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Closed Solution for Initial Post-Buckling Behavior Analysis of a Composite Beam with Shear Deformation Effect

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Abstract: This paper is focused on the post-buckling behavior of the fixed laminated composite beams with effects of axial compression force and the shear deformation. The analytical solutions are established for the original control equations (that is not simplified) by applying the Maclaurin series expansion, Chebyshev polynomials, the harmonic balance method and the Newton's method. The validity of the present method is verified via comparing the analytical approximate solutions with the numerical ones which are obtained by the shooting method. The present third analytical approximate solutions can give excellent agreement with the numerical solutions for a wide range of the deformation amplitudes. What's more, the effect of shear deformation on the post-bucking configuration of the sandwich beam is also proposed. It can be found that the shear angle has a great influence on the post-buckling load of composite beams. Therefore, the model simplifying the shear formation term as small quantity is not accurate for the case of sandwich beam with soft core.

Keywords: Analytical solution, post-buckling, sandwich beam, harmonic method.

1 Introduction

Sandwich structures, like beams, columns, and rods are commonly used in civil, aircraft, mechanicals and marine industries. Since changing the material and the thickness of the core and face sheets could possibly get various properties, the research of elastic instability of composite structures are becoming highly urgent in modern engineering structures. In fact, the problems of buckling and post-buckling behavior in sandwich structures were initially proposed by several academics, such as Hoff et al. [Hoff (1948); Sheinman and Adan (1987)]. Moreover, Sheinman et al. [Sheinman, Bass and Ishai (1989); Frostig, Baruch, Vilnay et al. (1992); Adan, Sheinman and Altus (1994)] did a series of studies on buckling and post-buckling behavior of sandwich structures based on nonlinear theory.

By using classical Euler theory, Barbero et al. [Barbero and Tomblin (1993, 1994)] developed an analytical model to discuss buckling problems of composite columns.

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Cheng et al. [Cheng, Lin and Wang (1997)] predicted local delamination buckling load on the face sheet of sandwich beams by using Fourier series and Stokes transformation. Huang et al. [Huang and Kardomateas (2002)] studied buckling and initial post-buckling behavior of sandwich beams via applying energy equivalency. A nonlinear beam equation included transverse shear was also established in their study. Considering transverse shear, Kardomateas et al. [Kardomateas and Huang (2003)] proposed a perturbation procedure to study buckling, and initial post-buckling behavior based on the nonlinear beam equations. Ganesan et al. [Ganesan and Kowda (2005)] conducted the local stochastic buckling analysis using the standard Perturbation analysis. Ovesy et al. [Ovesy, Loughlan, Ghannadpour et al. (2006)] predicted the post-buckling behavior of the simply supported box sections by employing three finite-strip methods. Based on the first order shear deformation theory (FSDT), Ghannadpour et al. [Ghannadpour, Ovesy and Zia-Dehkordi (2014)] investigated the buckling and initial post-buckling behavior of the cross-ply laminated plates by using semi-energy method. Then, the post-buckling behavior of the moderately thick composite plates was studied by Ovesy et al. [Ovesy, Zia-Dehkordi and Ghannadpour (2016)]. Under uniaxial compressive forces, the probabilistic characteristics of critical buckling loads for stochastic plate were determined. Based on higher order performing theories, Khdeir et al. [Khdeir and Redd (1997); Khdeir (2001)] developed buckling displacement analysis of laminated composite beams. By assuming the axial load being the eigenvalue, the buckling problem was formulated as an eigen boundary-value problem [Kardomateas (2010)]. Using Euler-Bernoulli and Timoshenko theories, Challamel et al. [Challamel and Girhammar (2011)] presented partial composite beam-columns buckling with shear and axial effects. Phan et al. [Phan, Kardomateas and Frostig (2012, 2013)] established and extended high-order sandwich panel theory for global buckling analysis of sandwich beams.

Considering partial interaction between the layers, Komijani et al. [Komijani, Reddy, Eslami et al. (2013)] studied the thermal buckling behavior of two layered Timoshenko composite beams. The Timoshenko kinematics was considered for both layers. In addition, the shear connection was also represented by a continuous relationship between the interface shear flow and the corresponding slip. Mohammadabadi et al. [Mohammadabadi, Daneshmehr and Homayounfard (2015)] figured out the governing equations of thermal buckling behavior could be solved numerically by using the generalized differential quadrature (GDQ) method. Based on the first-order shear deformation beam theory, Wu et al. [Wu, Kitipornchai and Yang (2017)] investigated thermal post-buckling behavior of functionally graded carbon nanotube-reinforced composite beams. Köllner et al. [Köllner and Völlmecke (2017)] established a geometrically nonlinear model for buckling and post-buckling behavior of composite struts under uniaxial compression.

He et al. [He, Wang and Yang (2016)] investigated the analytical solutions of the free vibration and buckling for composite beams. A new model was established by using higher order beam theory (HSDT), which was more accurate. Pagani et al. [Pagani, Yan and Carrera (2017)] studied the static response of simply supported cross-ply laminated and sandwich beams. A close-form solution based on a one-dimensional (1D) beam model was established. Huang et al. [Huang, Choe, Yang et al. (2019a)] proposed a novel 1D layer-wise model to investigate the post-buckling of simply supported sandwich beam under compressive loads. The post-buckling results were obtained by employing the asymptotic numerical method (ANM). The elastoplastic post-buckling analysis of the sandwich structures were investigated by Le Grognec et al. [Le Grognec and Sad Saoud (2015, 2017); Huang, Choe, Yang et al. (2019b)].

