Fractional Analysis of Viscous Fluid Flow with Heat and Mass Transfer Over a Flexible Rotating Disk

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Abstract: An unsteady viscous fluid flow with Dufour and Soret effect, which results in heat and mass transfer due to upward and downward motion of flexible rotating disk, has been studied. The upward or downward motion of non rotating disk results in two dimensional flow, while the vertical action and rotation of the disk results in three dimensional flow. By using an appropriate transformation the governing equations are transformed into the system of ordinary differential equations. The system of ordinary differential equations is further converted into first order differential equation by selecting suitable variables. Then, we generalize the model by using the Caputo derivative. The numerical result for the fractional model is presented and validated with Runge Kutta order 4 method for classical case. The compared results are presented in Table and Figures. It is concluded that the fractional model is more realistic than that of classical one, because it simulates the fluid behavior at each fractional value rather than the integral values.

Keywords: Caputo derivatives, similarity transformation, RK4, soret effect, Dufour effect.

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Nomenclature

Greek symbols

θ	Azimuthal direction
η	Scaled boundary layer coordinate
μ	Dynamic viscosity
ρ	Density
k	thermal conductivity
τ_r	Radial shear stress on the wall
τ_θ	Azimuthal shear stress on the wall
Ω	Angular speed of the disk
$\nu = \frac{\mu}{\rho}$	Ambient fluid kinematic viscosity
C_p	Specific heat at constant pressure
D	Molecular diffusion coefficient
K_T	Thermal diffusion ratio
σ	Fluid electric conductivity
μ_0	Free surface permeability
$\kappa*$	Mean absorption coefficient
T_m	Mean fluid temperature
$\sigma*$	Stefan-Boltzmann constant
$q_r = \frac{-4\sigma*}{3\kappa*}\frac{\partial T^4}{\partial z}$	Radiation heat flux
$S = \frac{\alpha}{\Omega_0}$	Unsteadiness parameter
$Rd = \frac{K*K}{4\sigma^* T_\infty^3}$	Radiation parameter
Pr	Prandtl number
$Sc = \frac{\nu}{D}$	Schmidt number
$So = \frac{D(T_0-T_\infty)}{\nu T_m(C_0-C_\infty)}$	Soret number
$Du = \frac{D(C_0-C_\infty)}{\nu CsCp(T_0-T_\infty)}$	Dufour number
$B = -\frac{P_0(1-\alpha t)}{\rho\nu\Omega_0}$	Pressure constant

1 Introduction

In the fluid dynamics the problem which has a remarkable application in various field is rotating disk phenomena. Keeping in view its application the research in this field got much attention of researchers in the areas such as, the extraction of energy from hydro power, solar collector and wind energy. The rotating disk phenomena was discussed by many researchers, Von Kármán [Von Kármán (1921)] was the first who studied the flow of fluid on the surface of rotating disk. Cochran [Cochran (1934)] presented more accurate result than that of Von Karman by the help of two series expansion, Benton [Benton (1966)] examined unsteady case and made some modification in Cochran's work. Early, the heat transfer was studied by Millsaps et al. [Millsaps (1951); Riley (1964)] on rotating disk. Kuiken [Kuiken (1971)] clarified blowing effect induced due to rotating porous disk. The influences of rotational deceleration was studied by Watson et al. [Watson and Wang (1979)], they showed a rotating disk in viscous fluid can rotate faster than the disk. The unsteady, laminar flow with decleration over a stretchable rotating disk has studied by Fang et al. [Fang and Tao (2012)]. Türkyılmazoglu et al. [Türkyılmazoglu and Uygun (2004)] derived Karman solution considering compressible flow. Where as surface of the disk partially slips in an infinite viscous fluid was studied by Miklavčič et al. [Miklavčič and Wang (2004)]. Karman's problem was further extended by Turkyilmazoglu [Turkyilmazoglu (2010)]. To examine the magneto hydro dynamics effects, the stretching disk effect is examined by Turkyilmazoglu [Turkyilmazoglu (2012)]. The numerical investigation was presented by Turkyilmazoglu [Turkyilmazoglu (2014)] , by considering the nanoparticle with the rotating disk. The fluid flow due to rotating disk was studied by Hayat et al. [Hayat, Qayyum, Imtiaz et al. (2017)].

The radiation effect and the phenomena of heat transfer has a significant role in space technology. In industries the radiation effect along with high temperature have great applications. Chen et al. [Chen, Armaly and Ali (1984)] considered the thermal radiation in the grey fluid. The study on the rotating disk problem is further extended to hydro magnetic. By using magnetic field [El-Mistikawy (1990)] presented important result. The momentum equation was solved by Rath et al. [Rath and Iyengar (1969)] on porous disk with time dependent flow.

The importance and application of fractional calculus (FC) cannot be ignored by someone in advanced technology and in the field of Science and engineering. Fractional calculus characterized those points, at which classical calculus is limited. At the present time fractional calculus is at a stage, where many real world application of science and techologies are still to be introduced. In previous time, FC was considered as esoteric mathematical theory of no applications, for the last decades, due to its wide applications in the field of advanced calculus variation, fractional control of engineering systems, fundamental explorations of the electrical, thermal, mechanical and other properties of several engineering objects like foam, animal tissues, viscoelastic polymers and their scientific importance, understanding of diffusion and wave phenomenas, their verifications and measurments, their role in plasma physics and numerical and analytical tools and techniques, FC has been bringing a tremendous change in research activates. Some researchers have presented several valuable results in monographs and articles [Rath and Iyengar (1969); Ortigueira (2011); Tarasov (2011)]. For special functions and integral transforms FC can be considered as "Laboratory". The applications of FC are mostly due to transcendental functions of the Wright type and Mittag Leffler. The most extensive property of FC in the field of viscoelasticity, in view of its role to model hereditary phenomenas with long memory. Some valuable discussion has been studied about fractional derivative by Benson et al. [Benson, Wheatcraft and Meerschaert (2000)]. Morales-Delgado et al. [Morales-Delgado, Gómez-Aguilar, Saad et al. (2019)] found the solution, using fractional approach, for external forces involving in oxygen diffusion from capillaries to tissue. Gul et al. [Gul, Khan, Noman et al. (2019)] studied the nanofluid passing over a thin needle with fraction order forced convection nanotube. The three dimensional flow of fractional order over rotating disk was studied by Gul et al. [Gul, Khan, Khan et al. (2018)]. Ali et al. [Ali, Jan, Khan et al. (2016)] discussed the solution for Brinkman type fluid with special functions for time free convection flow. Caputo derivative with respect to a kernal function

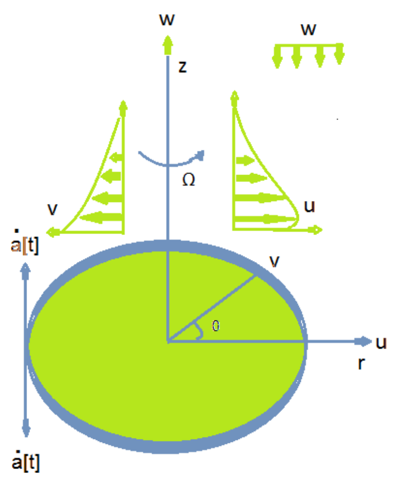


Figure 1: Geometry of the flow over a stretching rotating disk

and their application was examined by Almeida et al. [Almeida, Malinowska and Monteiro (2018)]. Turkyilmazoglu [Turkyilmazoglu (2018)] studied fluid flow over single vertically rotating disk with heat transfer. However, no work has been found so far in literature related to single vertically rotating disk with transfer of heat, in which the unsteadiness effect of temperature and concentration is taken.

This work is the extension of Turkyilmazoglu's work, in which the unsteadiness effect of concentration and temperature has been taken, which effect the fluid behavior. The fractional behavior of unsteady viscous fluid over a rotating disk has been investigated and presented. The governing equation of fluid flow are numerically investigated by Predictor Corrector method (FDE12) and for comparison Runge Kutta order 4 method is used for classical case. For comparison the result are graphically shown, upto a large extend which give a good agreement with $RK4$. Next, we formulate the problem with brief discussion.

