Solving the Optimal Control Problems of Nonlinear Duffing Oscillators By Using an Iterative Shape Functions Method

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Abstract: In the optimal control problem of nonlinear dynamical system, the Hamiltonian formulation is useful and powerful to solve an optimal control force. However, the resulting Euler-Lagrange equations are not easy to solve, when the performance index is complicated, because one may encounter a two-point boundary value problem of nonlinear differential algebraic equations. To be a numerical method, it is hard to exactly preserve all the specified conditions, which might deteriorate the accuracy of numerical solution. With this in mind, we develop a novel algorithm to find the solution of the optimal control problem of nonlinear Duffing oscillator, which can exactly satisfy all the required conditions for the minimality of the performance index. A new idea of shape functions method (SFM) is introduced, from which we can transform the optimal control problems to the initial value problems for the new variables, whose initial values are given arbitrarily, and meanwhile the terminal values are determined iteratively. Numerical examples confirm the high-performance of the iterative algorithms based on the SFM, which are convergence fast, and also provide very accurate solutions. The new algorithm is robust, even large noise is imposed on the input data.

1 Introduction

The aim of the paper is to provide a highly efficient method for solving the optimal control force u in the nonlinear Duffing oscillator:

$$\ddot{x}(t) + \gamma \dot{x}(t) + \alpha x(t) + \beta x^3(t) = u(t), \tag{1}$$

which is often appeared in the literature [Cveticanin (2013)], with widespread applications in science and engineering, from a nonlinear spring-mass system in mechanics to fault signal detection [Hu and Wen (2003)], and structures design [Suhardjo, Spencer and Sain (1992)]. The control of a Duffing oscillator has a seminal significance to the control problems of

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nonlinear dynamic responses of aerospace structures, such as beams, plates, and shells.

In the optimal control problem, one desires to control the response of a nonlinear structure remained within a specified safety limit, and thus one may encounter the problem that the external forces are not yet known, but service for a specific purpose of controlling the nonlinear structure to a desired state. The control forces are designed intentionally, such that a specified cost functional which weights the cost of control *vs.* the allowed response is minimized. The control of nonlinear aerospace structural systems has gained much attention in the past several decades, and different controllers were proposed for the applications to different areas [Suhardjo, Spencer and Sain (1992); Agrawal, Yang and Wu (1998)]. Van Dooren et al. [Van Dooren and Vlassenbroeck (1982); El-Gindy, El-Hawary, Salim et al. (1995); El-Kady and Elbarbary (2002)] have introduced the Chebyshev series expansion method to solve the controlled problem of Duffing oscillator. Razzaghi et al. [Razzaghi and Elnagar (1994)] have applied a pseudospectral method to solve this problem, Garg et al. [Garg, Patterson, Hager et al. (2010)] have provided a unified pseudospectral method to solve the optimal control problems, and Lakestani et al. [Lakestani, Razzaghi and Dehghan (2006)] have applied a semi-orthogonal spline wavelets to solve this problem. Previously, Rad et al. [Rad, Kazem and Parand (2012)] used the radial basis functions method to solve the optimal control problem of Duffing oscillator, and Elgohary et al. [Elgohary, Dong, Junkins et al. (2014)] applied a simple collocation method together with the radial basis functions method to solve the optimal control problem of Duffing oscillator with a simple performance index.

Liu [Liu (2012)] has applied the Lie-group adaptive method to solve the optimal control problem of nonlinear Duffing oscillator, while Liu [Liu (2014)] applied the Lie-group differential algebraic equation method to find a sliding control strategy for nonlinear system. Continuing this line, Tsai et al. [Tsai and Lee (2018)] proposed a Lie-group approach of the optimal control problem of nonlinear Duffing oscillator. In the paper we will solve the optimal control problem of Duffing oscillators under a complicated performance index without needing of the solution of nonlinear algebraic equations, of which the key point is that we can transform the optimal control problem into an initial value problem. When the initial values for the new variables are given freely, the terminal values require to be determined iteratively.