This paper is concerned with the post-buckling behavior of the sandwich beam. The solutions of the governing equation are presented by analytical and numerical methods. The effect of shear deformation on the post-bucking configuration of the sandwich beam is also proposed.

2 Formulation

In this section, an elastic sandwich is considered, a schematic view of a sysmmetric bcukled configuration of an initially straight beam is shown in Fig. 1. Here L, p, V, M are the length of the beam, axial force shear force and moment of the beam end. The rotation angle of neutral axis is $\theta, s \in [0, L]$ is the length coordinate system of the beam. The shear deformation is considered, so the slope β of the deflected beam axis line is not equal to the rotation angle θ of the cross section, for the sake of shear angle γ_{eq}. The expressions of equivalent rigidity modulus $(EI)_{eq}$, effective shear modulus $\overline{G}$, and shear effect coefficient α in detail, are calculated by energy equivalency, and the readers are suggested to view the Ref. [Huang and Kardomateas (2002)]. The Geometric and the physical parameters of the sandwich beam are shown in Fig. 2. The width is W, the height of the top layer, core layer, and bottom layer are f_1, c, f_2; the Young's and shear modulus of the top layer, core layer, and bottom layer are E_1, G_1, E_c, G_c, E_2, G_2, respectively. The initial post-buckling response of the sandwich beams with clamped-clampedcondition will be analyzed.

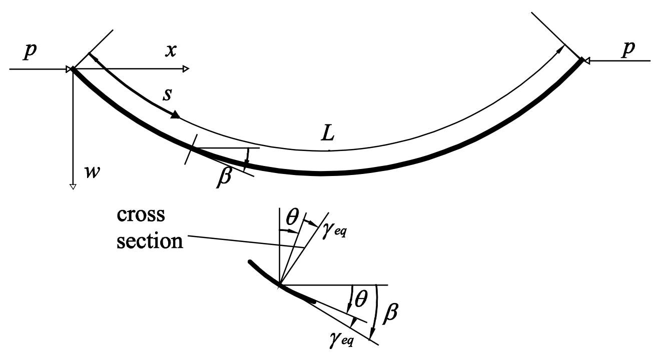

Figure 1: Sketch of sandwich beam under compression load

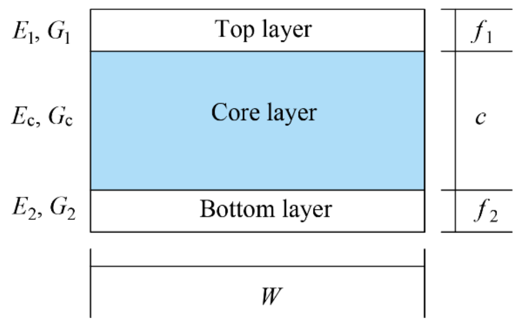

Figure 2: Sketch of cross section of sandwich beam

In this model, dimensional changes of the stresses in the cross-section are assumed to be negligible, that is, stresses are distributed uniformly over the section.The moment m can be calculated as [Huang and Kardomateas (2002)]

$$\frac{d\theta}{ds} = -\frac{m}{(EI)_{eq}} \tag{1}$$

where m=pw+M, v is deflection of the beam.

Obviously, the section shear angle γ_{eq} becomes [Huang and Kardomateas (2002)]

$$\gamma_{eq} = \beta - \theta = \frac{\alpha V}{\overline{G}A} \tag{2}$$

where shear force V at distance s is

$$V = p\sin\beta \tag{3}$$

Vertical displacement w satisfies

$$\frac{dw}{ds} = \sin\beta \tag{4}$$

From Eqs. (1) and (4), a new equation can be obtained as

$$\frac{d^2\theta}{ds^2} = -\frac{p}{(EI)_{eq}}\sin\beta \tag{5}$$

Using Eqs. (2) and (3), the following equations are given

$$\frac{d\theta}{ds} = \frac{d\beta}{ds} - \frac{\alpha}{\overline{G}A}p\cos\beta\frac{d\beta}{ds} \tag{6}$$

$$\frac{d^2\theta}{ds^2} = \frac{d^2\beta}{ds^2} - \frac{\alpha p}{\overline{G}A}\left[\cos\beta\frac{d^2\beta}{ds^2} - \sin\beta\left(\frac{d\beta}{ds}\right)^2\right] \tag{7}$$

Applying Eqs. (5) and (7), therefore, the post-buckling governing equation could be established as

$$-\frac{p}{(EI)_{eq}}\sin\beta = \frac{d^2\beta}{ds^2} - \frac{\alpha p}{\overline{G}A}\left[\cos\beta\frac{d^2\beta}{ds^2} - \sin\beta\left(\frac{d\beta}{ds}\right)^2\right] \tag{8}$$

