Computer Modeling in Engineering & Sciences CMES, vol.123, no.3, pp.957-972, 2020

RBF Based Localized Method for Solving Nonlinear Partial Integro-Differential Equations

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Abstract: In this work, a numerical scheme is constructed for solving nonlinear parabolic-type partial-integro differential equations. The proposed numerical scheme is based on radial basis functions which are local in nature like finite difference numerical schemes. The radial basis functions are used to approximate the derivatives involved and the integral is approximated by equal width integration rule. The resultant differentiation matrices are sparse in nature. After spatial approximation using RBF the partial integro-differential equations reduce to the system of ODEs. Then ODEs system can be solved by various types of ODE solvers. The proposed numerical scheme is tested and compared with other methods available in literature for different test problems. The stability and convergence of the present numerical scheme are discussed.

Keywords: RBFs, integral equations, local method.

1 Introduction

In the fields of sciences and engineering, physical systems which depends on time and space can be formulated by partial differential equations. Some time the physical system may not be accurately model by such formulations at a given time by ignoring effect of the system in some past times. Specifically in the fields of nuclear reactors, thermoplastic and heat flow there is need to include the memory effect to the system. This addition always occurs as an integral term in basic differential equation resulting a partial integro-differential equation. Various types of partial integro-differential equations available in the literature using for different physical systems. In the present work, we study the parabolic-type Volterra partial integro-differential equations [Avazzadeh, Rizi, Ghaini et al. (2012)]. These type of equations have application in reaction-diffusion problems [Engler (1983)], compression of viscoelastic media [Visser (1997)] and nuclear reactor dynamics [Pachpatte (1983)]. The analytical solution can be available in few cases, so researchers

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Received: 23 October 2019; Accepted: 17 February 2020.

have introduced different techniques for the approximating solutions of PIDEs. Some of these efficient numerical methods include the finite difference method [Tang (1993)], the Quintic B-spline collocation method [Zhang, Han and Yang (2013)], the Variational iteration method [Nawaz (2011)], the Quasi-wavelet with numerical method [Yang, Xu and Zhang (2011)], the spectral method [Fakhar-Izadi and Dehghan (2011)], the Method of lines [Kauthen (1992)], the Finite Element Methods [Yanik and Fairweather (1988)], the orthogonal spline collocation method [Yan and Fairweather (1992)], the Galerkin methods [Zhang, Lin, Lin et al. (2001)] and the θ-weighted with Radial Basis Functions [Avazzadeh, Rizi, Ghaini et al. (2012)]. Hardy [Hardy (1971)] introduced a numerical method using multiquadrics (MQ) radial basis functions. In year 1982 [Franke (1982)] a comparison had been made among different numerical methods and the Hardy's MQ radial basis functions method outclass all the other methods regarding accuracy, stability and ease of implementations. Tarwater [Tarwater (1985)] demonstrated the effect of shape parameter in MQ on the solution accuracy. In the work [Carlson and Foley (1991)] it has suggested that a small value of the shape parameter be used if the function varies rapidly, but a large value be used if the function has large curvature. In 1990 Kansa extended the method of Hardy's for solving numerically various types differential equations vary efficiently [Kansa and Multiquadrics (1990)]. The convergence analysis have been discussed by many authors (see for example [Micchelli (1984); Madych and Nelson (1990); Franke and Schaback (1998)]). The most advantage of utilizing the RBF for the approximation of PDEs is its effortlessness, pertinence to various PDEs, and viability in managing with multi-dimensional issues and complicated domains. In most of the cases the global method returned the differentiation matrices asymmetric and dense, which needed large amount of data and computations time. To overcome these issues the authors in Tolstykh et al. [Tolstykh (2000); Shu, Ding and Yeo (2003); Vertnik and Šarler (2006)] developed a local meshless technique where the kernel based interpolant in small sub-domains centered around each center is used for system matrices. This idea has been extended to develop different sorts of very efficient numerical methods and has been used effectively to a wide range of unsolved issues (see for example [Sarra (2012), Yao, Šarler and Chen (2011)]). Various other modification of local RBF methods and their application can be found in the literature for example local RBF method for Darcy flow by Kosec et al. [Kosec and Šarler (2008)], local RBF-based differential quadrature method for incompressible Navier-Stokes equations by Shu et al. [Shu, Ding and Yeo (2005)], H-adaptive local radial basis function collocation meshless method by Kosec et al. [Kosec and Šarler (2011)], the meshless local Petrov Galerkin (MLPG) method by Atluri et al. [Atluri and Shen (2005)], Stable calculation of Gaussian-based RBF-FD stencils by Fornberg et al. [Fornberg, Lehto and Powell (2013)], Scattered node compact finite difference-type formulas generated from radial basis functions by Wright et al. [Wright and Fornberg (2006)]. In this work, we use such type approach to solve the nonlinear parabolic-type Volterra partial integro-differential equations [Avazzadeh, Rizi, Ghaini et al. (2012)].