We arrange the paper as follows. The Hamiltonian formulation of the optimal control problem of nonlinear dynamical system and the Euler-Lagrange equations are introduced in Section 2. In Section 3, we introduce two types of shape functions for automatically satisfying the prescribed boundary conditions. In Section 4, the iterative algorithms based-on the shape functions method (SFM) are developed, considering different terminal conditions of state and co-state variables. Some examples are given in Section 5 to assess the performance of the SFM. Finally, the conclusions are drawn in Section 6.

2 A Hamiltonian formulation

For the nonlinear dynamical system depicted by a set of ordinary differential equations (ODEs), and subjected to external control force **u**:

$$\dot{\mathbf{x}} = \mathrm{f}(\mathbf{x}, t; \mathbf{u}), \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x}_0$ are prescribed initial values, we usually select an optimal control force $\mathbf{u}(t)$ by satisfying the following minimization of a specified performance index J:

$$\min\left\{J = g\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt\right\}, \tag{3}$$

where $t \in [t_0, t_f]$ is a time interval we interest, and $L(\mathbf{x}(t), \mathbf{u}(t), t)$ is the Lagrange function.

Let H be the Hamiltonian:

$$H = L(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{f}, \tag{4}$$

from which the augmented performance index is given by

$$\min\left\{J = g\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} (H - \boldsymbol{\lambda}^{\mathrm{T}}\dot{\mathbf{x}})dt\right\}. \tag{5}$$

The variation of the above performance index expressed in terms of the variations of $\mathbf{x}$, $\boldsymbol{\lambda}$ and $\mathbf{u}$ is

$$\delta J = (g_\mathbf{x} - \boldsymbol{\lambda})^{\mathrm{T}}\delta\mathbf{x}\big|_{t_f} + \boldsymbol{\lambda}^{\mathrm{T}}\delta\mathbf{x}\big|_{t_0} + \int_{t_0}^{t_f}\left[\left(H_\mathbf{x} + \dot{\boldsymbol{\lambda}}\right)^{\mathrm{T}}\delta\mathbf{x} + (H_{\boldsymbol{\lambda}} - \dot{\mathbf{x}})^{\mathrm{T}}\delta\boldsymbol{\lambda} + H_\mathbf{u}^{\mathrm{T}}\delta\mathbf{u}\right]dt, \tag{6}$$

where the subscript denotes the partial differential. Thus, the minimization of Eq. (5) leads to a triple of the Euler-Lagrange equations:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial\boldsymbol{\lambda}}, \tag{7}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial\mathbf{x}}, \tag{8}$$

$$\frac{\partial H}{\partial\mathbf{u}} = 0. \tag{9}$$

Depending on what are prescribed for the states of $\mathbf{x}(t_0)$ and $\mathbf{x}(t_f)$, some complementary boundary conditions for $\boldsymbol{\lambda}$ at t_0 and t_f can be obtained from the vanishing of

$$(g_\mathbf{x} - \boldsymbol{\lambda})^{\mathrm{T}}\delta\mathbf{x}\big|_{t_f} + \boldsymbol{\lambda}^{\mathrm{T}}\delta\mathbf{x}\big|_{t_0} \text{ in Eq. (6).}$$

For most cases, Eq. (9) renders an explicit form of $\mathbf{u}$ in terms of state and co-state variables, which is thus being inserted into Eqs. (7) and (8) to obtain a set of two-point boundary value problems. In general, Eqs. (7)-(9) constitute a two-point boundary value problem of nonlinear differential algebraic equations (DAEs), which is difficult to be solved.

3 A shape function approach

Let us consider the Duffing equation with a control force in the following Hamiltonian:

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2(u - \gamma x_2 - \alpha x_1 - \beta x_1^3), \tag{10}$$

where $x_1 := x$ and $x_2 := \dot{x}$, and λ_1 and λ_2 are two Lagrange multipliers. Then, we can derive the following Euler-Lagrange equations:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\gamma x_2 - \alpha x_1 - \beta x_1^3 - \lambda_2,$$
$$\dot{\lambda}_1 = (\alpha + 3\beta x_1^2)\lambda_2,$$
$$\dot{\lambda}_2 = -\lambda_1, \tag{11}$$

$$u = -\lambda_2. \tag{12}$$

Depending on the form of J, there are different terminal conditions of x_1, x_2, λ_1, and λ_2. It is utmost important to exactly satisfy the terminal conditions, in order that the value of J obtained is really the minimal one.