And the clamped-clamped boundary conditions are

$$\beta(0) = \beta(L) = 0 \tag{9}$$

3 Solution methodology

In general, the governing equations are deformed in a dimensionless manner. In order to facilitate the discussion of calculations and results, another independent variable, $\tau = 2\pi s/L$, is introduced. Then, the dimensionless equations are shown as:

$$(1-2\Omega\lambda\cos\beta)\frac{d^2\beta}{d\tau^2}+2\Omega\lambda\sin\beta\left(\frac{d\beta}{d\tau}\right)^2+\lambda\sin\beta=0 \tag{10}$$

$$\beta(0)=\beta(2\pi)=0 \tag{11}$$

where the dimensionless parameters are

$$\lambda=\frac{pL^2}{4\pi^2(EI)_{eq}},\quad \Omega=\frac{2\alpha\pi^2(EI)_{eq}}{L^2\bar{G}A},\quad \theta=\beta-2\lambda\Omega\sin\beta \tag{12}$$

and the angle of rotation at $\tau=\pi/2$ is

$$\beta(\pi/2)=a \tag{13}$$

where a is the amplitude of the angle of rotation β. Introducing $u=\beta/a$, and then using the Maclaurin series expansion and the Chebyshev polynomials (neglecting the terms $T_i(i>3)$) [Belendez, Rodes, Belendez et al. (2009)], the functions $\sin(au)$ and $\cos(au)$ are reduced to polynomial equation, the new nonlinear equations with no circular functions can be obtained as

$$\left(1-2\Omega\lambda\left(D_0+D_1u^2\right)\right)\frac{d^2u}{d\tau^2}+2\Omega\lambda\left(B_1u+B_2u^3\right)\left(\frac{du}{d\tau}\right)^2+\lambda\left(B_1u+B_2u^3\right)=0 \tag{14}$$

$$u(0)=u(2\pi)=0,\ u(\pi/2)=1 \tag{15}$$

where B_1, B_2, D_0 and D_1 are the coefficients of the polynomials and listed in the Appendix A.

The analytical approximate solution to Eqs. (14) and (15) will be established in terms of a. Following the harmonic balance method [Sun, Lim, Wu et al. (2009)] and Eq. (11), an initial approximation which satisfies condition in Eq. (15) can be given as

$$u_1(\tau)=\sin\tau,\ \tau\in[0,2\pi] \tag{16}$$

where, $u_1(\tau)$ is a periodic function of τ, of period 2π. Since the shape function is arbitrary, the coefficient of $\sin\tau$ must be zero to ensure that the equation could always be correct. Therefore, the coefficient of $\sin\tau$ equation is shown as

$$-8a+8B_1\lambda_1+6B_2\lambda_1+16D_0a\lambda_1\Omega+12D_1a\lambda_1\Omega+4B_1a^2\lambda_1\Omega+2B_2a^2\lambda_1\Omega=0 \tag{17}$$

Using Eq. (17), $\lambda_1(a)$ could be obtained as

$$\lambda_1(a)=\frac{4}{4B_1+3B_2a^2+8D_0\Omega+2B_1a^2\Omega+6D_1a^2\Omega+B_2a^4\Omega} \tag{18}$$

The first approximate solution could be written as

$$\beta_1(\tau)=a\sin\tau,\ \tau\in[0,2\pi] \tag{19}$$

In order to obtain the second analytical approximate solution, the variables are expressed as

$$u_2(\tau)=u_1(\tau)+\Delta u_1(\tau),\tau\in[0,2\pi] \tag{20}$$

$$\lambda_2=\lambda_1+\Delta\lambda_1 \tag{21}$$

Applying Eqs. (20) and (21), Eqs. (14) and (15) could be made linearization with respect to the correction $(\Delta u_1(\tau),\Delta\lambda_1)$ leads to

$$\{1-2\Omega[(\lambda_1+\Delta\lambda_1)(D_0+D_1u_1^2)+2\lambda_1u_1\Delta u_1]\}\frac{d^2u_1}{d\tau^2}+[1-2\Omega\lambda_1(D_0+D_1u_1^2)]$$
$$\times\frac{d^2\Delta u_1}{d\tau^2}+2\Omega[(\lambda_1+\Delta\lambda_1)(B_1u_1+u_1^3B_2)+\lambda_1(B_1\Delta u_1+3B_2u_1^2\Delta u_1)]\left(\frac{du_1}{d\tau}\right)^2$$
$$+4\Omega\lambda_1(B_1u_1+u_1^3B_2)\frac{du_1}{d\tau}\frac{d\Delta u_1}{d\tau}+(\lambda_1+\Delta\lambda_1)(B_1u_1+u_1^3B_2)$$
$$+\lambda_1(B_1\Delta u_1+3B_2u_1^2\Delta u_1)=0 \tag{22}$$
$$\Delta u_1(0)=\Delta u_1(2\pi)=0, \Delta u_1(\pi/2)=0 \tag{23}$$

The correction term $\Delta u_1(\tau)$ can be taken as

$$\Delta u_1(\tau)=z_1(\sin\tau+\sin3\tau) \tag{24}$$

Since the shape function is arbitrary, the coefficients of $\sin\tau$, $\sin3\tau$ must be zero to ensure that the equation could always be correct.