2 Development of the proposed method

The nonlinear parabolic-type Volterra partial integro-differential equation is of the type

$$u_t(\mathbf{x},t) = \mathcal{L}u(\mathbf{x},t) + \int_0^t \kappa(\mathbf{x},t,s,u(\mathbf{x},s))ds + f(\mathbf{x},t), \tag{1}$$

where $\mathbf{x}\in\Omega\subset R^d, d\geq 1$, and $t\in[0,\ T]$, and subject to the following initial and and the boundary conditions respectively

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \mathbf{x}\in\Omega, \tag{2}$$

$$\mathcal{B}u(\mathbf{x},t) = g(\mathbf{x},t), \mathbf{x}\in\partial\Omega, \tag{3}$$

where $\mathcal{L}$ and $\mathcal{B}$ are spatial operators. It is assume that the functions κ and f are continuous on $\{(\mathbf{x},t)$: $\mathbf{x}\in\Omega$, $0\leq t\leq T\}$. Let N be the number of points in the set $\{\mathbf{x}_i, i=1, \ldots, N\}\in\Omega$ and Ω_k be a sub-domain in the domain Ω contains m points for each center $\mathbf{x}_k$ in Ω. The unknown function u is approximated by the linear combination of radial basis functions in each sub-domain Ω_k, $k=1, \ldots, N$ (see for example [Uddin, Ali and Ali (2015); Uddin, Minullah, Ali et al. (2015)]) is given by

$$u(\mathbf{x},t) = \sum_{j=1}^{m} \varphi(r_j)\lambda_j, \tag{4}$$

where $r_j=\|\mathbf{x}-\mathbf{x}_j\|$, and $\mathbf{x}, \mathbf{x}_j\in\Omega_k$ is the Euclidean norm between two centers $\mathbf{x}$ and $\mathbf{x}_j$ and φ is a radial basis function defined for $r\geq 0$ and λ_j are the expansion coefficients. From Eq. (4), N number of $m\times m$ systems of linear equations in the matrix form are given by

$$\mathbf{u}^k = A^k\lambda^k, k=1,\ldots,N, \tag{5}$$

where $A^k=[\varphi(\|\mathbf{x}_i-\mathbf{x}_j\|)]_{m\times m}$, $\mathbf{x}_i,\mathbf{x}_j\in\Omega_k$, is the system matrix for each $k=1,\ldots,N$, and

$$\lambda^k = (A^k)^{-1}\mathbf{u}^k, k=1,\ldots,N. \tag{6}$$

and we can also approximate $\mathcal{L}u(\mathbf{x},t)$ by

$$\mathcal{L}u(\mathbf{x},t) = \sum_{j=1}^{m} \mathcal{L}\varphi(r_j)\lambda_j, \tag{7}$$

which can be represented by the matrix-vector form

$$\mathcal{L}\mathbf{u}(\mathbf{x},t)^k = A_L^k\lambda^k, k=1,\ldots,N, \tag{8}$$

where $A_L^k = [\mathcal{L}\varphi(\|\mathbf{x}_i-\mathbf{x}_j\|)]_{m\times m}, \mathbf{x}_i,\mathbf{x}_j\in\Omega_k, k=1,\ldots,N.$