2 Mathematical formulation of the problem

Let us consider the impermeable, moving upward/downward rotating disk, which caused the fluid flow as shown in Fig. 1. At $t = 0$, the location of disk is at $a(0) = h$, where at some time t the vertical distance is $Z = a(t)$ and the vertical velocity $w = a^*(t)$, about z-axis the rotating disk has the angular velocity $\Omega(t)$. The velocity components $V(u, v, w)$ are taken along the radial, azimuthal and axial direction respectively. The speed of rotating disk must not be kept constant at any time, otherwise the phenomena will be same as the wall blowing problem, which regarded to a classical rotating disk with a steady fluid flow. The governing equation for the fluid flow motion is as follow [Turkyilmazoglu (2018); Shah, Shuaib and Khan (2017)]:

$$\frac{u}{r} + u_r + w_z = 0, \tag{1}$$

$$u_t + uu_r + wu_z - \frac{v^2}{r} = -\frac{1}{\rho}p_r + \nu(u_{rr} + \frac{1}{r}u_r + u_{zz} - \frac{u}{r^2}), \tag{2}$$

$$v_t + uv_r + wv_z + \frac{uv}{r} = \nu(v_{rr} + \frac{1}{r}v_r + v_{zz} - \frac{v}{r^2}), \tag{3}$$

$$w_t + uw_r + ww_z = -\frac{1}{\rho}p_z + \nu(w_{rr} + \frac{1}{r}w_r + w_{zz}), \tag{4}$$

$$T_t + uT_r + wT_z = \frac{k}{\rho C_p}(T_{rr} + \frac{1}{r}T_r + T_{zz}) + \frac{16\sigma^* T_\infty^3}{3\rho K^* C_p}T_{zz} + \frac{DK_T}{C_s C_p}(C_{rr} + \frac{1}{r}C_r + C_{zz}), \tag{5}$$

$$\rho(uC_r + wC_z) = D(C_{rr} + \frac{1}{r}C_r + C_{zz}) + \frac{DK_T}{T_m}(T_{rr} + \frac{1}{r}T_r + T_{zz}), \tag{6}$$

where u, v, w, T, P, and C represent the radial velocity, tangential velocity, axial velocity, Temperature, pressure and concentration respectively. The time dependent temperature is $T_w(t)$, while the fluid had ambient temperature T_∞ above the disk. The temperature difference between the ambient and the wall is $\Delta T = T_w(t) - T_\infty$.
The imposed boundary condition for infinite rotating disk are as follows:

$$\begin{aligned} &u = 0,\ v = r\Omega_0(t),\ w = \omega_0(t),\ p = p_0,\ T = T_0,\ C = C_0, \quad \text{at } z = a(t) \\ \text{and } &u \longrightarrow 0,\ v \longrightarrow 0,\ T \longrightarrow T_\infty,\ C \longrightarrow C_\infty \quad \text{as } z \longrightarrow \infty. \end{aligned} \tag{7}$$

The thermal radiation term is defined as [Shah, Shuaib and Khan (2017)]

$$q_r = \frac{4\sigma^*}{3K^*}\frac{\partial T^4}{\partial z}. \tag{8}$$

$$\begin{aligned} &u = \frac{rv}{a^2(t)}F(\eta), \quad v = \frac{rv}{a^2(t)}G(\eta), \quad w = \frac{v}{a(t)}H(\eta), \quad p = \frac{\rho v^2}{a^2(t)}p(\eta), \\ &T = T_\infty + \Delta T\theta, \quad \eta = \frac{Z}{a(t)} - 1, \quad \eta_z = \frac{1}{a(t)}\text{and} \quad \eta_t = \frac{-a(t)}{a(t)}(\eta + 1). \end{aligned} \tag{9}$$

The following system of ODE is formed by using Eq. (9) in Eqs. (1)-(7):

$$F'' - HF' - F^2 + G^2 + S(\frac{\eta + 1}{2}F' + F) = 0, \tag{10}$$

$$G'' - HG' - 2FG + S(\frac{\eta + 1}{2}G' + G) = 0, \tag{11}$$

$$H'' - HH' + S(\frac{\eta + 1}{2}H' + H) = 0, \tag{12}$$

$$\frac{\theta''}{Pr}(1 + \frac{4}{3Rd}) + \frac{1}{2}(S(\eta + 1) - H)\theta' + Du\phi'' = 0, \tag{13}$$

$$\frac{1}{Sc}\phi'' + (\frac{1}{2}S(\eta + 1) - H)\phi' + So\,\theta'' = 0. \tag{14}$$

The transformed boundary conditions are given by

$$F(0) = 0,\ G(0) = w,\ H(0) = B\frac{S}{2},\ \theta(0) = 1,\ \phi(0) = 1, \quad \text{at } \eta = 0, \tag{15}$$

and $F(\infty) \longrightarrow 0,\ G(\infty) \longrightarrow 0,\ H(\infty) \longrightarrow 0,\ \theta(\infty) \longrightarrow 0,\ \phi(\infty) \longrightarrow 0$ (16)

$$\text{as } \eta \longrightarrow \infty, \tag{17}$$

The physical controlling parameters S of the upward/downward movement of the disk is expressed as:

$$S = 2\frac{a(t)a^*(t)}{v}, \tag{18}$$

The parameter ω, which represents the constant rotation of the disk is defined as

$$\omega = \frac{\Omega(t)a^2(t)}{v}. \tag{19}$$

3 Basics of Caputo derivative

Definition 3.1. *The fractional integral of order $\alpha > 0$ for a function $g : R^+ \to R$ is defined by*

$$I_t^\alpha(g(t)) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-\chi)^{\alpha-1} g(\chi) d\chi.$$

Here and elsewhere Γ denotes the Gamma function.

Definition 3.2. *The Caputo fractional order derivative for function $g \in C^n$ of order α is given below:*

$${}^C D_t^\alpha(g(t)) = I^{n-\alpha} D^n g(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{g^n(\chi)}{(t-\chi)^{\alpha+n-1}} d\chi,$$

which is well defined for absolutely continuous functions and $n - 1 < \alpha < n \in N$. Note that the value of the Caputo fractional derivative of the function g at point t involves all the values of $g^n(\chi)$, for $\chi \in [0, t]$. Clearly ${}^C D_t^\alpha(g(t))$ tends to $g'(t)$ as $\alpha \to 1$.

4 Solution procedure

The equations that describes the system given by (10-14) and their initial condition (15) are transformed into a system of first-order differential equations by using the following variables as

$$\text{Let } \eta = y_1,\ F = y_2,\ F^{'} = y_3,\ G = y_4,\ G^{'} = y_5,\ H = y_6,\ H^{'} = y_7,$$
$$\theta = y_8,\ \theta^{'} = y_9,\ \Phi = y_{10} \text{ and } \Phi^{'} = y_{11}. \tag{20}$$

$$\left.\begin{aligned}
y_1' &= 1,\\
y_2' &= y_3,\\
y_3' &= y_6 y_3 + y_2^2 - y_4^2 - S(\frac{(y_1+1)y_3}{2} + y_2),\\
y_4' &= y_5,\\
y_5' &= y_6 y_5 + 2y_2 y_4 - S(\frac{(y_1+1)y_5}{2} + y_4),\\
y_6' &= y_7,\\
y_7' &= -2y_6 y_3 + S((y_1+1)y_2 + y_6),\\
y_8' &= y_9,\\
y_9' &= \frac{3P_r R_d(D_u y_{11}(2y_6-(1+y_1)S)S_c + (-y_6+S+y_1 S)y_9}{-8+6R_d(-1+D_u P_r S_0 S_c)},\\
y_{10}' &= y_{11},\\
y_{11}' &= \frac{S_c((1+y_1)S(y_{11}(4+3R_d) - 3P_r R_d S_0 y_9) + y_6(-2y_{11}(4+3R_d) + 3P_r R_d S_0 y_8)}{-8+6R_d(-1+D_u P_r S_0 S_c)}.
\end{aligned}\right\} \tag{21}$$

The transform equations given in (19) represents a non-linear system in classical derivative. To generalize the model equations given by (19), we apply the Caputo derivative and re write these equation as follows:

$$\left.\begin{aligned}
D_\eta^\alpha y_1 &= 1,\\
D_\eta^\alpha y_2 &= y_3,\\
D_\eta^\alpha y_3 &= y_6 y_3 + y_2^2 - y_4^2 - S(\frac{(y_1+1)y_3}{2} + y_2),\\
D_\eta^\alpha y_4 &= y_5,\\
D_\eta^\alpha y_5 &= y_6 y_5 + 2y_2 y_4 - S(\frac{(y_1+1)y_5}{2} + y_4),\\
D_\eta^\alpha y_6 &= y_7,\\
D_\eta^\alpha y_7 &= -2y_6 y_3 + S((y_1+1)y_2 + y_6),\\
D_\eta^\alpha y_8 &= y_9,\\
D_\eta^\alpha y_9 &= \frac{3P_r R_d(D_u y_{11}(2y_6-(1+y_1)S)S_c + (-y_6+S+y_1 S)y_9}{-8+6R_d(-1+D_u P_r S_0 S_c)},\\
D_\eta^\alpha y_{10} &= y_{11},\\
D_\eta^\alpha y_{11} &= \frac{S_c((1+y_1)S(y_{11}(4+3R_d) - 3P_r R_d S_0 y_9) + y_6(-2y_{11}(4+3R_d) + 3P_r R_d S_0 y_8)}{-8+6R_d(-1+D_u P_r S_0 S_c)}.
\end{aligned}\right\} \tag{22}$$

The transformed conditions for the first-order fractional differential equations are as

$$y_2(0) = 0,\ y_4(0) = 0,\ y_6(0) = B\frac{S}{2},\ y_8(0) = 1,\ y_{10}(0) = 1,$$
$$y_2(\infty) \longrightarrow 0,\ y_4(\infty) \longrightarrow 0,\ y_6(\infty) \longrightarrow 0\ \ y_8(\infty) \longrightarrow 0 \text{ and } y_{10}(\infty) \longrightarrow 0. \quad (23)$$

5 Results and discussion

In this section, the non-linear system of differential Eqs. (19)-(20) for varies physical parameter are discussed with two different method, Runge Kutta order 4 and the Predictor Corrector method and their results are shown with brief discussion. Runge Kutta order 4 method is used for classical order and Predictor Corrector method for frictional order and also compared both techniques and the results are depicted in Figs. 13(a)-13(b).