The coefficient equations of $\sin\tau$, and $\sin3\tau$ are

$$-a+C_1\lambda_1+C_1\Delta\lambda_1+z_1(-a+C_2)=0 \tag{25}$$
$$C_3\lambda_1+C_3\Delta\lambda_1+z_1(-9a+C_4)=0 \tag{26}$$

The terms z_1 and $\Delta\lambda_1$ can be obtained as

$$z_1=\frac{M_0}{M_m}(4B_1+3B_2+2a^2B_1\Omega+a^2B_2\Omega+8aD_0\Omega+6aD_1\Omega)^2 \tag{27}$$
$$\Delta\lambda_1=2a\frac{M_0}{M_m}(-6B_2+12a^2B_1\Omega+7a^2B_2\Omega+20aD_1\Omega) \tag{28}$$

The formulas of C_1, C_2, C_3, C_4, M_0, and M_m are listed in the Appendix A. Therefore, the second approximate solution could be written as

$$\beta_2(\tau)=a[\sin\tau+z_1(\sin\tau+\sin3\tau)],\ \tau\in[0,2\pi] \tag{29}$$

In order to obtain a more accurate analytical approximate solution, solutions are expressed as

$$u_3(\tau)=u_1(\tau)+\Delta u_2(\tau),\ \tau\in[0,2\pi] \tag{30}$$
$$\lambda_3=\lambda_1+\Delta\lambda_2 \tag{31}$$

Applying Eqs. (30) and (31), Eqs. (14) and (15) could be made linearization with respect to the correction $(\Delta u_2(\tau),\ \Delta\lambda_2)$ leads to

$$\{1-2\Omega[(\lambda_1+\Delta\lambda_2)(D_0+D_1u_1^2)+2\lambda_1u_1\Delta u_2]\}\frac{d^2u_1}{d\tau^2}+[1-2\Omega\lambda_1(D_0+D_1u_1^2)]$$
$$\times\frac{d^2\Delta u_2}{d\tau^2}+2\Omega[(\lambda_1+\Delta\lambda_2)(B_1u_1+u_1^3B_2)+\lambda_1(B_1\Delta u_2+3B_2u_1^2\Delta u_2)]\left(\frac{du_1}{d\tau}\right)^2$$
$$+4\Omega\lambda_1(B_1u_1+u_1^3B_2)\frac{du_1}{d\tau}\frac{d\Delta u_2}{d\tau}+(\lambda_1+\Delta\lambda_1)(B_1u_1+u_1^3B_2)$$
$$+\lambda_1(B_1\Delta u_2+3B_2u_1^2\Delta u_2)=0 \tag{32}$$
$$\Delta u_2(0)=\Delta u_2(2\pi)=0, \Delta u_2(\pi/2)=0 \tag{33}$$

The correction term $\Delta u_2(\tau)$ can be taken as

$$\Delta u_2(\tau) = z_2(\sin\tau + \sin 3\tau) + z_3(\sin 3\tau + \sin 5\tau) \tag{34}$$

Since the shape function is arbitrary, the coefficients of $\sin\tau$, $\sin 3\tau$ and $\sin 5\tau$ must be zero to ensure that the equation could always be correct. The coefficient equations of $\sin\tau$, $\sin 3\tau$ and $\sin 5\tau$ are shown as

$$-a + F_2 + z_3F_1 + z_2F_2 + F_3\Delta\lambda_2 = 0 \tag{35}$$
$$\lambda_1F_5 + z_3F_4 + z_2F_4 + F_5\Delta\lambda_2 = 0 \tag{36}$$
$$\lambda_1F_6 + z_3(-25a + F_8) + z_2F_7 + F_5\Delta\lambda_2 = 0 \tag{37}$$

The terms z_2, z_3 and $\Delta\lambda_2$ can be obtained as

$$z_2 = \frac{M_1}{M_n}(4B_1 + 3B_2 + 2a^2B_1\Omega + a^2B_2\Omega + 8aD_0\Omega + 6aD_1\Omega) \tag{38}$$
$$z_3 = \frac{M_2}{M_n}(4B_1 + 3B_2 + 2a^2B_1\Omega + a^2B_2\Omega + 8aD_0\Omega + 6aD_1\Omega) \tag{39}$$
$$\Delta\lambda_2 = \frac{M_3}{M_n}\frac{1}{(4B_1+3B_2+2a^2B_1\Omega+a^2B_2\Omega+8aD_0\Omega+6aD_1\Omega)} \tag{40}$$

The formulas of F_1-F_6, M_1, M_2, M_3 and M_n are listed in the Appendix B.
Then, the third approximate solution could be written as

$$\beta_3(\tau) = a[\sin\tau + z_2(\sin\tau + \sin 3\tau) + z_3(\sin 3\tau + \sin 5\tau)], \tau \in [0, 2\pi] \tag{41}$$

From Eqs. (12), (19), (29) and (41), the first three approximations to the rotation angle θ can be written as

$$\theta_1 = \beta_1 - 2\lambda_1\Omega\sin\beta_1 \tag{42}$$
$$\theta_2 = \beta_2 - 2\lambda_2\Omega\sin\beta_2 \tag{43}$$
$$\theta_3 = \beta_3 - 2\lambda_3\Omega\sin\beta_3 \tag{44}$$

Once $\beta(s)$ and p are achieved from Eq. (8), the lateral deflection of the beam $w(s)$ and the coordinate $x(s)$ of a point on the beam which is at length coordinate system of the beam s can then be obtained

$$x(s) = \int_0^s \cos\beta \; d\xi \tag{45}$$
$$w(s) = \int_0^s \sin\beta \; d\xi \tag{46}$$

4 Results and discussion

In this section, the accuracy of the proposed analytical approximations will be illustrated by comparing the presented solutions with numerical solutions obtained via the shooting method [Yu, Wu and Lim (2012)]. In order to obtain the numerical solutions, we transform Eq. (10) into the first-order system

$$\frac{d\beta}{d\tau} = \varphi \tag{47}$$
$$(1 - 2\Omega\lambda\cos\beta)\frac{d\varphi}{d\tau} + 2\Omega\lambda\sin\beta\varphi^2 + \lambda\sin\beta = 0 \tag{48}$$

with boundary conditions:

$$\beta(0) = \beta(2\pi) = 0,\ \beta(\pi/2) = a \tag{49}$$

The boundary value problem Eqs. (47)-(49) can be solved by the shooting method. The numerical solutions for the total deflection angle β_s, dimensionless axial force λ_s and dimensionless lateral deflection w_s are obtained; all of them are dependent on the total rotation angle amplitude a.