In order to demonstrate the shape function method, we first confine ourselves to a specific example with

$$J = \frac{1}{2}x^2(t_f) + \frac{1}{2}\dot{x}^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\, dt. \tag{13}$$

From Eqs. (6) and (13) we have $\mathbf{x}^f = \boldsymbol{\lambda}^f$ where $\mathbf{x}^f = \mathbf{x}(t_f)$ and $\boldsymbol{\lambda}^f = \boldsymbol{\lambda}(t_f)$, which leads to

$$\lambda_1(t_f) = x_1^f := a, \ \lambda_2(t_f) = x_2^f := b. \tag{14}$$

Both $x_1^f = x(t_f)$ and $x_2^f = \dot{x}(t_f)$ are unknown constants, which are denoted by a and b. Because the initial values of λ_1 and λ_2 are unknown, we cannot directly integrate Eq. (11), which must satisfy the following constraints:

$$\lambda_1(t_f) = a, \ \lambda_2(t_f) = b. \tag{15}$$

To guarantee the numerical solutions of λ_1 and λ_2 can exactly satisfy Eq. (15), we introduce

$$s_1(t) = s_{10} + \frac{(1-s_{10})(t-t_0)}{t_f-t_0}, \ s_1(t_0) = s_{10}, s_1(t_f) = 1, \tag{16}$$

$$s_2(t) = s_{20} + \frac{(1-s_{20})(t-t_0)}{t_f-t_0}, \ s_2(t_0) = s_{20}, s_2(t_f) = 1, \tag{17}$$

where s_{10} and s_{20} are constant parameters. The usefulness of s_{10} and s_{20} will be explained below.

Theorem 1: For any functions $\Lambda_1(t), \Lambda_2(t) \in C\,[t_0, t_f]$, $\lambda_1(t)$ and $\lambda_2(t)$ given by

$$\lambda_1(t) = \Lambda_1(t) + s_1(t)[a - \Lambda_1(t_f)], \tag{18}$$

$$\lambda_2(t) = \Lambda_2(t) + s_2(t)[b - \Lambda_2(t_f)], \tag{19}$$

satisfy

$$\lambda_1(t_f) = a, \ \lambda_2(t_f) = b. \tag{20}$$

Proof: Inserting $t = t_f$ into Eq. (18) leads to

$$\lambda_1(t_f) = \Lambda_1(t_f) + s_1(t_f)[a - \Lambda_1(t_f)],$$

which using Eq. (16) becomes

$$\lambda_1(t_f) = \Lambda_1(t_f) + s_1(t_f)[a - \Lambda_1(t_f)] = \Lambda_1(t_f) + a - \Lambda_1(t_f) = a.$$

Thus, we proved the first one in Eq. (20). Similarly, inserting $t = t_f$ into Eq. (19) leads to

$$\lambda_2(t_f) = \Lambda_2(t_f) + s_2(t_f)[b - \Lambda_2(t_f)],$$

which using Eq. (17) becomes

$$\lambda_2(t_f) = \Lambda_2(t_f) + s_2(t_f)[b - \Lambda_2(t_f)] = \Lambda_2(t_f) + b - \Lambda_2(t_f) = b.$$

Thus, we proved the second one in Eq. (20).

Sometimes we may consider the optimal orbit control problem, which would bring the

initial point $\left(x(t_0), \dot{x}(t_0)\right) = (A_0, B_0)$ to a desired point $\left(x(t_f), \dot{x}(t_f)\right) = (C_0, D_0)$ in the phase space by using the optimal control force, such that we face the over-specified conditions of $x(t)$:

$$x(t_0) = A_0, \ \dot{x}(t_0) = B_0, x(t_f) = C_0, \dot{x}(t_f) = D_0. \tag{21}$$