To show the behavior of function such as velocity, temperature and concentration with variation in different physical parameter (see Tab. 1) and Figs. 2-13 are obtained. The Figs. a2-a12 on the left side show RK4 result. Figs. b2-b12 show classical order result at α=1.0, while in rest of each Figs. c and d represent frictional result with α= 0.9, 0.8.

Figs. 2(a)-2(d) represents the effect of parameter S on the axial velocity. It can be seen that upto some extend axial velocity decreases, but then it start increasing with enhancement of unsteadiness parameter S, because boundary layer thickness increases with S.

The decrease of radial velocity with increase in S is presented in Figs. 3(a)-3(d). Parameter S increased the fluid velocity partially, that shows some velocity and gradually the velocity get reduced. The variation in tangential velocity with parameter S have been examined in Figs. 4(a)-4(d). The tangential velocity increase with enhancement in S.

The effect of Prandtl number Pr on the temperature profile is illustrated in Figs. 5(a)-5(d). $Pr = \rho\nu Cp/K$, the thermal diffusion has inverse relation with Prandtl fluid, the fluid show anomalous behavior at classical order αn=1.0, but when $\alpha$is reduced to 0.9 and 0.8, the temperature start decline as the Prandtl number increase.

Rd =$K^*K/4\sigma * T_\infty^3$, with an increase in thermal radiation Rd the temperature of fluid increases illustrated in Figs. 6(a)-6(d). Physically, the fluid which have greater thermal diffusivity have high temperature. Therefore when parameter Rd increases the temperature of fluid also increases.

Figs. 7(a)-7(d) illustrate the effect of Dufour Du on temperature distribution profile. It can be seen that the temperature decreases with increase of parameter Du, because as Du increases the viscosity of fluid reduced and concentration of the fluid increased, as a result the temperature of fluid also decreases.

Figs. 8(a)-8(d) presents the increase in temperature as we slightly increase the Schmidth number Sc. $Sc = \nu/D$, the enhancement of temperature is due to increase of viscosity of fluid. The concentration of fluid reduced as the Prandtl Pr and thermal radiation term Rd increase shown in Figs. 9(a)-9(d) and Figs. 10(a)-10(d) respectively. As we discussed in Figs. 5(a)-5(d), that physically the higher thermal diffusivity is possessed in less Prandtl fluid, and less thermal diffusivity is possessed in high Prandtl fluid. Therefore, when the thermal radiation Rd and Prandtl Pr increases the concentration of fluid reduced.

Figs. 11(a)-11(d) depict the Dufour effect on concentration profile of a viscous fluid. $Du = D(C_0 - C_\infty)/\nu C_s C_p(T_0 - T_\infty)$, clearly the viscosity of fluid decrease as Dufour

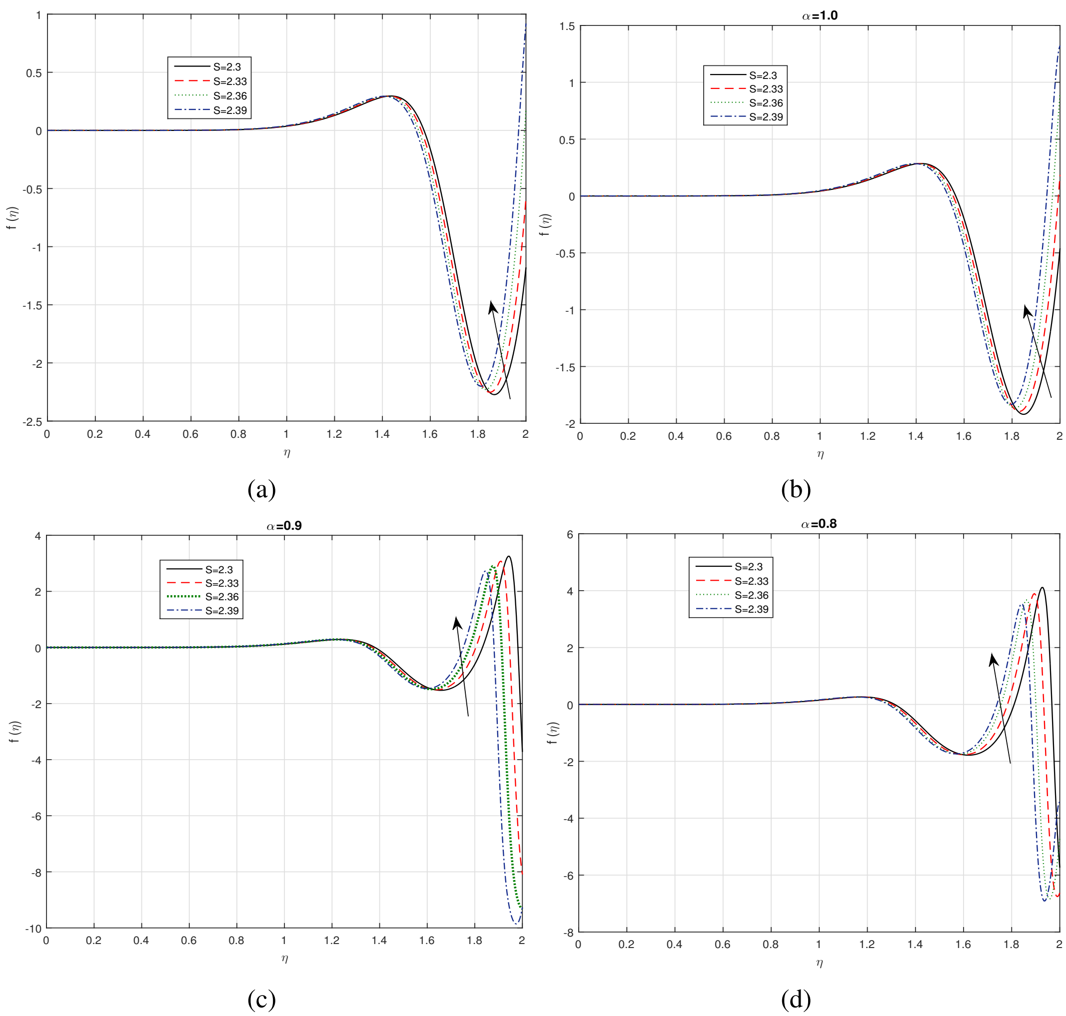


Figure 2: The effect of parameter S on the axial velocity profile (a) RK4 results (b) Fractional order α = 1.0 (c) fractional order α = 0.9 (d) fractional order α = 0.8. Where Pr = 6.7, Sc = 0.13, Rd = 13, Du = 0.4, So = 0.1, $B = 1$