By eliminating the expansion coefficients λ^k from Eqs. (6)-(8), we get

$$\mathcal{L}\mathbf{u}^k = W^k\mathbf{u}^k, k=1,\ldots,N, \tag{9}$$

where $W^k = A_L^k(A^k)^{-1}$, denote the corresponding weight for the kth center, hence for all centers k=1, …, N, we have from Eq. (9)

$$\mathcal{L}U = \mathbf{W}U, \tag{10}$$

where $\mathbf{W}$ is a $N\times N$ sparse differentiation matrix. Using these kernel based approximation of function u and $\mathcal{L}u$ in the model Eq. (1), we get the system of ODEs of Eq. (1),

$$\frac{dU}{dt} = F + \mathbf{W}U + K, \tag{11}$$

where

$$K = \int_0^t k(\mathbf{x}_i,t,s,u(\mathbf{x}_i,s))ds, F = f(\mathbf{x}_i,t), i=1,\ldots,N$$

are $N\times 1$ matrices. Advancing the solution in time, we write the solution of Eq. (11) in the form

$$U_{n+1} = \{1+\delta t\mathbf{W}\}U_n + \delta t\{F_n + K_n\}; \tag{12}$$

where U_n, F_n and K_n denote the values at ($\mathbf{x}_i$,t_n=n δt, i=1, …, N), with step size δt and the ith element of K_n can be computed by using the numerical trapezoidal rule as

$$K_n = \frac{t_n}{2n}\{\kappa(\mathbf{x}_i,t_n,s_0,u(\mathbf{x}_i,s_0)) + 2\sum_{q=1}^{n-1}\kappa(\mathbf{x}_i,t_n,s_q,u(\mathbf{x}_i,s_q)) + \kappa(\mathbf{x}_i,t_n,s_n,u(\mathbf{x}_i,s_n))\}. \tag{13}$$

This is the required scheme for obtaining the numerical solution at any time level n. Initially we take U_0=$h_0(\mathbf{x})$ from the given initial condition $u(\mathbf{x},0)$=$h_0(\mathbf{x})$. In the next section we will discuss the stability and convergence of the this scheme.

3 Stability and convergence of the scheme

The scheme in Eq. (12) is a recurrence relation that allows us to advance the solution in time from t_n=$n\delta t$ to t_{n+1}=$(n+1)\delta t$. The value of the amplification matrix B=1+$\delta t\mathbf{W}$ depend on the ratio $\delta t/h^r$, here r is the order of largest space derivative and h denote the distance between two nodes. Suppose u_n be the exact solution of Eq. (1) at time t_n=n δt, then it follows that $|D^Nu(x)-D^NU(x)|\leq Ch^N|u|$ see [Fasshauer (2007); Uddin and Haq (2011)]. Further assume the numerical scheme in Eq. (12) is of order p in space, so we get

$$u_{n+1} = Bu_n + \delta t\{F_n + K_n\} + O((\delta t)^2 + h^p), \delta t \to 0, h \to 0. \tag{14}$$

We define the error at nth time level by $\varepsilon_n=u_n-U_n$, By subtracting Eq. (12) from Eq. (14) we get

$$\varepsilon_{n+1} = B\varepsilon_n + O((\delta t)^2 + h^p), \delta t \to 0, h \to 0. \tag{15}$$

By Lax-Richtmyer definition of stability the scheme Eq. (15) is stable if

$$\|B\| \leq 1, \tag{16}$$

The result $B\leq\rho(B)$ is always true and $B=\rho(B)$ if B is normal. Let us assume that the initail condition and the solution of the given integro-differential equation must be sufficiently smoothand h to be enough small. For keeping the values of $\delta t/h^r$ to be constant we must have $\delta t \to 0$. so there exist a constant C such that

$$\|\varepsilon_{n+1}\| \leq \|B\|\varepsilon_n\| + C((\delta t)^2 + h^p), n=0, 1, 2, \ldots, T. \tag{17}$$

Since initailly at $n=0$, $U_0=u_0$ therefore $\varepsilon_0=0$ and so by mathematical induction, we have

$$\|\varepsilon_{n+1}\| \leq (1 + \|B\|^2 + \ldots + \|B\|^{n-1})C((\delta t)^2 + h^p), n=0, 1, 2, \ldots, T. \tag{18}$$

By using Eq. (16), we have

$$\|\varepsilon_{n+1}\| \leq nC((\delta t)^2 + h^p), n=0, 1, 2, \ldots, T. \tag{19}$$

This shows the scheme is convergent.

