N} k_{pq}x^p t^q \right| < \epsilon$$

where

$$k_{pq} = \frac{1}{p!q!}\frac{\partial^{p+q}K(0,0)}{\partial x^p \partial t^q}, \qquad p,q{=}0,1,\ldots,N.$$

Proof. The theorem follows from the Maclaurin series expansion of $K(x,t)$.

For $\epsilon > 0$, one can accept

$$K(x,t) = \sum_{p=0}^{N} \sum_{q=0}^{N} k_{pq} x^p t^q \tag{11}$$

for a sufficiently large N.

The expression (11) can be converted into the matrix form

$$K(x,t) = \mathbf{X}(x)\mathbf{K}\mathbf{X}^{\mathrm{T}}(t), \quad \mathbf{K} = \left[\mathrm{k}_{pq}\right]. \tag{12}$$

Substituting relations (12) and (10) in the Fredholm part, we obtain

$$I(x) = \int_a^b \mathbf{X}(x)\mathbf{K}\mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)\mathbf{MSA}dt. \tag{13}$$

Proposition 2.4. If $\mathbf{X}(\mathrm{x}) = [1 \;\; x \;\; \ldots \;\; x^N]$ and $x \in R$, then

$$\int_a^b \mathbf{X}^{\mathrm{T}}(\mathrm{t})\mathbf{X}(\mathrm{t})dt = \left[\mathrm{q_{ij}}\right]_{(N+1)\times(N+1)},$$

where

$$\mathrm{q_{ij}} = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1}, \quad i,j{=}0,1,2,\ldots,N.$$

Let $\mathbf{Q} = \left[\mathrm{q_{ij}}\right]$. By substituting the expression (13) into Eq. (6), we get the matrix relation for Fredholm integral part $I(x){=}\mathbf{X}(\mathrm{x})\mathbf{KQMSA}$.

2.3 Matrix relation for the conditions

We can denote the matrix form of the initial condition Eq. (2) with the help of (8) as follows

$$\left\{\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{X}(a)\mathbf{M}^k\mathbf{S} + b_{jk}\mathbf{X}(b)\mathbf{M}^k\mathbf{S}\right)\right\}\mathbf{A} = \lambda_j, \; j{=}0,1,\ldots,m-1. \tag{14}$$

3 Solution method

We can give the solution steps as follows:

A) Express the Eq. (1) in matrix form by combining the matrix relations in Section 2 and to derive the augmented matrix by using collocation points. B) Express the initial conditions as augmented matrix. C) Combine augmented matrix of conditions with augmented matrix by collocation points and find the solution.

A) In an attempt to set a fundamental matrix equation, replacing the matrix relations (10) and (13) with (1) we derive that

$$\sum_{k=0}^{m} \mathbf{P}_k(x)\mathbf{X}(x)\mathbf{M}^k\mathbf{SA} = g(x) + \mathbf{X}(\mathrm{x})\mathbf{KQMSA} \tag{15}$$

The collocation points $\mathrm{x_i}$ are defined by

$$x_i = a + \frac{b-a}{N}i, \quad i{=}0, 1, \ldots, N \tag{16a}$$

or

$$x_i = \frac{b+a}{2} - \frac{b-a}{2}cos\left(\frac{\pi i}{N}\right) \quad (Chebyshev - Lobatto). \tag{16b}$$

Using the points (16), the following system of the matrix equations is obtained:

$$\sum_{k=0}^{m} \mathbf{P}_k(x_i)\mathbf{X}(x_i)\mathbf{M}^k\mathbf{SA} = g(x) + \lambda\mathbf{X}(x_i)\mathbf{KQSA} \tag{17}$$

or shortly

$$\left\{\sum_{k=0}^{m} \mathbf{P}_k\mathbf{XM}^k\mathbf{S} - \lambda\mathbf{XKQS})\right\}\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{P_k} = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ 1 & x_2 & \cdots & x_2^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}$$