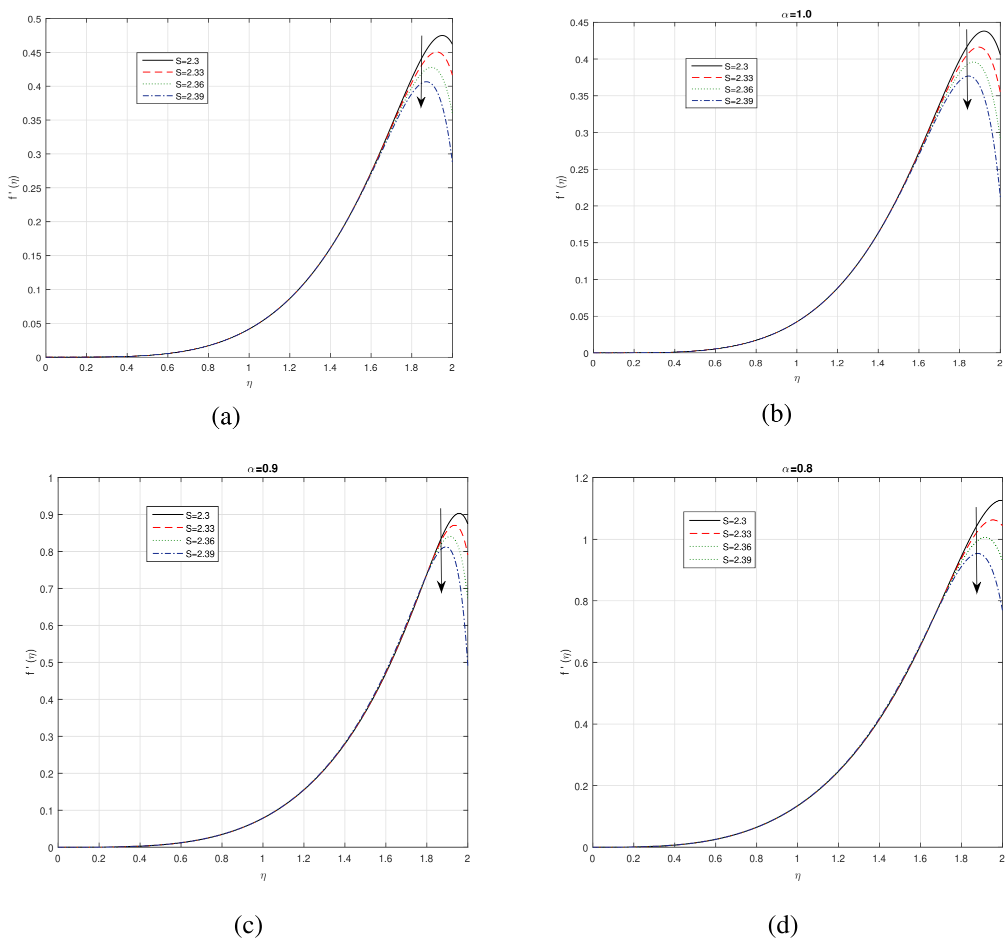


Figure 3: The effect of unsteadiness parameter S on the radial velocity profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where Pr =6.7, Sc = .10, Rd =13, Du =0.5, So =0.1, $B = 1$

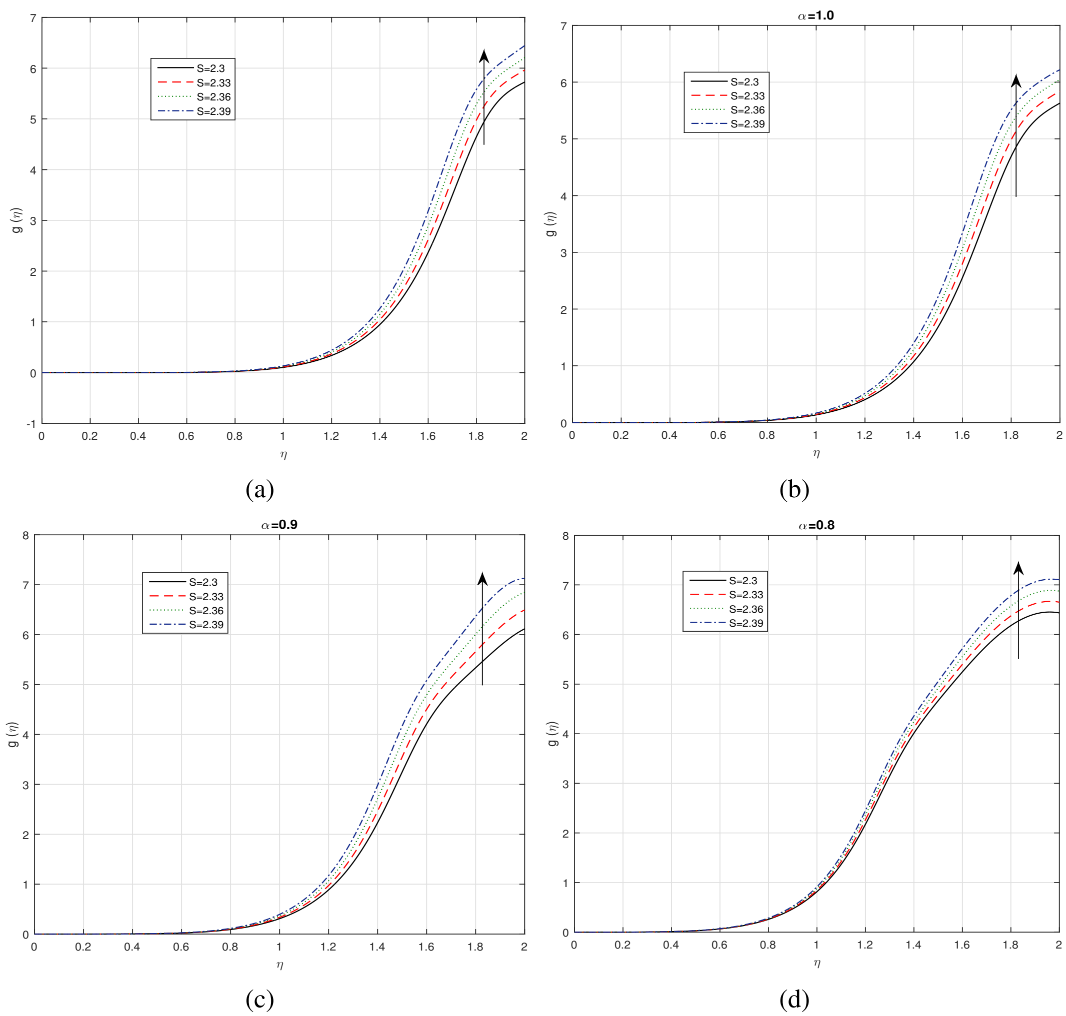


Figure 4: The effect of unsteadiness parameter S on the tangential velocity profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where Pr =5.7, Sc = .12, Rd =11, Du =0.5, So =0.2, $B = 1$

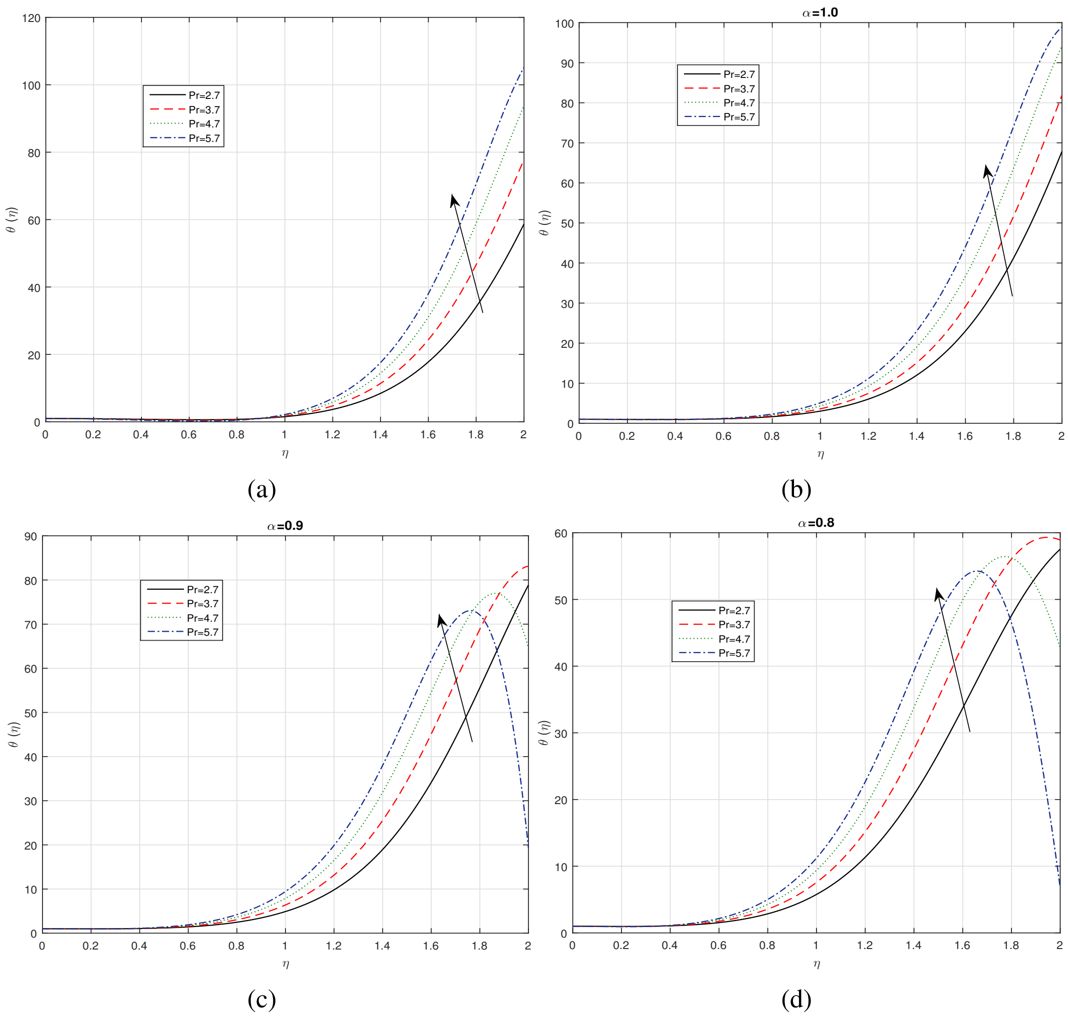


Figure 5: The effect of parameter Pr on the Temperature profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where $Sc = 0.6$, Rd =13, Du =0.3, S =2.2, So =0.5, $B = 1$