In order to guarantee that the numerical solution of x can exactly satisfy Eq. (21), the four shape functions $f_k(t)$, $k = 1, \dots, 4$ have to satisfy

$$f_1(t_0) = 1, \ \dot{f}_1(t_0) = 0, f_1(t_f) = 0, \ \dot{f}_1(t_f) = 0, \tag{22}$$

$$f_2(t_0) = 0, \ \dot{f}_2(t_0) = 1, f_2(t_f) = 0, \ \dot{f}_2(t_f) = 0, \tag{23}$$

$$f_3(t_0) = 0, \ \dot{f}_3(t_0) = 0, f_3(t_f) = 1, \ \dot{f}_3(t_f) = 0, \tag{24}$$

$$f_4(t_0) = 0, \ \dot{f}_4(t_0) = 0, f_4(t_f) = 0, \ \dot{f}_4(t_f) = 1. \tag{25}$$

Through some operations we can derive

$$f_1(t) = 1 - 3\left(\frac{t-t_0}{t_f-t_0}\right)^2 + 2\left(\frac{t-t_0}{t_f-t_0}\right)^3, \tag{26}$$

$$f_2(t) = (t_f - t_0)\left[\frac{t-t_0}{t_f-t_0} - 2\left(\frac{t-t_0}{t_f-t_0}\right)^2 + \left(\frac{t-t_0}{t_f-t_0}\right)^3\right], \tag{27}$$

$$f_3(t) = 3\left(\frac{t-t_0}{t_f-t_0}\right)^2 - 2\left(\frac{t-t_0}{t_f-t_0}\right)^3, \tag{28}$$

$$f_4(t) = (t_f - t_0)\left[\left(\frac{t-t_0}{t_f-t_0}\right)^3 - \left(\frac{t-t_0}{t_f-t_0}\right)^2\right]. \tag{29}$$

Theorem 2: For any function $y(t) \in C^1 [t_0, t_f]$, $x(t)$ given by

$$x(t) = y(t) + f_1(t)[A_0 - y(t_0)] + f_2(t)[B_0 - \dot{y}(t_0)] + f_3(t)[C_0 - y(t_f)] + f_4(t)[D_0 - \dot{y}(t_f)], \tag{30}$$

satisfies all the conditions in Eq. (21).

Proof: Inserting $t = t_0$ into Eq. (30) leads to

$x(t_0) = y(t_0) + f_1(t_0)[A_0 - y(t_0)] + f_2(t_0)[B_0 - \dot{y}(t_0)] + f_3(t_0)[C_0 - y(t_f)] + f_4(t_0)[D_0 - \dot{y}(t_f)],$

which using the first ones in Eqs. (22)-(25) becomes

$x(t_0) = y(t_0) + A_0 - y(t_0) = A_0.$

Thus, we proved the first one in Eq. (21).

Taking the time derivative of Eq. (30) and inserting $t = t_0$ leads to

$\dot{x}(t_0) = \dot{y}(t_0) + \dot{f}_1(t_0)[A_0 - y(t_0)] + \dot{f}_2(t_0)[B_0 - \dot{y}(t_0)] + \dot{f}_3(t_0)[C_0 - y(t_f)] + \dot{f}_4(t_0)[D_0 - \dot{y}(t_f)],$

which using the second ones in Eqs. (22)-(25) becomes

$\dot{x}(t_0) = \dot{y}(t_0) + B_0 - \dot{y}(t_0) = B_0.$

Thus, we proved the second one in Eq. (21). The proofs of the third and fourth conditions in Eq. (21) can be done similarly, and we omit them. $\square$

In terms of the new variable $y(t)$, we can recast the Duffing equation to

$$\ddot{y} + \gamma\dot{y} + \alpha y + \beta(y - G)^3 = u + \ddot{G} + \gamma\dot{G} + \alpha G, \tag{31}$$

where
$$G(t) = f_1(t)[y(t_0) - A_0] + f_2(t)[\dot{y}(t_0) - B_0] + f_3(t)[y(t_f) - C_0] + f_4(t)[\dot{y}(t_f) - D_0]. \quad (32)$$
Solving Eq. (31) and inserting $y(t)$ into Eq. (30), we can guarantee that $x(t)$ exactly satisfies all the conditions in Eq. (21).