The fundamental matrix Eq. (17) for (1) corresponds to an equation system with $N+1$ algebraic equations for the $N+1$ unknown coefficients a_0, a_1, a_2, … , a_N. Concisely we can write as

$$\mathbf{WA} = \mathbf{G} \quad \text{or} \quad [\mathbf{W}; \mathbf{G}] \tag{18}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k\mathbf{XM}^k\mathbf{S} - \lambda\mathbf{XKQS}$$

B) At the same time, the matrix form (17) for the conditions can be expressed by

$$\mathbf{U}_j\mathbf{A} = \lambda_j \text{ or } [\mathbf{U}; \lambda_j],\ j = 0,\ 1, 2,\ \ldots, m-1 \tag{19}$$

where

$$\mathbf{U}_j = [u_{j0}\ \ u_{j1}\ \ \ldots\ \ u_{jN}] = \sum_{k=0}^{m-1} \left(a_{jk}\mathbf{X}(a)\mathbf{M}^k\mathbf{S} + b_{jk}\mathbf{X}(b)\mathbf{M}^k\mathbf{S}\right)$$

C) To attain the solution of (1) under conditions (2), by substituting the rows in matrix Eq. (18) for the last m rows of matrix Eq. (19), we obtain the new augmented matrix system

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}} \text{ or} [\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] \tag{20}$$

where the new augmented matrix system can be written as

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & u_{(m-1)2} & \cdots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix}$$

If $rank(\tilde{\mathbf{W}}) = rank[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = N + 1$, then we can deduce

$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}.$

One can uniquely determine the matrix **A** (whence the coefficients $a_0,\ a_1, a_2, \ldots, a_N$). Thus the Eq. (1) under the coefficient Eq. (2) has unique solution, which is expressed by truncated Bell series

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x).$$

4 Error estimation

The accuracy of the approximate solutions can easily be estimated as follows. Since the truncated Bell series (3) is the approximate solution of (1), when $y_N(x)$ and its derivatives are replaced in (1), the following equation should be satisfied approximately; i.e., for $x{=}x_q{\in}[a, b],\ q{=}0, 1, \ldots, N,$

$$R_N(x_q) = \sum_{k=0}^{m} P_k(x_q) y_N^{(k)}(x_q) - \lambda \int_a^b K(x_q, t_q) y_N^{(k)}(t_q) dt - g(x_q) \cong 0$$

or

$$R_N(x_q) \le 10^{-k_q}, (k_q \text{ is any positive integer}).$$

If $\max 10^{-k_q} = 10^{-k}$ (k is a positive integer) is assigned, then the truncation limit N is increased until the difference $R_N(x_q)$ at each point gets smaller than the assigned 10^{-k}. Thus, if $R_N(x_q) \to 0$ when N gets larger enough, then the error diminishes.

Moreover, with help of the residual function denoted by $R_N(x)$ and the mean value of the function $|R_N(x)|$ on $[a, b]$, the accuracy of the solution can be checked and the error could be predicted [Mollaoğlu and Sezer (2017); Oğuz and Sezer (2015); Balcı and Sezer (2015)]. Hence, one can reckon the upper bound for the mean error $\overline{R_N}$ as described below:

$$\left| \int_a^b R_N(x) dx \right| \le \int_a^b |R_N(x)| dx$$

and

$$\int_a^b |R_N(x)| dx = (b-a)|R_N(c)|, \quad a \le c \le b$$

$$\Rightarrow \left| \int_a^b R_N(x) dx \right| = (b-a)|R_N(c)| \Rightarrow (b-a)|R_N(c)| \le \int_a^b |R_N(x)| dx$$

$$|R_N(c)| \le \frac{\int_a^b |R_N(x)| dx}{b-a} = \bar{R}_N.$$

5 Numerical examples

Using the exact solution $y(x)$ and the approximate solution $y_N(x)$, the error function e_N is calculated as described below

$$e_N = y(x) - y_N(x).$$

Example 1. Let us examine the second order linear Fredholm type integro-differential equation

$$y''(x) + xy'(x) - y(x) = -\frac{5x}{6} - 1 + \int_0^1 xty(t) dt, \quad 0 \le x, t \le 1$$

with the initial conditions $y(0)=1$, $y'(0)=1$. Let the approximate solution $y(x)$ by the truncated Bell series