4 Application of the method to problems

In this section, we apply the present numerical scheme to various problems 1-D and 2-D of the type given in Eq. (1) to confirm the accuracy, efficiency and validity of the present numerical scheme. The L_2 and L_∞ error norms are used to measure the error in solutions of the of Eq. (1) whose special 1D case is given by

$$u_t(x,t) = \alpha(x,t)u_{xx}(x,t) + \int_0^t \kappa(x,t,s,u(x,s))ds + f(x,t), x \in [a,b], t \in [0,T]. \tag{20}$$

Problem 4.1 In this problem, we solved the problem in Eq. (20) with the following initial and boundary conditions,

$$u(x,0)=x, 0\leq x\leq 1, u(0,t)=0, u(1,t)=e^{-t}, 0\leq t\leq T. \tag{21}$$

For the choice of the functions $\alpha(x,t)=x^2/2$, and $\kappa(x, s, t, u)=e^{s-t}u$, the function $f(x,t)$ can be found when the exact solution is used $u(x,t)=xe^{-xt}$, see for example [Avazzadeh, Rizi, Ghaini et al. (2012)]. We tested the accuracy of our numerical scheme by calculating the error norms L_2 and L_∞ respectively, and these numerical results are shown in Tab. 1 for all time $t\in[0,1]$. The solution is advanced in time with a step size $\delta t=0.0001$. The present numerical scheme performed well and more accurate results are obtained as compared to

Table 1: L_2 and L_∞ error norms for δt=0.0001, N=40, MQ: $\varphi(r)=\sqrt{r^2+c^2}$, c=5.2, and comparison with Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)] (Method-I)

Time		Present Method		Method-I
t	L_2	L_∞	L_2	L_∞
0.1	8.1660e-8	9.1909e-7	7.2262e-5	1.8818e-5
0.2	1.3134e-7	1.3936e-6	1.0648e-4	2.6480e-5
0.3	1.7393e-7	1.7837e-6	1.2475e-4	3.0188e-5
0.4	2.1714e-7	2.173e-6	1.3405e-4	3.1915e-5
0.5	2.6589e-7	2.6139e-6	1.3791e-4	3.2470e-5
0.6	3.2399e-7	3.1429e-6	1.3840e-4	3.2421e-5
0.7	3.9458e-7	3.7823e-6	1.3681e-4	3.2001e-5
0.8	4.8030e-7	4.5658e-6	1.3397e-4	3.1393e-5
0.9	5.8336e-7	5.5011e-6	1.3047e-4	3.0699e-5
1.0	7.0563e-7	6.6145e-6	1.2669e-4	2.9994e-5

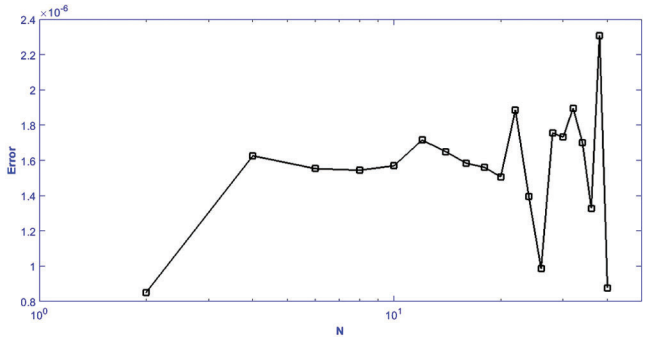

Figure 1: Approximate solution: Error versus number of nodes at time 0.1 in spatial domain [0, 1], corresponding to problem 1

results for the same problem in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)], and these results are shown in Tab. 1 and Fig. 1 respectively.

Problem 4.2 In this second problem, we consider Eq. (20) subject to the following initial and boundary conditions

$$u(x,0)=x(1-x), x\in[0,1], u(0,t)=u(1,t)=0, 0\le t\le T, \tag{22}$$

and given selected function $\alpha(x, t)=x^2(1-x)/2$, and $\kappa(x, s, t, u)=(1-2x)e^{s-t}u(x, s)^2$, where the exact solution, $u(x, t)=x(1-x)e^{-xt}$ is used from Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)] for the purpose of comparison. The problem is solved by the present

numerical method and the results are compared with results discussed in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)] (Method-I), and the same are shown in Tabs. 2-3 and Fig. 2.