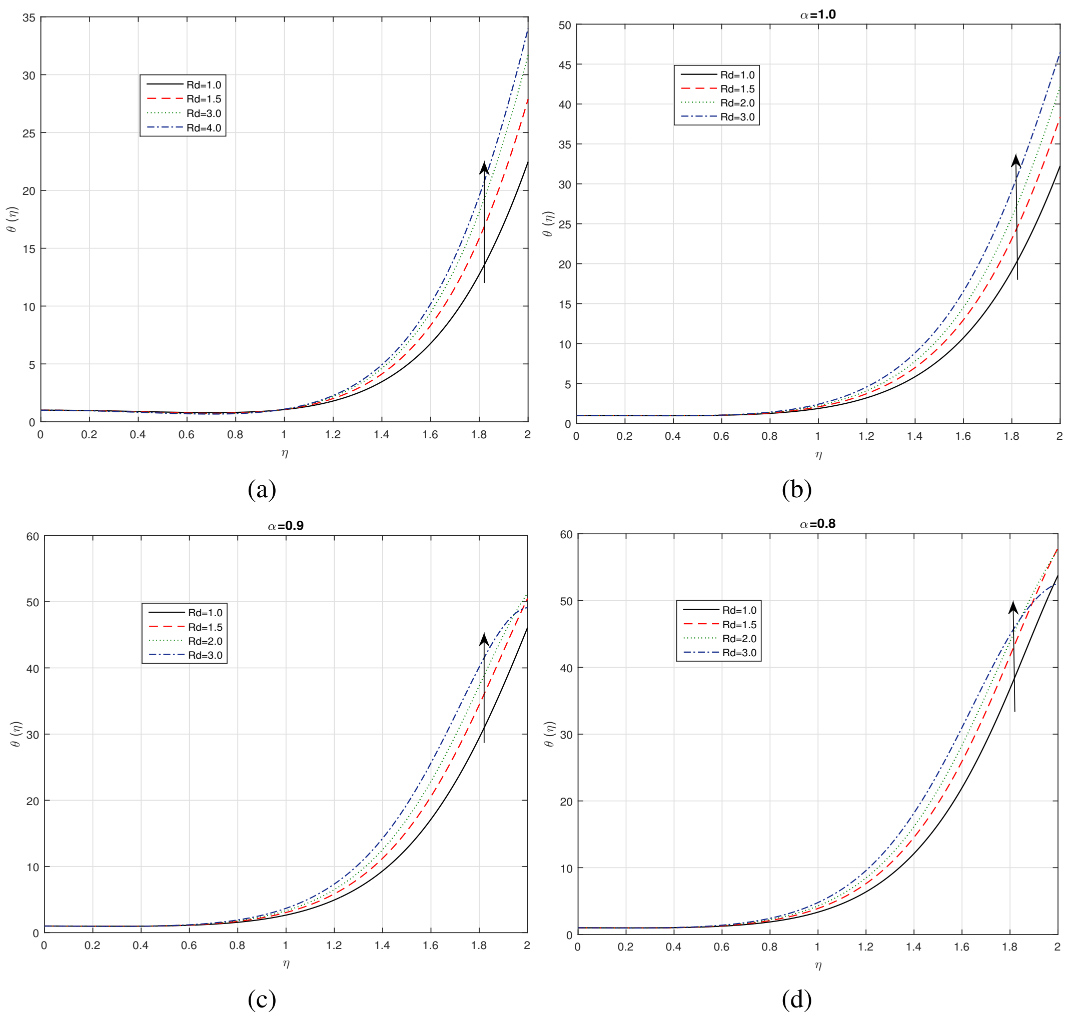


Figure 6: The effect of thermal radiation parameter Rd on the Temperature profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where Pr =2.7, S = 2.05, So = 0.2, Sc = 0.1, Du =0.3, $B = 1$

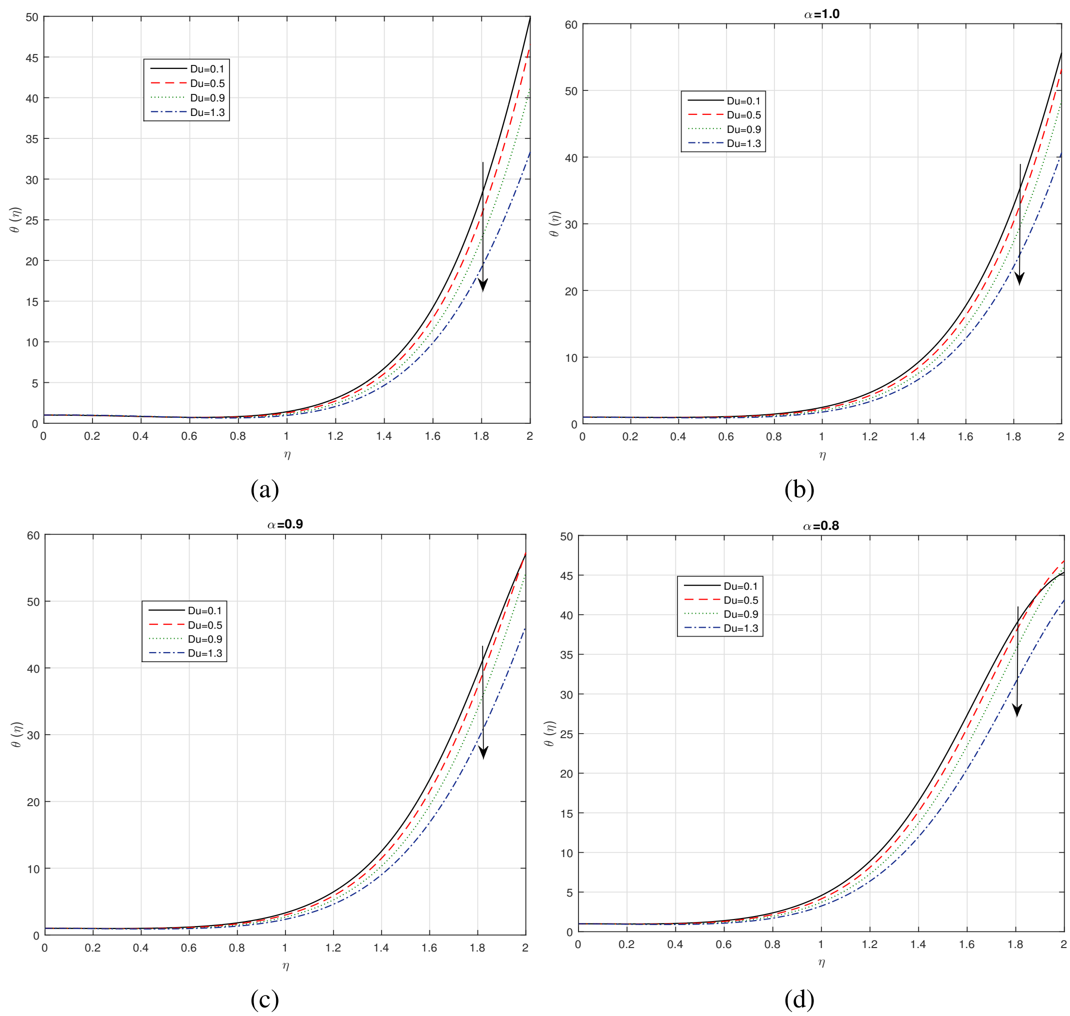


Figure 7: The parameter Dufour Du effect on the Temperature profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =2.2, Pr =2.7, So =0.2, Sc = 0.1, Rd =3.0, $B = 1$