In this paper, a numerical example is given to consider a three-layer composite beam, the top and bottom layers are Glass Fiber Reinforced Polymer (GFRP), and the core layer is Polyvinyl chloride (PVC). The parameters of the PVC-GFRP composite beam are shown in Tab. 1. The physical parameters can be obtained by applying the energy equivalency in Ref. [Huang and Kardomateas (2002)], and then the dimensionless parameter Ω can be calculated as $\Omega = 1.8617$.

Table 1: The parameters of the PVC-GFRP composite beam

Material	Young's modulus	Shear modulus	Thickness (mm)	Width (mm)	Length (mm)
Top layer	E_1=26 GPa	G_1=3 GPa	f_1=5		
Core layer	E_c=93 MPa	G_c=35 MPa	c=30	W=40	L=1000
Bottom layer	E_2=26 GPa	G_2=3 GPa	f_2=5		

The analytical solutions of the dimensionless axial force λ_1, λ_2, λ_3 are derived from Eqs. (18), (21) and (31), respectively. The ratios of λ_1, λ_2, λ_3, λ_s to the critical load λ_b (λ_b can be calculated by solving the linearized equations of Eqs. (10) and (11)) change with the total deflection angle amplitude a, as shown in Fig. 3.

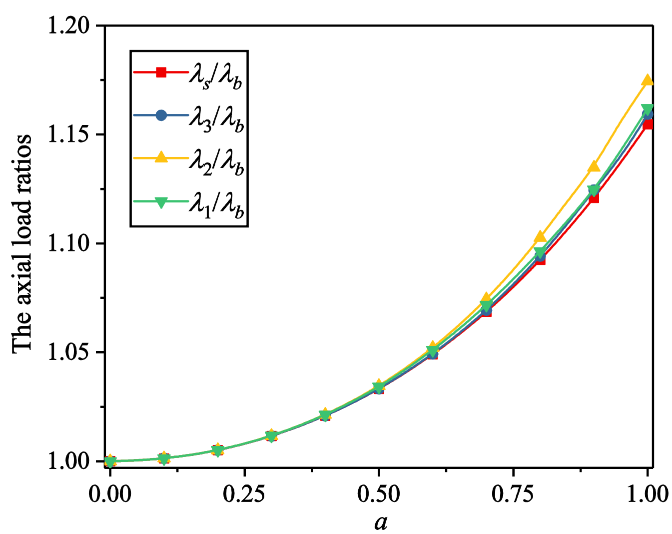

Figure 3: The axial load ratios λ_i/λ_b with respect to a

The analytical solutions of the total deflection angle β_1, β_2, β_3 are calculated by Eqs. (19), (29) and (41), respectively. The comparisons of the numerical solutions β_s and analytical approximate solutions β_1, β_2, β_3 at various positions on composite beams are shown in Figs. 4-6, for the total deflection angle amplitude a=0.3, 0.6 and 1.0, respectively.

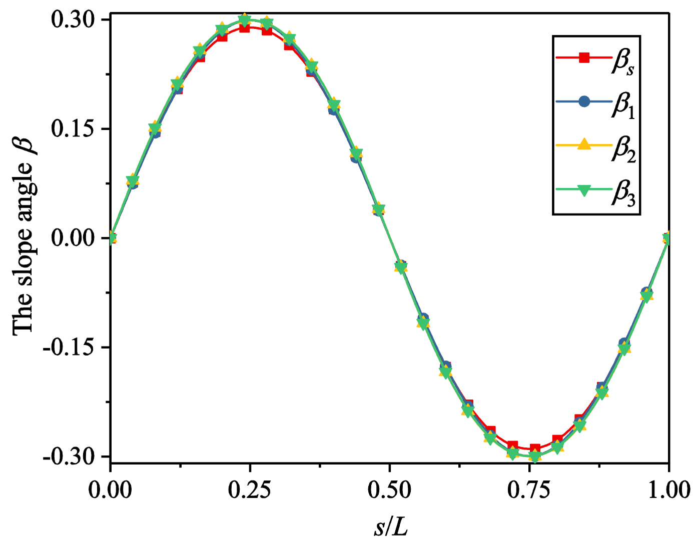

Figure 4: The tangential deflection angles at various positions on composite beams (a=0.3)