4 Iterative algorithms based on the SFM

Theorem 1 is crucial that the new shape function method guarantees that the terminal conditions of λ_1 and λ_2 can be exactly satisfied by Eqs. (18) and (19). Based on the concept of shape functions we can develop an iterative algorithm, namely the shape functions method (SFM), to solve Eqs. (11) and (14). For this purpose we can consider the variables transformations from $[\lambda_1, \lambda_2]$ to $[\Lambda_1, \Lambda_2]$:
$$\lambda_1(t) = \Lambda_1(t) - G_1(t), \lambda_2(t) = \Lambda_2(t) - G_2(t), \quad (33)$$
where
$$G_1(t) := s_1(t)[\Lambda_1(t_f) - a], \quad (34)$$
$$G_2(t) := s_2(t)[\Lambda_2(t_f) - b]. \quad (35)$$
In terms of $\Lambda_1(t)$ and $\Lambda_2(t)$ and from Eq. (11), we have a new system of ODEs:
$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\gamma x_2 - \alpha x_1 - \beta x_1^3 - \Lambda_2 + G_2,$$
$$\dot{\Lambda}_1 = \dot{G}_1 + (\alpha + 3\beta x_1^2)(\Lambda_2 - G_2),$$
$$\dot{\Lambda}_2 = \dot{G}_2 - \Lambda_1 + G_1, \quad (36)$$
which can be deemed as an initial value problem (IVP), whose initial values are given by $x_1(t_0) = A_0, x_2(t_0) = B_0$, and $\Lambda_1(t_0)$ and $\Lambda_2(t_0)$ can be given arbitrarily, say $\Lambda_1(t_0) = \Lambda_2(t_0) = 0$.

Unfortunately, $\Lambda_1(t_f)$ denoted by c and $\Lambda_2(t_f)$ denoted by d in the functions $G_1(t)$ and $G_2(t)$ are unknown constants. At the same time, $a = x_1(t_f)$ and $b = x_2(t_f)$ are also unknown constants. If a, b, c and d are available, we can apply the fourth-order Runge-Kutta method (RK4) to integrate the above ODEs and then $\lambda_1(t), \lambda_2(t)$ and $u(t) = -\lambda_2(t)$ can be obtained from Eq. (33), of which all the specified conditions are satisfied exactly. Specially we can obtain the initial values of the co-state variables as follows:
$$\lambda_1(t_0) = \Lambda_1(t_0) - G_1(t_0) = s_{10}[x_1(t_f) - \Lambda_1(t_f)], \quad (37)$$
$$\lambda_2(t_0) = \Lambda_2(t_0) - G_2(t_0) = s_{20}[x_2(t_f) - \Lambda_2(t_f)], \quad (38)$$
if we set $\Lambda_1(t_0) = \Lambda_2(t_0) = 0$. Here we can observe that the usefulness of s_{10} and s_{20}. In general, $\lambda_1(t_0) \neq 0$ and $\lambda_2(t_0) \neq 0$. Therefore, if we give suitable values of $s_{10} \neq 0$ and $s_{20} \neq 0$, which can generate the correct values of $\lambda_1(t_0) \neq 0$ and $\lambda_2(t_0) \neq 0$, by using the convergent values of $x_1(t_f), x_2(t_f), \Lambda_1(t_f)$ and $\Lambda_2(t_f)$.

The iterative algorithm SFM for solving the optimal control problem is summarized as follows.

(i) Give $x_1(t_0)$, $x_2(t_0)$, $\Lambda_1(t_0) = 0$, $\Lambda_2(t_0) = 0$, an initial guess of a_0, b_0, c_0 and d_0 and the convergence criterion ϵ, and then compute $\Delta t = (t_f - t_0)/N$ with N given.

(ii) For $k = 0, 1, 2, ...$, we repeat the following iterations:

Applying the RK4 to integrate the following ODEs with N steps to $t = t_f$:

$\dot{x}_1 = x_2$,

$\dot{x}_2 = -\gamma x_2 - \alpha x_1 - \beta x_1^3 - \Lambda_2 + s_2(d_k - b_k)$,

$\dot{\Lambda}_1 = \dot{s}_1(c_k - a_k) + (\alpha + 3\beta x_1^2)(\Lambda_2 - s_2(d_k - b_k))$,

$\dot{\Lambda}_2 = \dot{s}_2(d_k - b_k) - \Lambda_1 + s_1(c_k - a_k)$.