Problem 4.3 In this problem, Eq. (20) is solved with the present method for the following values of the functions $\alpha(x, t)=1+x^2$, and $\kappa(x, s, t, u)=x^2+tu^2$, subject to the following initial condition

$$u(x, 0) = \sin hx + 1, x \in [0, 1], \tag{23}$$

Table 2: L_2 and L_∞ error norms for δt=0.0001, N=40, c=8, MQ: $\varphi(r)=\sqrt{r^2+c^2}$, corresponding to problem 2

Time		Present Method		Method-I
t	L_2	L_∞	L_2	L_∞
0.1	1.8022e-8	2.2784e-7	2.5238e-5	7.3486e-6
0.2	3.121e-8	4.1113e-7	4.4124e-5	1.2618e-5
0.3	4.8468e-8	6.2691e-7	5.8625e-5	1.6448e-5
0.4	8.1653e-8	9.3944e-7	6.9800e-5	1.9233e-5
0.5	1.3738e-7	1.4026e-6	7.8325e-5	2.1268e-5
0.6	2.1761e-7	2.0756e-6	8.4760e-5	2.2710e-5
0.7	3.2293e-7	2.9844e-6	8.9535e-5	2.3722e-5
0.8	4.535e-7	4.1341e-6	9.2955e-5	2.4371e-5
0.9	6.0915e-7	5.5204e-6	9.5293e-5	2.4779e-5
1.0	7.8944e-7	7.1405e-6	9.6750e-5	2.4925e-5

Table 3: Absolute error for different t, and x, MQ: $\varphi(r)=\sqrt{r^2+c^2}$, corresponding to problem 2

x	t	Exact solution	Absolute Error
0.1	0.01	0.08991	1.9703e-11
0.2	0.05	0.15841	1.3513e-8
0.3	0.1	0.20379	8.1282e-8
0.4	0.15	0.22602	1.5278e-7
0.5	0.2	0.22621	1.1920e-7
0.7	0.5	0.14798	1.3717e-6
0.9	0.9	0.040037	2.4739e-6

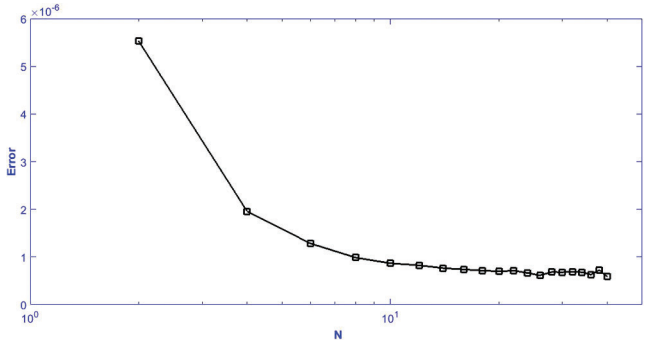

Figure 2: Approximate solution: error versus of nodes at time t=0.1 in the spatial domain [0,1], corresponding to problem 2

and boundary conditions

$$u(0,t) = 1, u(1,t) = \sin hx + e^{-t}, 0 \le t \le T, \tag{24}$$

where the exact solution $u(x,t)$=sinhx+e^{-xt} of the above problem is used from the earlier work [Avazzadeh, Rizi, Ghaini et al. (2012)]. The error norms L_2 and L_∞, when the solution is advanced with step size δt=0.0001 are compared our results with obtained results in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)] are shown in Tab. 4 and Fig. 3 respectively. The proposed method gives more accurate results than the method in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)].

Problem 4.4 In this problem, we solved Eq. (20) with the exact solution $u(x,t)$=$-x^3e^{-x^2t}$ and with the initial and boundary conditions

$$u(x,0) = -x^3, x \in [0,1], u(0,t) = 0, u(1,t) = -e^{-t}, 0 \le t \le T, \tag{25}$$

The values of the functions $\alpha(x, t)$=$x^2/2$ and $\kappa(x, s, t, u)$=$(1-2x)e^{s-2t}u^3$ are used. The results of the present kernel based local method is represented in the form norms L_2 and L_∞ in Tab. 5 and Fig. 4. These results compared with the results of an other method in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)].