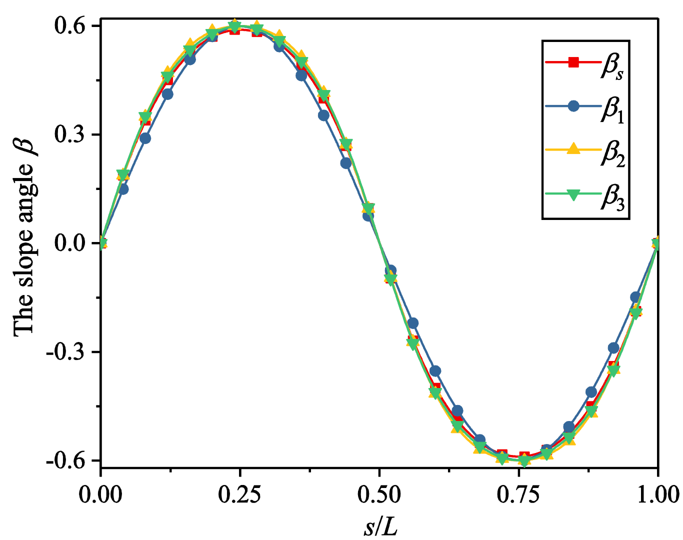

Figure 5: The tangential deflection angles at various positions on composite beams (a=0.6)

It can be seen from the above comparison figures that the first three-order analytical approximation solutions presented in this paper have good precision, and the third-order analytical approximation solution has the highest precision.

The third analytical approximate solutions for the lateral deflection of the composite beam w_3 are determined by Eq. (46). The comparisons of the analytical approximate solutions w_3 and numerical solutions w_s for the total deflection angle amplitude $a =$ 0.4 and 0.8 are demonstrated in Figs. 7 and 8, respectively.

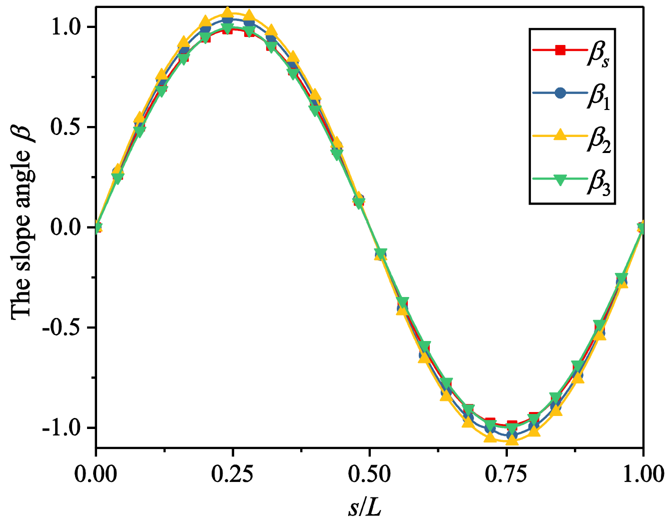

Figure 6: The tangential deflection angles at various positions on composite beams (a=1.0)

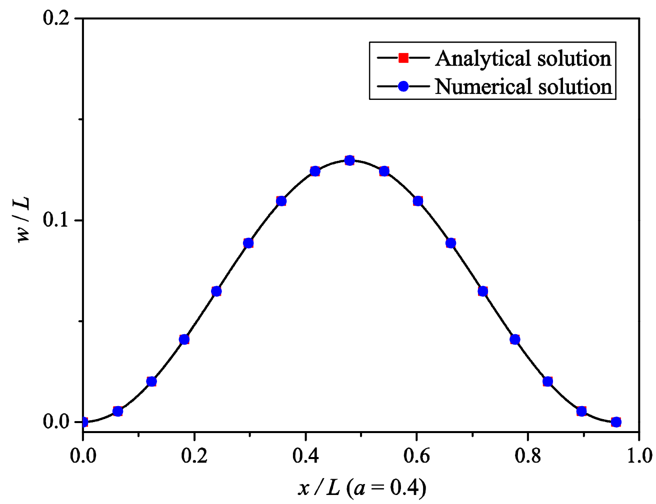

Figure 7: Post-buckling deformation of the composite beam with clamped supports (a=0.4)

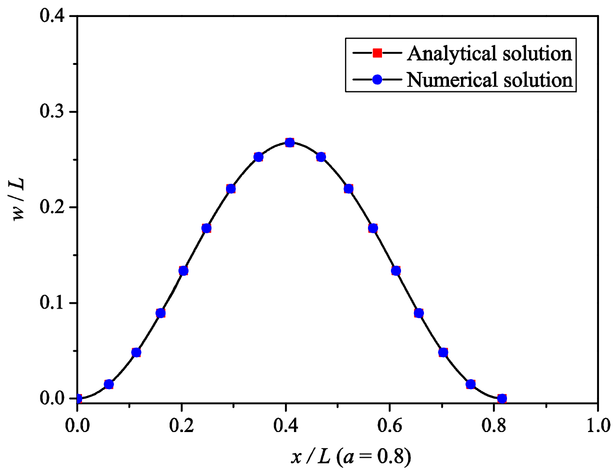

Figure 8: Post-buckling deformation of the composite beam with clamped supports (*a*=0.8)

The shear strain of the composite beam could be ignored in Huang et al. [Huang and Kardomateas (2002)], namely, $\sin\gamma_{eq} = \gamma_{eq}$, $\cos\gamma_{eq} = 1$. The post-buckling governing Eq. (5) can be simplified as follows:

$$\frac{d^2\theta}{d\tau^2} + \Omega\lambda^2 \sin 2\theta + \lambda \sin\theta = 0 \tag{50}$$

The boundary condition is:

$$\theta(0) = 0,\ \theta'(0) = \xi,\ \theta(\pi/2) = \bar{a} \tag{51}$$

where

$$\bar{a} = a - 2\lambda\Omega \sin a \tag{52}$$

Using the simplified model Eqs. (50)-(52), the dimensionless axial force λ_0 can be numerically derived. The present axial force λ_s from the original model and λ_0 from the simplified model Eqs. (50)-(52) with respect to total rotation angle amplitude *a*, as shown in Fig. 9. For the same *a*, the axial force λ_s is less than λ_0. By comparing the calculated results from the present original and simplified model, it can be found that the shear angle has a great influence on the post-buckling load of composite beams. What's more, the shear formation term in governing equation could not be simplified as small quantity for the sandwich beam with soft core.