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x).$$

where $P_0(x)=-1$, $P_1(x)=x$, $P_2(x)=1$, $g(x)=-\dfrac{5}{6}-1$, $\lambda=1$ and kernel $K(x,t)=xt$. Then for $N=2$, the collocation points are

$$\left\{x_0 = 0,\ \ x_1 = \frac{1}{2},\ \ x_2 = 1\right\}$$

and from Eq. (15), the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_0\mathbf{X}\mathbf{M}^0\mathbf{S} + \mathbf{P}_1\mathbf{X}\mathbf{M}^1\mathbf{S} + \mathbf{P}_2\mathbf{X}\mathbf{M}^2\mathbf{S} - \lambda\mathbf{X}\mathbf{K}\mathbf{Q}\mathbf{S}\right\}\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{P}_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} -1 \\ -17/12 \\ -11/6 \end{bmatrix}$$

The augmented matrix for fundamental matrix equation is found as

$$[\mathbf{W}; \mathbf{G}] = \begin{bmatrix} -1 & 0 & 2 & ; & -1 \\ -5/4 & -1/6 & 47/24 & ; & -17/12 \\ -3/2 & -1/3 & 29/12 & ; & -11/6 \end{bmatrix}$$

From Eq. (14), the matrix forms for the initial conditions are

$[\mathbf{U}_0; \lambda_0] = [1 \quad 0 \quad 0 \quad ; \quad 1]$ and $[\mathbf{U}_1; \lambda_1] = [0 \quad 1 \quad 1 \quad ; \quad 1]$

From system (20), the new augmented matrix based on conditions can be obtained as follows:

$$\left[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}\right] = \begin{bmatrix} -1 & 0 & 2 & ; & -1 \\ 1 & 0 & 0 & ; & 1 \\ 0 & 1 & 1 & ; & 1 \end{bmatrix}$$

Solving this system, the undetermined Bell coefficient matrix is obtained as

$\mathbf{A} = [1 \quad 1 \quad 0]^{T}$

By replacing the above Bell coefficient matrix with Eq. (7), we derive the approximate solution $y(x)=x+1$ which is also the exact solution.

Example 2. Let us consider

$$y'(x) - y(x) = \frac{1 - e^{(x+1)/4}}{x+1} + \int_0^{1/4} e^{tx} y(t)dt$$

with $y(0)=1$.

The exact solution of problem is $y=e^x$ and $P_0(x)=-1,\ P_1(x)=1,\ K(x,t)=e^{tx}$ and $g(x) = \dfrac{1 - e^{(x+1)/4}}{x+1}$. For $N=3,\ 4$ and 5 the obtained approximate solutions are

$$y_3(x) = 0.18136x^3 + 0.498741x^2 + 1.0x + 1.0$$
$$y_4(x) = 0.045799x^4 + 0.166021x^3 + 0.500032x^2 + 0.999999x + 1.0$$
$$y_5(x) = 0.0092155x^5 + 0.04146967x^4 + 0.1666854x^3 + 0.4999992x^2 + 1.0x + 1.0$$

The absolute errors of approximate solutions above on some points are shown in Tab. 1.

Example 3. Let us consider

$$y''(x) + (x+1)y'(x) - 2y(x) = xe^x - e + 1 + \int_0^1 y(t)dt$$

with $y(0)=1$ and $y'(0)=1$.

The exact solution of problem is $y=e^x$ and $P_0(x)=-2,\ P_1(x)=x+1,\ P_2(x)=1,\ K(x,t)=1$ and $g(x)=xe^x-e+1$. For $N=4,\ 8$ and 9 the obtained approximate solutions are