Problem 4.5 We consider an other set of function for approximate the problem in Eq. (20) by the current numerical scheme with the initial condition and boundary conditions given by

$$u(x,0) = \sin x, x \in [0,1], u(0,t) = 0, u(1,t) = e^{-t}\sin(1), 0 \le t \le T, \tag{26}$$

where $\alpha(x, t)$=x^5+$4x^2$ and $\kappa(x,s,t,u)$=$(x^3\ t^3+1)\ s^2\ u^2$ and $f(x,t)$ can be found from the exact solution $u(x, t)$=e^{-t}sin x, given in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)]. The present localized kernel based method is used and the solution is advanced in time with the time step δt. The results are shown in form of error norms L_2 and L_∞ respectively and are shown in Tabs. 6-7 and Fig. 5. The proposed method performed more accurate results than the method in Avazzadeh et al. [Avazzadeh, Rizi, Ghaini et al. (2012)].

Table 4: L_2 and L_∞ error norms for δt=0.0001, N=40, c=4.3, MQ: $\varphi(r)=\sqrt{r^2+c^2}$, corresponding to problem 3

Time		**Present Method**		**Method-I**
t	L_2	L_∞	L_2	L_∞
0.1	5.7009e-8	6.2154e-7	2.8762e-5	5.6669e-6
0.2	8.3627e-8	8.7320e-7	7.1767e-5	1.4970e-5
0.3	1.1948e-7	1.1814e-6	1.1479e-4	2.4301e-5
0.4	1.661e-7	1.5692e-6	1.5411e-4	3.2854e-5
0.5	2.2597e-7	2.0635e-6	1.8929e-4	4.0506e-5
0.6	3.0259e-7	2.6969e-6	2.2070e-4	4.7335e-5
0.7	4.0021e-7	3.5096e-6	2.4891e-4	5.3505e-5
0.8	5.2369e-7	4.546e-6	2.7449e-4	5.9117e-5
0.9	6.7837e-7	5.8545e-6	2.9796e-4	6.4264e-5
1.0	8.6992e-7	7.4851e-6	3.1977e-4	6.9050e-5

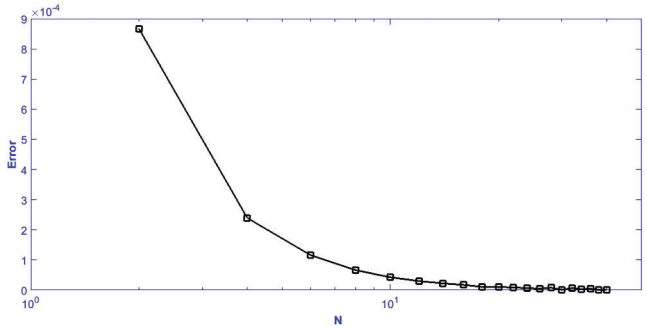

Figure 3: Approximate solution: error *vs.* number of nodes in spatial domain at time t=0.1 of problem 3

Problem 4.6 Here we applied the local kernel based method for solving Eq. (20) having the exact solution given by $u(x, t)=x\,cos(t)$ with initial condition $u(x,0)=x$, $x\in[0,1]$, and with the boundary conditions $u(0,t)=0$, $u(1,t)=cos(t)$, $0\le t\le T$. The functions $\alpha(x, t)=cos^2x+sin(x)$ and $\kappa(x, s, t, u)=e^{x+s+t}u^2$ are used and the function $f(x,t)$ can be found from the exact solution. The error norms L_2 and L_∞ are shown in Tab. 8 and Fig. 6.

Problem 4.7 In this last test problem, we find the numerical solution by present numerical scheme for a 2D problem in the form of Eq. (1) given by

Table 5: L_2 and L_∞ error norms for δt=0.0001, N=40, c=3.2, corresponding to problem 4

Time		Present Method		Method-I
t	L_2	L_∞	L_2	L_∞
.1	5.6637e-8	8.1797e-7	2.9221e-4	8.9279e-5
0.2	6.4247e-7	7.2644e-6	3.8885e-4	1.1055e-4
0.3	1.4046e-6	1.4915e-5	4.2964e-4	1.1711e-4
0.4	2.1522e-6	2.215e-5	4.4344e-4	1.1751e-4
0.5	2.8162e-6	2.8506e-5	4.4216e-4	1.1473e-4
0.6	3.3698e-6	3.3817e-5	4.3204e-4	1.1021e-4
0.7	3.8078e-6	3.8205e-5	4.1670e-4	1.0478e-4
0.8	4.1361e-6	4.1696e-5	3.9825e-4	9.9165e-5
0.9	4.3669e-6	4.4425e-5	3.7824e-4	9.3371e-5
1.0	4.5149e-6	4.6522e-5	3.5750e-4	8.7583e-5