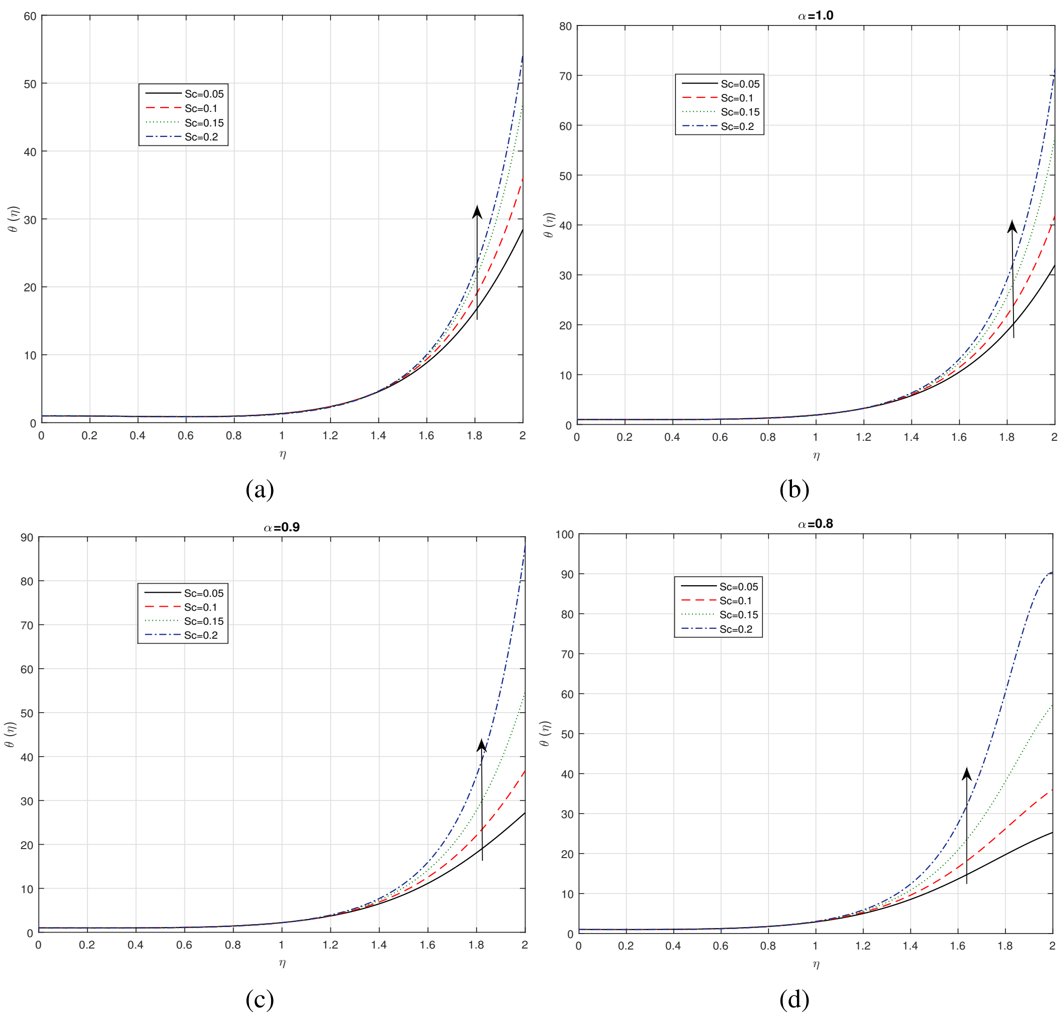


Figure 8: The effect of parameter Sc on the Temperature distribution profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =2.2, Pr =2.7, So = 0.3, Rd =0.8, Du =0.2, $B = 1$

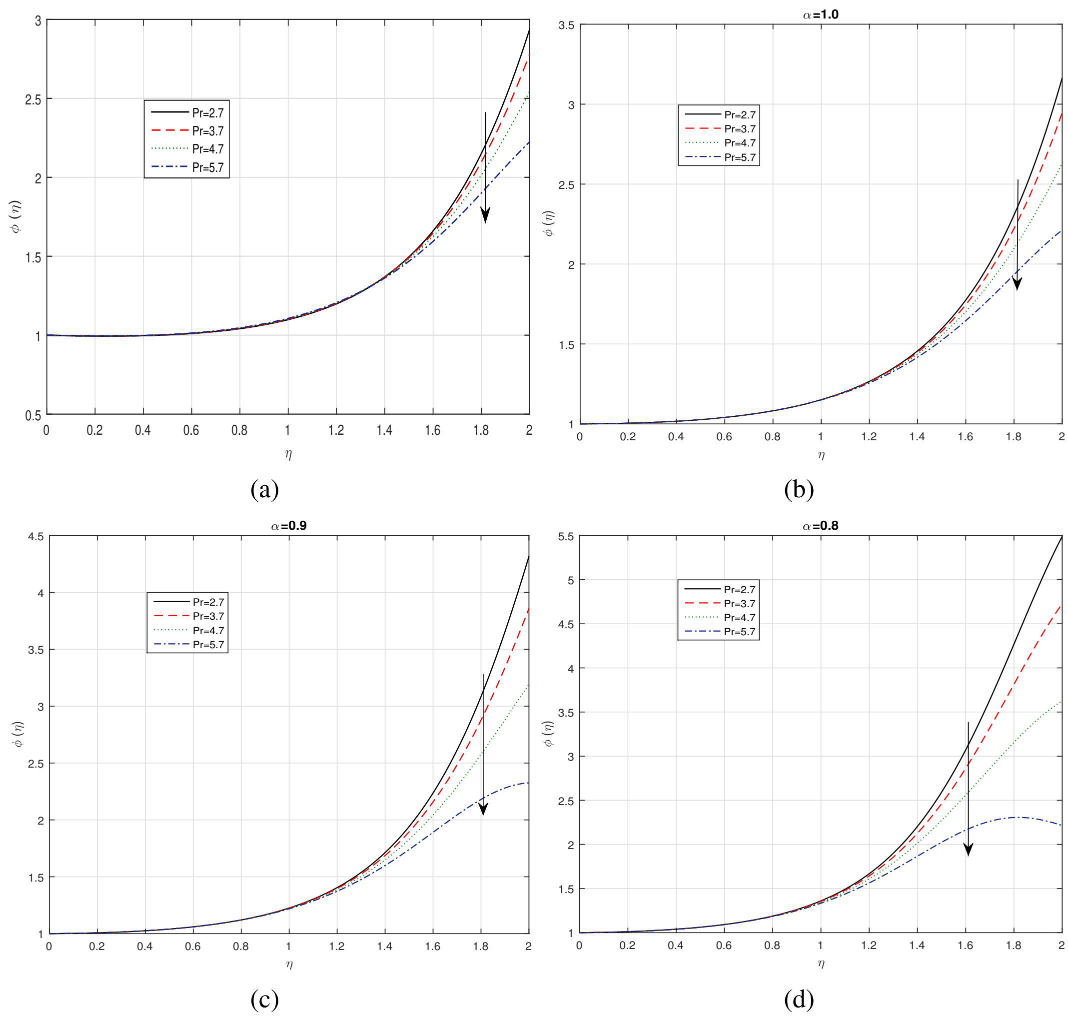
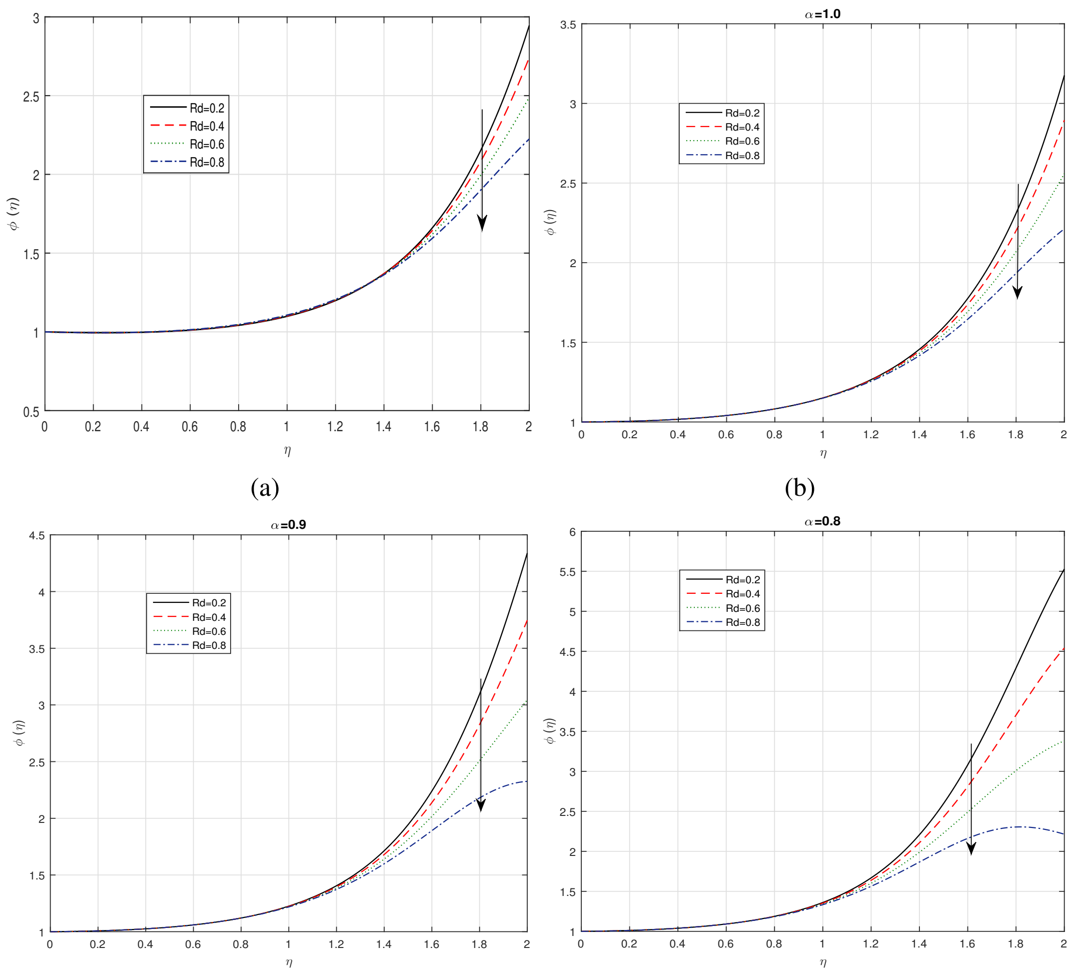