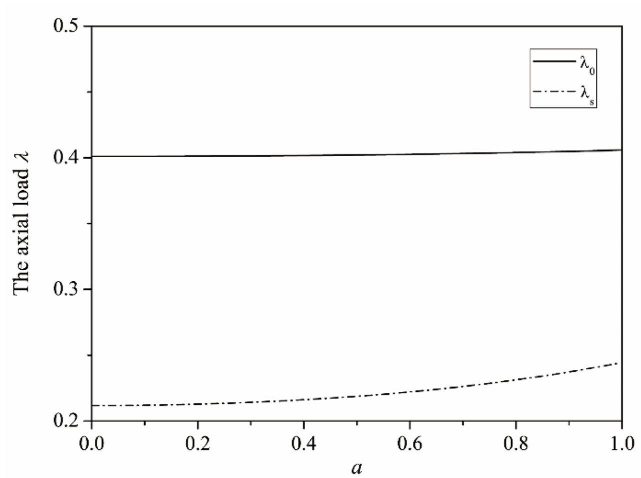

Figure 9: Axial force with respect to max deflection

5 Conclusion

The post-buckling behavior of the fixed laminated composite beams under effects of axial compression force and the shear deformation, is mainly discussed in this paper. The post-buckling governing equation is reduced as a simple nonlinear system by applying the Maclaurin series expansion and Chebyshev polynomials. The analytical approximate solutions for the post-buckling of composite beams are derived by combining the harmonic balance method with the Newton's method. The numerical solutions are obtained by the shooting method. The comparisons between the analytical approximate solutions and the numerical solutions are given in graphs. The results show that the present third analytical approximate solutions are verified with excellent precision, for a wide range of the deformation amplitudes. In summary, the analytical approximate solutions presented here for the post-buckling deformation of composite beams are precise and easy to apply. Compared with the case without simplifying the shear deformation term in the governing equation, the influence of shear deformation on the post-buckling load of composite beams is analyzed. The shear formation could not be simplified as small quantity for sandwich beam with soft core.

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Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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Appendix A

$$B_1 = 1 - \frac{a^4}{384} + \frac{a^6}{11520} - \frac{a^8}{737280}$$

$$B_2 = -\frac{a^2}{6} + \frac{a^4}{96} - \frac{a^6}{3840} + \frac{a^8}{276480}$$

$$D_0 = 1 - \frac{a^6}{23040} + \frac{a^8}{737280} - \frac{a^{10}}{51609600}$$

$$D_1 = -\frac{a^2}{2} + \frac{a^6}{1280} - \frac{a^8}{46080} + \frac{a^{10}}{3440640}$$

$$C_1 = B_1 + \frac{3B_2}{4} + 2D_0 a\Omega + \frac{3D_1 a\Omega}{2} + \frac{1}{2}B_1 a^2\Omega + \frac{1}{4}B_2 a^2\Omega$$

$$C_2 = \lambda_1\left(B_1 + \frac{3B_2}{2} + 2D_0 a\Omega - D_1 a\Omega - B_1 a^2\Omega - \frac{5}{8}B_2 a^2\Omega\right)$$

$$C_3 = -\frac{B_2}{4} - \frac{D_1 a\Omega}{2} + \frac{1}{2}B_1 a^2\Omega + \frac{1}{8}B_2 a^2\Omega$$

$$C_4 = \lambda_1\left(B_1 + \frac{3B_2}{4} + 18D_0 a\Omega + \frac{19D_1 a\Omega}{2} + \frac{5}{2}B_1 a^2\Omega + \frac{11}{8}B_2 a^2\Omega\right)$$

$$M_0 = \frac{2\left(-2B_2 + 2a^2B_1\Omega + a^2B_2\Omega - 4aD_1\Omega\right)}{(4B_1 + 3B_2 + 2a^2B_1\Omega + a^2B_2\Omega + 8aD_0\Omega + 6aD_1\Omega)}$$

$$\begin{aligned} M_m = {} & 512{B_1}^2 + 768B_1B_2 + 276{B_2}^2 + (a^2(384{B_1}^2 + 520B_1B_2 + 158{B_2}^2) \\ & + a(1024B_1 + 768B_2)D_0 + a(1024B_1 + 784B_2)D_1)\Omega \\ & + (a^4(16{B_1}^2 + 20B_1B_2 + 7{B_2}^2) + a^3(288B_1 + 156B_2)D_1 + 464a^2{D_1}^2 \\ & + D_0(a^3(256B_1 + 112B_2) + 512a^2D_1))\Omega^2 \end{aligned}$$