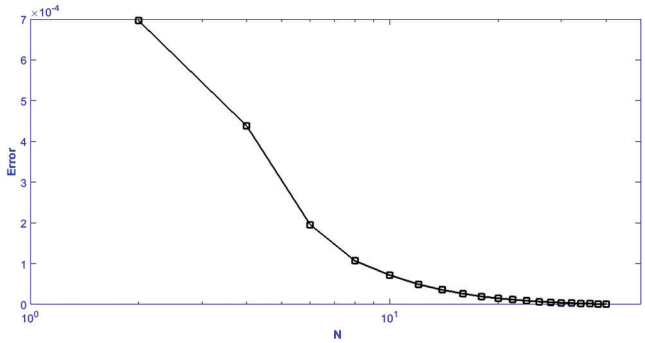

Figure 4: Approximate solution: Error versus number of nodes in space domain [0, 1], at t=0.1 to problem 4

$$u_t(x,y,t) + \int_0^t \kappa(x,y,t,s,u(x,s))ds = (\Delta - \beta)u(x,y,t) + f(x,y,t). \tag{27}$$

where $(x, y)\in[0,1]\times[0,1]$, and $t\in[0, T]$, subject to the conditions: $u(0, y, t)=y(y-1)t$ and $u(x, 0, t)=u(x, 1, 0)=0$ for all $t\in(0, T]$, and $u(x, y, 0)=0$ and $u(x, y, t) \to 0$ as $x \to +\infty$ The function $\kappa(x, y, s, t, u)=\exp(-\beta (t-s))u(x, s)$ is selected, where as the function $f(x, y, t)$ is calculated using the exact solution taken from Han et al. [Han, Zhu, Brunner et al. (2006)] as $u(x, y, t)=y(y-1)t \exp(-\beta x)$. The results of the present kernel based local method is represented in the form norms L_2 and L_∞ are presented in Tab. 9 and Fig. 7

Table 6: L_2 and L_∞ error norms for δt=0.0001, N=10, c=3.6, MQ: of problem 5

Time		**Present Method**		**Method-I**
t	L_2	L_∞	L_2	L_∞
0.1	2.0737e-7	9.5969e-7	7.4402e-5	1.8045e-5
0.2	3.1607e-7	1.4274e-6	8.6495e-5	2.0161e-5
0.3	3.7380e-7	1.6866e-6	8.7941e-5	2.0125e-5
0.4	4.0106e-7	1.8103e-6	8.5265e-5	1.9312e-5
0.5	4.0873e-7	1.845e-6	8.0760e-5	1.8165e-5
0.6	4.0354e-7	1.8216e-6	7.5505e-5	1.6893e-5
0.7	3.9002e-7	1.7604e-6	7.0018e-5	1.5621e-5
0.8	3.7130e-7	1.6758e-6	6.4581e-5	1.4378e-5
0.9	3.4959e-7	1.5776e-6	5.9339e-5	1.3192e-5
1.0	3.2641e-7	1.4728e-6	5.4369e-5	1.2075e-5

Table 7: Absolute error for different t, and x, MQ: $\varphi(r)=\sqrt{r^2+c^2}$, corresponding to problem 5

x	t	**Exact**	**Absolute Error**
0.1	0.1	0.09033	4.4754e-7
0.2	0.2	0.16266	1.2312e-6
0.4	0.4	0.26103	1.7712e-6
0.5	0.5	0.29079	1.6049e-6
0.7	0.7	0.31991	9.4521e-7
0.9	0.9	0.31848	2.4739e-6

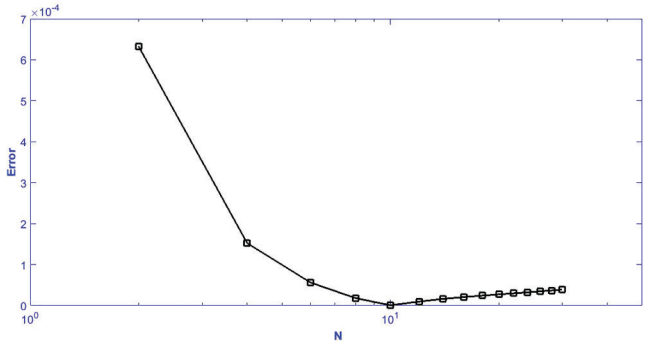

Figure 5: Approximate solution: Error versus number of nodes in spatial domain [0, 1], at time t=0.1 to problem 4