Figure 9: The effect of Prandtl number Pr on the concentration distribution profile (a) RK4 results (b) Fractional order α =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =2.0, Rd =0.8, So = 0.2, Sc = 0.05, Du =0.5, B = 1.2



(a) (b) (c) (d)

Figure 10: The effect of thermal radiation parameter Rd on the concentration distribution profile (a) RK4 results (b) Fractional order α =1.0 result (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =2.0, Pr =4.7, Sc = 0.05, So =0.2, $B = 1.2$

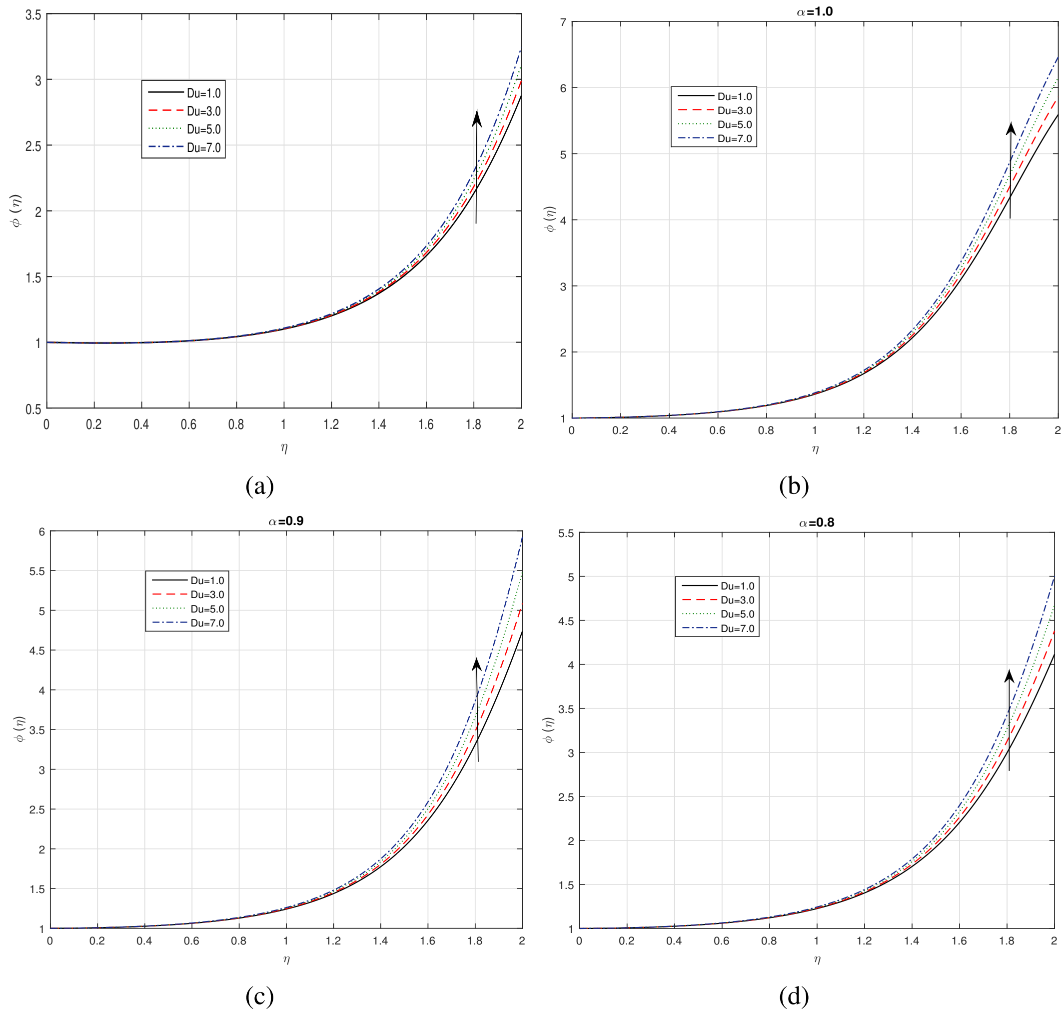


Figure 11: The parameter Dufour Du effect on the concentration distribution profile (a) RK4 results (b) Fractional order α =1.0 result (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =0.2, Pr =6.7,Sc =0.1, Rd =0.2, Du =0.3, $B = 1$

Du increase, as a result the concentration of fluid also show inclination. The effect of Soret number So on the concentration of fluid is illustrated in Figs. 12(a)-12(d). It can be observed from Figs. 12(a)-12(d) that by increasing So the concentration of fluid is increased and viscosity of fluid is reduced.

Fig. 13(a) shows the comparison between Runge Kutta order 4 and frictional method $FDE12$ at α=1.0, both show increasing in temperature θ with enhancement of parameter Pr. The frictional solution shows fast convergence than Runge Kutta of order 4, because it can characterize the function at each frictional value of α. Therefore the frictional model show best results.

In Fig. 13(b) the comparison of Runge Kutta order 4 method and $FDE12$ on concentration

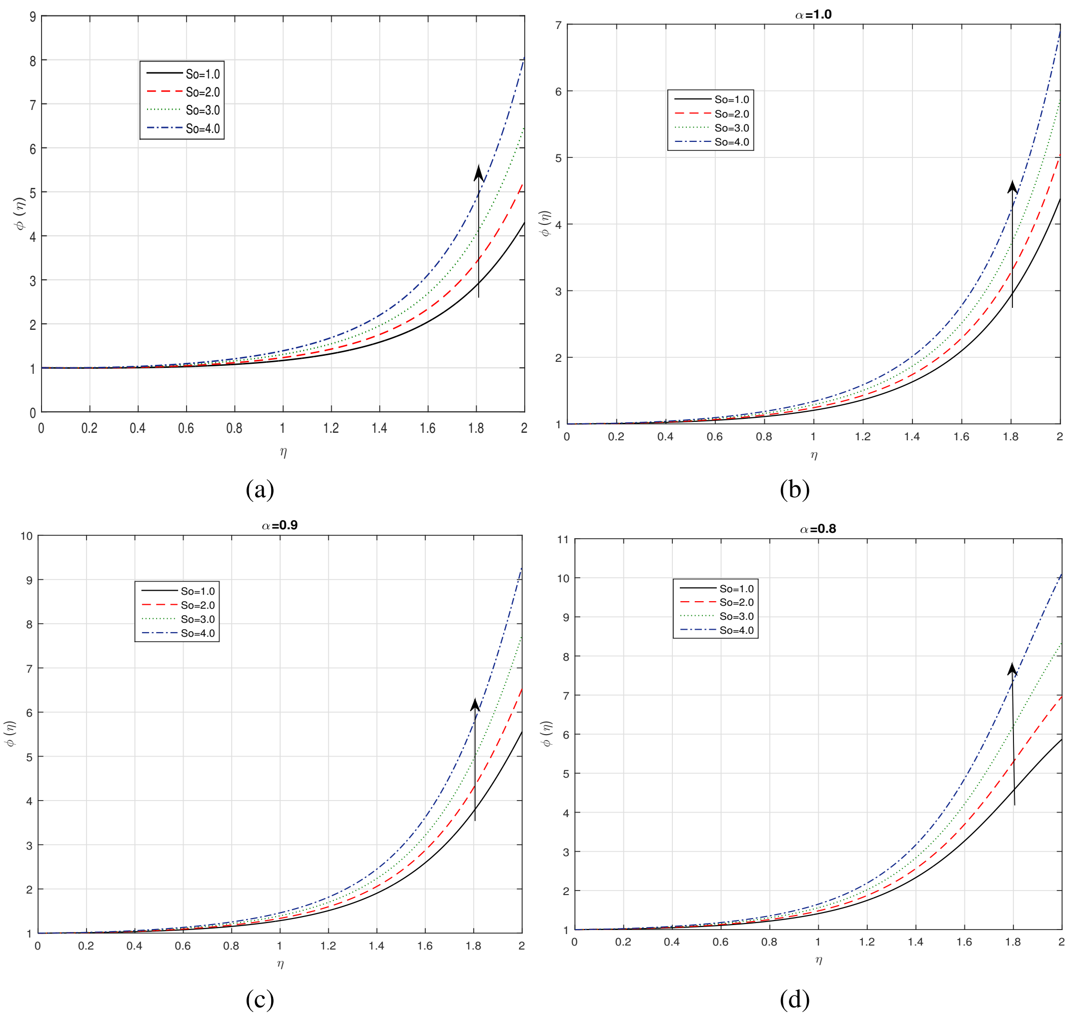


Figure 12: The parameter So effect on the concentration distribution profile (a) RK4 results (b)Fractional order α result =1.0 (c) fractional order α = 0.9 (d) fractional order α =0.8. Where S =2.1 Sc = .05,Pr = 6.7, Rd =0.2, Du =1.0, $B = 1$