Appendix B

$$F_1 = \lambda_1(-\frac{3B_2}{4} - \frac{5}{2}a^2B_1\Omega - a^2B_2\Omega - \frac{11aD_1\Omega}{2})$$

$$F_2 = -a + \lambda_1(B_1 + \frac{3B_2}{4} + \frac{1}{2}a^2B_1\Omega + \frac{1}{4}a^2B_2\Omega + 2aD_0\Omega + \frac{3aD_1\Omega}{2})$$

$$F_3 = B_1 + \frac{3B_2}{4} + \frac{1}{2}a^2B_1\Omega + \frac{1}{4}a^2B_2\Omega + 2aD_0\Omega + \frac{3aD_1\Omega}{2}$$

$$F_4 = -9a + \lambda_1(B_1 + \frac{3B_2}{4} - \frac{7}{2}a^2B_1\Omega - \frac{7}{4}a^2B_2\Omega + 18aD_0\Omega - \frac{5aD_1\Omega}{2})$$

$$F_5 = -\frac{B_2}{4} + \frac{1}{2}a^2B_1\Omega + \frac{1}{8}a^2B_2\Omega - \frac{aD_1\Omega}{2}$$

$$F_6 = \frac{1}{8}a^2 B_2 \Omega$$

$$F_7 = \lambda_1\left(-\frac{3B_2}{4} + \frac{7}{2}a^2 B_1 \Omega + \frac{7}{8}a^2 B_2 \Omega - \frac{11aD_1\Omega}{2}\right)$$

$$F_8 = \lambda_1\left(B_1 + \frac{3B_2}{4} + \frac{9}{2}a^2 B_1 \Omega + \frac{9}{4}a^2 B_2 \Omega + 50aD_0\Omega + \frac{43aD_1\Omega}{2}\right)$$

$$\begin{aligned} M_1 = {} & -12B_1B_2 - 9{B_2}^2 + \Omega\big(24a^2{B_1}^2 + 22a^2B_1B_2 + 4a^2{B_2}^2 \\ & - 24aB_1D_1 - 26aB_2D_1\big) + \Omega^2\big(8a^4{B_1}^2 + 8a^4B_1B_2 + 2a^4{B_2}^2 + 8a^3B_1D_1 \\ & + 4a^3B_2D_1 - 16a^2{D_1}^2\big) \end{aligned}$$

$$\begin{aligned} M_2 = {} & -3{B_2}^2 + \big(36a^2B_1B_2 + 17a^2{B_2}^2 - 28aB_2D_1\big)\Omega + (-28a^4{B_1}^2 - 10a^4B_1B_2 \\ & + 72a^3B_1D_1 + 26a^3B_2D_1 - 44a^2{D_1}^2)\Omega^2 \end{aligned}$$

$$\begin{aligned} & M_3 = 4a(288B_1{B_2}^2 + 225{B_2}^3 + (a^2(-1152{B_1}^2B_2 - 1374B_1{B_2}^2 - 387{B_2}^3) \\ & +a(-384B_1B_2 + 54{B_2}^2)D_1)\Omega + (a^4(1152{B_1}^3 + 1260{B_1}^2B_2 + 332B_1{B_2}^2 \\ & -4{B_2}^3) + a^3(768{B_1}^2 - 1080B_1B_2 - 844{B_2}^2)D_1 + a^2(-1920B_1 - 948B_2) \\ & {D_1}^2)\Omega^2 + (a^6(664{B_1}^3 + 820{B_1}^2B_2 + 360B_1{B_2}^2 + 56{B_2}^3) + a^5(920{B_1}^2 \\ & +728B_1B_2 + 168{B_2}^2)D_1 - a^4(1272B_1 + 524B_2){D_1}^2 - 312a^3{D_1}^3)\Omega^3) \end{aligned}$$

$$\begin{aligned} & M_n = 12288{B_1}^3 + 27264{B_1}^2B_2 + 19872B_1{B_2}^2 + 4743{B_2}^3 + (a^2(15104{B_1}^3 \\ & +30144{B_1}^2B_2 + 19518B_1{B_2}^2 + 4059{B_2}^3) + a((24576{B_1}^2 + 36096B_1B_2 \\ & +13248{B_2}^2)D_0 + (29952{B_1}^2 + 43968B_1B_2 + 15786{B_2}^2)D_1))\Omega \\ & +(a^4(6144{B_1}^3 + 10484{B_1}^2B_2 + 5732B_1{B_2}^2 + 1012{B_2}^3) + a^3((17920{B_1}^2 \\ & +20352B_1B_2 + 5736{B_2}^2)D_0 + (23296{B_1}^2 + 28216B_1B_2 + 7900{B_2}^2)D_1) \\ & +a^2((23040B_1 + 15744B_2)D_0D_1 + (17664B_1 + 11604B_2){D_1}^2))\Omega^2 \\ & +(a^6(744{B_1}^3 + 1036{B_1}^2B_2 + 440B_1{B_2}^2 + 56{B_2}^3) + a^5((5632{B_1}^2 \\ & +4608B_1B_2 + 896{B_2}^2)D_0 + (5736{B_1}^2 + 4648B_1B_2 + 856{B_2}^2)D_1) \\ & +a^4\left((9728B_1 + 2816B_2)D_0D_1 + (7800B_1 + 2252B_2){D_1}^2\right) \\ & +a^3(-3072D_0{D_1}^2 - 1992{D_1}^3))\Omega^3 \end{aligned}$$