Table 8: L_2 and L_∞ error norms for problem 7 with δt=0.0001, N=40, c=1.83 using MQ

Time		Present Method		Method-I
t	**L_2**	**L_∞**	**L_2**	**L_∞**
0.1	1.03610e-9	1.4038e-8	3.3649e-7	2.0870e-7
0.2	1.0384e-9	1.4140e-8	3.9640e-7	2.1980e-7
0.3	1.02538e-9	1.3986e-8	4.1812e-7	2.2400e-7
0.4	1.0018e-9	1.3689e-8	4.2893e-7	2.2640e-7
0.5	9.7049e-10	1.3285e-8	4.3757e-7	2.2860e-7
0.6	9.3274e-10	1.2793e-8	4.4727e-7	2.3130e-7
0.7	8.8991e-10	1.2225e-8	4.5903e-7	2.3460e-7
0.8	8.4356e-10	1.1596e-8	4.7297e-7	2.3830e-7
0.9	7.9563e-10	1.0919e-8	4.8973e-7	2.4280e-7
1.0	7.4872e-10	1.0209e-8	5.0876e-7	2.4780e-7

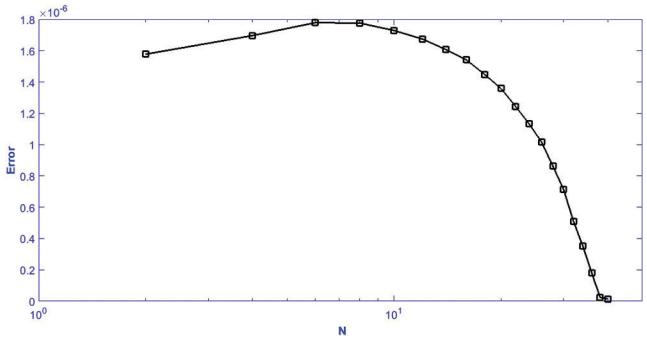

Figure 6: Approximate solution with N=40

Table 9: L_2 and L_∞ error norms for δt=0.0001 Using IMQ $\varphi(r)=1/\sqrt{r^2+c^2}$

t	I×J	Shape parameter	L_2	L_∞
0.1	4×4	0.66	5.0351e-5	7.6403e-4
0.5			4.1577e-4	5.5003e-3
0.1	8×8	0.42	7.9110e-5	2.2450e-3
0.5			5.5718e-4	1.2990e-2
0.1	16×16	0.14	6.9801e-5	3.2969e-3
0.5			3.7400e-4	1.7412e-2
0.1	32×32	0.066	7.3087e-5	6.7045e-3
0.5			3.7354e-4	3.4084e-2

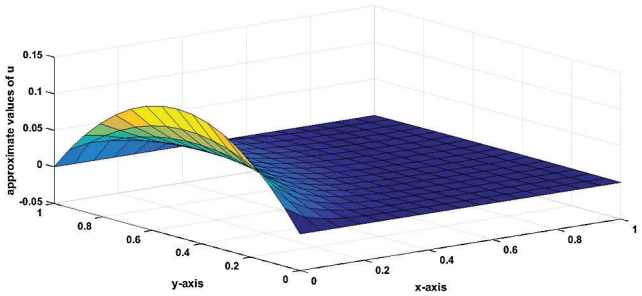

Figure 7: Approximate solution: when $N \times N = 16 \times 16$

respectively. These solutions are well comparable with exact and numerical results obtained in Han et al. [Han, Zhu, Brunner et al. (2006)].

5 Conclusion

In this work, a local kernel based numerical scheme is developed for solving the nonlinear Volterra type integro-differential equations. The numerical scheme is local in nature like finite differences numerical scheme and the system matrices are small and well conditioned. The numerical scheme is stable for a large range of RBF shape parameters. Convergence and stability of the proposed numerical scheme have been established. A number of 1D and 2D integral equations have been solved to validate the present numerical scheme. The efficiency and applicability of the proposed method is well demonstrated and compared with some available results. The developed numerical scheme is an alternative for solving such types of models effetely and accurately.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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