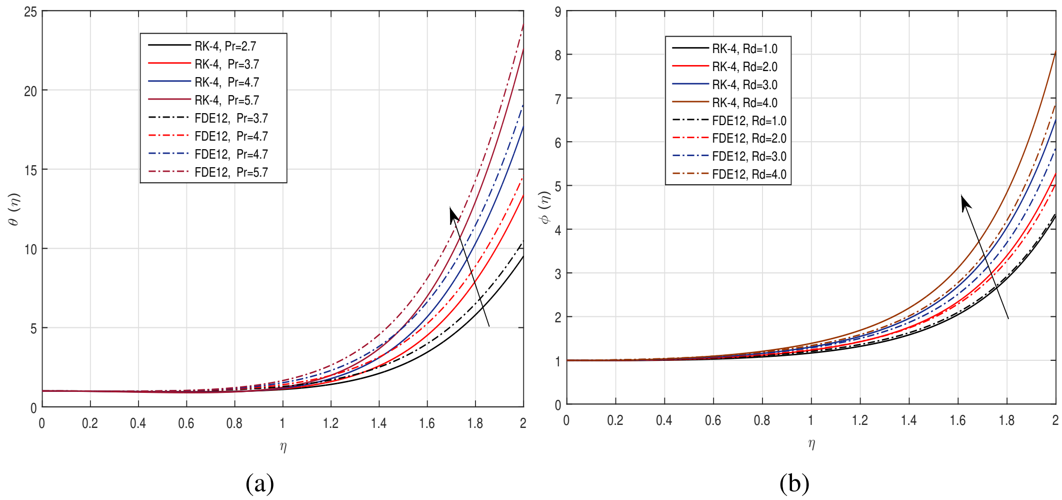


Figure 13: Comparison of RK-4 and fractional method taking $\alpha = 1$ on temperature and concentration profile respectively

profile with the variation of thermal radiation parameter Rd has been shown. It is clear from Fig. 13(b) that at initial value of Rd the $RK4$ show fast convergence. But as we increasing the value greater or equal to 4 the $FDE12$ change the behavior of fluid concentration and reduces the viscosity of fluid. As a result the concentration increases.

Table 1: Numerical comparison of RK4 method with Caputo frictional model for the physical parameters $= 2.1$, $Pr = 6.7$, $Rd = 0.2$, $Du = 1.0$, $Sc = 0.05$, $So = 4.0$, and $B = 0.02$

	RK4				Caputo			
η	$F(\eta)$	$G(\eta)$	$\theta(\eta)$	$\phi(\eta)$	$F(\eta)$	$G(\eta)$	$\theta(\eta)$	$\phi(\eta)$
0	0	0	1.0000	1.0000	0	0	1.0000	1.0000
0.5	0.0002	0.0020	0.8908	0.0588	0.0003	0.0021	0.8906	0.0575
1.0	0.343	0.1635	1.2825	1.3886	0.341	0.1654	1.2830	1.3884
1.5	0.2676	2.3909	6.0259	2.5887	0.2653	2.3901	6.0284	2.5870
2.0	-3.0836	7.1733	28.1336	8.0863	-3.0825	7.1728	28.1339	8.0823

Figs. 14(a)-14(e) are plotted for the purpose to reduce error possibilities and get as less error as possible by Caputo derivatives. From observation of Figs. 14(a)-14(e), we can examine that its maximum error is 10^{-10} and its minimum error can be reduced up to 10^{-4}. That is why error versus different physical parameter's graphs such as S, Pr, Rd, Du and $Sc$are plotted. if the findings of this results and figures are used for practical phenomena like engineering purposes, the desired result would be drawn. Tab. 1 shows the behavior of functions such as concentration, temperature and velocities for different values of η and their comparison.

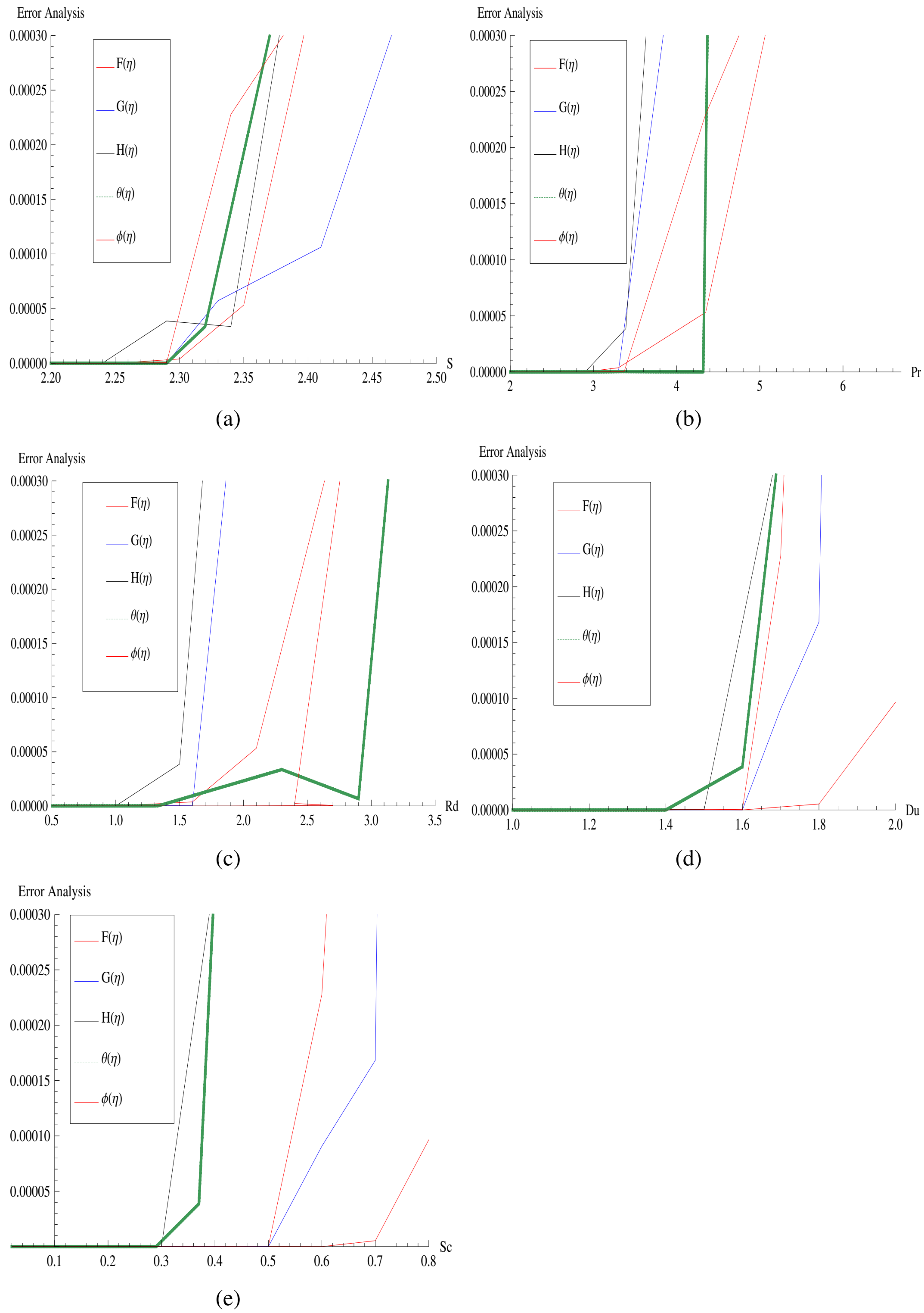


Figure 14: This Figure shows the range of different parameters for different functions

6 Conclusion

We analyzed, the unsteady viscous fluid on combined with heat and mass transfer with Soret and Dufour effect over a impermeable rotating disk, the following results are concluded:

- The fractional solution are more fast convergence than that of classical method.
- The increase of axial and tangential velocity and decrease of radial velocity with enhancement of S.
- By the increasing of prandtl number the fluid show anomalous behavior at integer case, but then gradually increase the temperature.
- The enhancement of Dufour parameter the fluid temperature decrease and with Soret So the temperature of fluid increase.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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