Table 1: Comparison of the absolute errors of Example 2 for N=3, 4, 5

x_i	$y(x_i) = e^{x_i}$	$\lvert e_3(x_i)\rvert$	$\lvert e_4(x_i)\rvert$	$\lvert e_5(x_i)\rvert$
0	1.0	0	0	0
0.05	1.05127	1.57388 E-6	2.75073 E-8	6.35555 E-10
0.1	1.10517	2.14808 E-6	9.71756 E-8	1.55065 E-9
0.15	1.16183	4.80228 E-7	1.66110 E-7	3.66563 E-9
0.2	1.22140	2.23816 E-6	2.31760 E-7	6.48017 E-9
0.25	1.28403	2.03542 E-5	6.86219 E-7	1.67854 E-8

$$y_4(x) = 1 + x + 0.4999x^2 + 0.1628x^3 + 0.0534x^4$$
$$y_6(x) = 1.941413812E - 3x^6 + 7.827919267E - 3x^5 + 4.186911067E - 2x^4 + 0.1666348620x^3 + 0.4999996243x^2 + 1.0x + 1$$
$$y_8(x) = 1 + x + 0.5x^2 + 0.1667x^3 + 0.0417x^4 + 0.0083x^5 + 0.0014x^6 + 1.817\mathrm{E} - 4x^7 + 3.6163\mathrm{E} - 5x^8$$

The absolute errors of approximate solutions above on some points are shown in Tab. 2.

On this example, let us see the effect of usage of different collocation points. In Tab. 2 absolute errors are given for the solution with the help of uniformly distributed (equidistant mesh points) x_k points given in (16a). We are presenting the absolute errors for the solution with the help of Chebyshev-Lobatto points (16b) in Tab. 3.

Example 4. Let us study second order linear Fredholm type integro-differential equation having the Bell series solution that is given by

Table 2: Comparison of the absolute errors of Example 3 for N=4, 6, 8

x_i	$y(x_i) = e^{x_i}$	$\lvert e_4(x_i)\rvert$	$\lvert e_6(x_i)\rvert$	$\lvert e_8(x_i)\rvert$
0	1	0	0	0
0.2	1.2214	2.0556E-5	7.4523E-8	2.6160E-9
0.4	1.4918	6.0813E-5	1.6743E-7	1.3178E-8
0.6	1.8221	8.2091E-5	2.9110E-7	4.0508E-8
0.8	2.2255	3.9652E-4	5.4930E-7	9.7239E-8
1	2.7183	2.1991E-3	8.8984E-6	2.1946E-7

Table 3: Comparison of the absolute errors of for N=4, 8 according to distribution of points

x_i	$y(x_i) = e^{x_i}$	$N=4$		$N=8$	
		For the points in (16a)	**For the points in (16b)**	**For the points in (16a)**	**For the points in (16b)**
0	1	0	0	0	0
0.2	1.2214	2.0556E-5	8.7954E-6	2.61602E-9	2.49100E-8
0.4	1.4918	6.0813E-5	4.9948E-6	1.31778E-8	1.80840E-7
0.6	1.8221	8.2091E-5	5.7088E-5	4.05084E-8	5.38540E-7
0.8	2.2255	3.9652E-4	2.5576E-4	9.72386E-8	1.07846E-6
1	2.7183	2.1991E-3	2.2878E-3	2.19459E-7	1.65554E-6

$$y''(x) + (x+1)y'(x) - 2y(x) = xe^x - e^3 + 1 + \int_0^3 y(t)dt$$

with initial condition $y(0)=1$ and $y'(0)=1$. The exact solution of problem is $y(x)=e^x$ and we seek the approximate solution $y_N(x)$ as a truncated Bell series:

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x)$$

where $P_0(x)=-2$, $P_1(x)=x+1$, $P_2(x)=1$, $g(x)=xe^x-e^3+1$, $\lambda=1$ and kernel $K(x,t)=1$. For $N=4,\ 8$ and 10, the approximate solutions are obtained as:

$$y_4(x)=1+x+0.6139361x^2+0.0859716x^3+0.0935607x^4$$

$$y_8(x)=1+x+0.500048624x^2+0.166504207x^3+0.042173056x^4 +0.007585005x^5+0.001982272x^6-0.000056044x^7+0.000077314x^8$$

$$y_{10}(x)=1+x+0.50000050244x^2+0.16666345457x^3+0.04168103175x^4 +0.00830203277x^5+0.00142860232x^6+0.0001673193x^7 +0.00003997532x^8-0.00000164461x^9+0.00000092818x^{10}$$

These approximate solutions of the equation are visualized in Fig. 1. It is seen that the graphs of $y_8(x)$ and $y_{10}(x)$ almost coincide. The exact solution is not showed in Fig. 1 because it is covered by the graphs of $y_8(x)$ and $y_{10}(x)$ because of precision problems in visualization.

The absolute errors for $N=4,\ 8,\ 10$ are shown in Tab. 4. According to Tab. 4, it can be said that when N increases, the absolute error gets smaller. For each N value, the

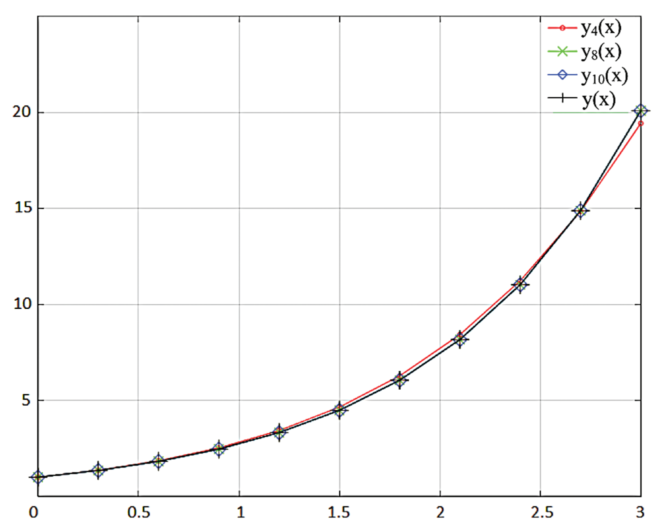

Figure 1: $y_N(x)$ for $N=4,\ 8,\ 10$

absolute error for the points close to zero is relatively less than other points since zero is initial (condition) point.

Furthermore, the numerical results for residual functions are shown in Tab. 5. Also graphs of the residual error functions are given in Fig. 2. According to Tab. 5, it can be said that when N increases, the residual error will decrease. Also for each N value, the residual error on the points close to zero is relatively less than other points. In Fig. 2, graphs of $R_8(x)$ and $R_{10}(x)$ almost coincides with x-axis, i.e., residual errors get closer to zero.

Example 5. Let us study the following second order linear Fredholm type integro-differential equation having the Bell series solution that is given by

$$y''(x) + xy'(x) - xy(x) = e^x - 2\sin(x) + \int_{-1}^{1} \sin(x)e^{-t}y(t)dt$$

with initial condition $y(0)=1$ and $y'(0)=1$. The exact solution of problem is $y(x)=e^x$ and we seek the approximate solution $y_N(x)$ as a truncated Bell series:

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x)$$

where $P_0(x)=-x$, $P_1(x)=x$, $P_2(x)=1$, $g(x)=e^x-2\sin(x)$, $\lambda=1$ and kernel $K(x,t)=\sin(x)e^{-t}$. A solution to this example is given in Yalçınbaş et al. [Yalçınbaş and Sezer (2000)] by means of Taylor polynomials. Akyüz-Daşçıoğlu et al. [Akyüz-Daşçıoğlu and Sezer (2007)] provided a solution by giving another Taylor polynomial approach. Also Yalçinbaş et al. [Yalçinbaş, Sezer and Sorkun (2009)] used Legendre collocation matrix method via Legendre polynomials to solve this example. Thus we can make a comparison of our proposed Bell polynomial approach on this example.

Table 4: Comparison of the absolute errors of Example 4 for $N=4,\ 8,\ 10$

x_i	$y(x_i)=e^{x_i}$	$\lvert e_4(x_i)\rvert$	$\lvert e_8(x_i)\rvert$	$\lvert e_{10}(x_i)\rvert$
0	1.0	0	0	0
0.3	1.349858808	0.0084745163	2.653680784 E-6	2.209950588 E-8
0.6	1.8221188	0.0295935279	1.126619532 E-5	1.121597453 E-7
0.9	2.459603111	0.0617436015	2.638981446 E-5	2.878937693 E-7
1.2	3.320116923	0.1065174536	4.698802199 E-5	6.196286365 E-7
1.5	4.481689070	0.1634723484	7.498748225 E-5	1.308472130 E-6
1.8	6.049647464	0.2230546751	1.126399804 E-4	2.794784547 E-6
2.1	8.166169913	0.2570491257	1.651829221 E-4	5.922822344 E-6
2.4	11.02317638	0.2056864341	2.413512380 E-4	1.222886161 E-5
2.7	14.87973172	0.0397593562	2.458278482 E-4	2.368430265 E-5
3	20.08553692	0.6604621232	5.467501877 E-4	3.212657233 E-5

Table 5: Comparison of $R_N(x_i)$ of Example 3 for N=4, 8, 10

x_i	$\lvert R_4(x_i)\rvert$	$\lvert R_8(x_i)\rvert$	$\lvert R_{10}(x_i)\rvert$
0	3.692318767E-5	1.630768123E-4	6.307681233E-5
0.3	4.361162589E-2	1.730831115E-4	6.373208341E-5
0.6	2.632650585E-2	1.386992010E-4	6.446963933E-5
0.9	3.231000235E-2	1.366793461E-5	6.474132843E-5
1.2	0.0797259151	8.687016312E-5	6.326585560E-5
1.5	2.406768057E-5	2.179274130E-5	5.742142341E-5
1.8	0.4290891256	8.403601647E-5	4.229604937E-5
2.1	1.597309010	3.946343543E-4	9.264301183E-6
2.4	4.147948963	1.788094064E-4	5.607134169E-5
2.7	9.099541009	9.892363935E-2	5.875317578E-5
3	18.01702685	6.046351838E-2	9.505659953E-4

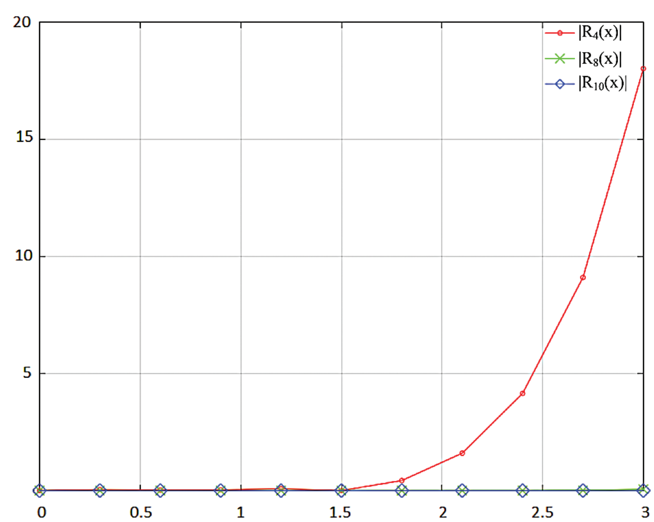

Figure 2: Residual error functions of Example 3 for N=4, 8, 10

In Tab. 6, comparison of the absolute errors in the proposed method and the Legendre method in Yalçinbaş et al. [Yalçinbaş, Sezer and Sorkun (2009)] is given. It is seen that for each N values, the results of proposed method are closer to the exact solution than Legendre method.

In Tab. 7, the proposed Bell polynomial approach is compared with the two methods based on Taylor polynomials given by Yalçınbaş et al. [Yalçınbaş and Sezer (2000);

Table 6: Comparison of the numerical results with Legendre Method for Example 5

		N=3		N=6		N=9	
x_i	e^{x_i}	**Legendre method**	**Proposed method**	**Legendre method**	**Proposed method**	**Legendre method**	**Proposed method**
-1	0.36787944	0.36801200	0.36801433	0.36795047	0.36787028	0.36784656	0.36787636
-0.8	0.44932896	0.44621670	0.44621801	0.44939678	0.44932605	0.44929744	0.44932737
-0.6	0.54881164	0.54450947	0.54451010	0.54887061	0.54880910	0.54878097	0.54881096
-0.4	0.67032005	0.66689869	0.66689892	0.67037064	0.67031815	0.67028984	0.67031984
-0.2	0.81873075	0.81739274	0.81739278	0.81877398	0.81873030	0.81870073	0.81873073
0	1.00000000	1.00000000	1.00000000	1.00003513	1.00000000	1.00000000	1.00000000
0.2	1.22140276	1.21872886	1.21872889	1.22143042	1.22140351	1.22137279	1.22140279
0.4	1.49182470	1.47758771	1.47758776	1.49184806	1.49182898	1.49179492	1.49182492
0.6	1.82211880	1.78058493	1.78058493	1.82212371	1.82211200	1.82208957	1.82211958
0.8	2.22554093	2.13172890	2.13172871	2.22542162	2.22541674	2.22551275	2.22554282
1	2.71828183	2.53502800	2.53502741	2.71766127	2.71766268	2.71828047	2.71828527

Table 7: Comparison of the numerical results with Akyüz-Sezer and Yalcinbas-Sezer methods for Example 5

		N=6			N=9		
x_i	$y(x_i)=e^{x_i}$	**Yalcinbas-Sezer method**	**Akyüz-Sezer method**	**Proposed method**	**Yalcinbas-Sezer method**	**Akyüz-Sezer method**	**Proposed method**
-1	0.368050	0.368050	0.368045	0.367870	0.367879	0.367879	0.367876
-0.8	0.449329	0.449363	0.449361	0.449326	0.449328	0.449329	0.449327
-0.6	0.548812	0.548815	0.548814	0.548809	0.548811	0.548812	0.548811
-0.4	0.670320	0.670319	0.670320	0.670318	0.670320	0.670320	0.670320
-0.2	0.818731	0.818730	0.818731	0.818730	0.818730	0.818731	0.818731
0	1.00000000	1.000000	1.000000	1.00000000	1.000000	1.000000	1.00000000
0.2	1.221403	1.22140	1.221403	1.221404	1.22140	1.221403	1.221403
0.4	1.491825	1.49182	1.491825	1.491829	1.49182	1.491825	1.491825
0.6	1.822119	1.82211	1.822116	1.822112	1.82211	1.822119	1.822120
0.8	2.225541	2.22549	2.225501	2.225417	2.22554	2.225541	2.225543
1	2.718282	2.71805	2.718067	2.717663	2.71828	2.718282	2.718285

Akyüz-Daşçıoğlu and Sezer (2007)]. For the sake of clarity, we call these methods Yalcinbaş-Sezer and Akyüz-Sezer methods, respectively. It is seen in Tab. 7 that on some points the proposed method has better approximation, on other points, Akyüz-Sezer and Yalcinbaş-Sezer methods have better approximations We can conclude that these three methods do not significantly differ from each other.

6 Conclusions

Efficiency of the proposed method provided by using Bell polynomials for the solution of high order linear Fredholm integro differential equations with variable coefficients is shown on the examples. To sum up briefly, in Example 1, it is seen that the method gives the exact solution which is a polynomial. In case that the exact solution is polynomial, method can give much better results. The absolute error calculations and the effect of the choice of the collocation points are given in Example 2, it is observed that the choice of collocation points does not affect the results significantly. In Example 3, absolute and residual errors are analyzed and showed graphically. Method gives better results for the points closer to initial points. The proposed method is compared to Legendre collocation method and other two methods based on Taylor polynomials on Example 4. Bell polynomial collocation method gives remarkably better results than Legendre collocation method. The superiority over Legendre method is clear. But on the other side, the results do not show a significant difference among two methods based on Taylor polynomials.

As in the other methods, the main advantage the proposed method is a solution of the integro differential equations by means of matrix representations. This makes the problem easy programmable for computers and simulation. Also this can give better approximate results on short time and also testing errors easy. Computationally, the coefficients matrix of Bell polynomials for the solutions whose entries are $S(n,k)$ is always nonsingular matrix, which always allows us to look for the solution in polynomial form of the solution. However, it may not be able to provide solution smoothly since the determinant of coefficient matrix $\tilde{\mathbf{W}}$ of the augmented matrix $\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]$ can be zero. But this disadvantage can be handled by replacing suitable rows of augmented matrix with the row matrices of equations obtained from initial conditions.

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