Take

$a_{k+1} = x_1(t_f), b_{k+1} = x_2(t_f), c_{k+1} = \Lambda_1(t_f), d_{k+1} = \Lambda_2(t_f)$,

and if a_{k+1}, b_{k+1}, c_{k+1} and d_{k+1} converge according to a given stopping criterion:

$r_k := \sqrt{(a_{k+1} - a_k)^2 + (b_{k+1} - b_k)^2 + (c_{k+1} - c_k)^2 + (d_{k+1} - d_k)^2} < \epsilon$,

then stop; otherwise, go to step (ii).

According to Theorem 2, the second iterative algorithm SFM for solving the optimal orbit control problem is summarized as follows.

(i) Give $y_1(t_0) = 0$, $y_2(t_0) = 0$, $\lambda_1(t_0)$, $\lambda_2(t_0)$, an initial guess of a_0, b_0, and the convergence criterion ϵ, and then compute $\Delta t = (t_f - t_0)/N$ with N given.

(ii) For $k = 0, 1, 2, ...$, we repeat the following iterations:

Applying the RK4 to integrate the following ODEs with N steps to $t = t_f$:

$\dot{y}_1 = y_2$,

$\dot{y}_2 = -\gamma y_2 - \alpha y_1 - \beta(y_1 - G)^3 - \lambda_2 + \ddot{G} + \gamma \dot{G} + \alpha G$,

$\dot{\lambda}_1 = [\alpha + 3\beta(y_1 - G)^2]\lambda_2$,

$\dot{\lambda}_2 = -\lambda_1$,

where G was given by Eq. (32). Taking

$a_{k+1} = y_1(t_f), b_{k+1} = y_2(t_f)$,

if a_{k+1} and b_{k+1} converge according to a given stopping criterion:

$r_k := \sqrt{(a_{k+1} - a_k)^2 + (b_{k+1} - b_k)^2} < \epsilon$,

then stop; otherwise, go to step (ii).

When y_1 and y_2 are available, we can solve x_1 and x_2 by

$x_1(t) = y_1(t) + f_1(t)[A_0 - y_1(t_0)] + f_2(t)[B_0 - y_2(t_0)] + f_3(t)[C_0 - y_1(t_f)] + f_4(t)[D_0 - y_2(t_f)]$,

$x_2(t) = y_2(t) + \dot{f}_1(t)[A_0 - y_1(t_0)] + \dot{f}_2(t)[B_0 - y_2(t_0)] + \dot{f}_3(t)[C_0 - y_1(t_f)] + \dot{f}_4(t)[D_0 - y_2(t_f)]$,

where $y_1(t_f)$ and $y_2(t_f)$ are the convergent values of the sequence $(a_k, b_k), k = 1, 2, ...$, and meanwhile $y_1(t_0)$ and $y_2(t_0)$ are the given arbitrary initial values, say $y_1(t_0) = y_2(t_0) = 0$.

5 Numerical examples

5.1 Example 1

First, we consider the following performance index for an undamped Duffing oscillator:

$$J = \frac{1}{2}x^2(t_f) + \frac{1}{2}\dot{x}^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\, dt, \tag{39}$$

where we fix $t_0 = 0$, $t_f = 2$, $x(t_0) = A_0 = 0.5$ and $\dot{x}(t_0) = B_0 = 0.5$.

Under the following parameters $\gamma = 0$, $\alpha = 1$, and $\beta = 0.9$, we apply the SFM with $s_{10} = -1$, $s_{20} = -0.2$, $N = 200$, $a_0 = b_0 = c_0 = d_0 = 0$, to solve this problem, which is convergent with 24 iterations under the convergence criterion $\epsilon = 10^{-10}$ as shown in Fig. 1(a), and the responses of x_1, x_2 and the control force $u = -\lambda_2$ are plotted in Fig. 1(b). The computed results by Eqs. (37)-(39) are $\lambda_1(t_0)$=0.27213155, $\lambda_2(t_0)$=0.28945859, and J=0.14780793. The results computed from the Lie-group (LG) approach [Tsai and Lee (2018)] were also plotted in Fig. 1(b) for the purpose of comparison, and they are close besides u. The results obtained by the LG are $\lambda_1(t_0)$=0.293177, $\lambda_2(t_0)$=0.343054, and J=0.15044, which is not better than the new result J=0.14780793. According to the minimality of J, smaller one is better.
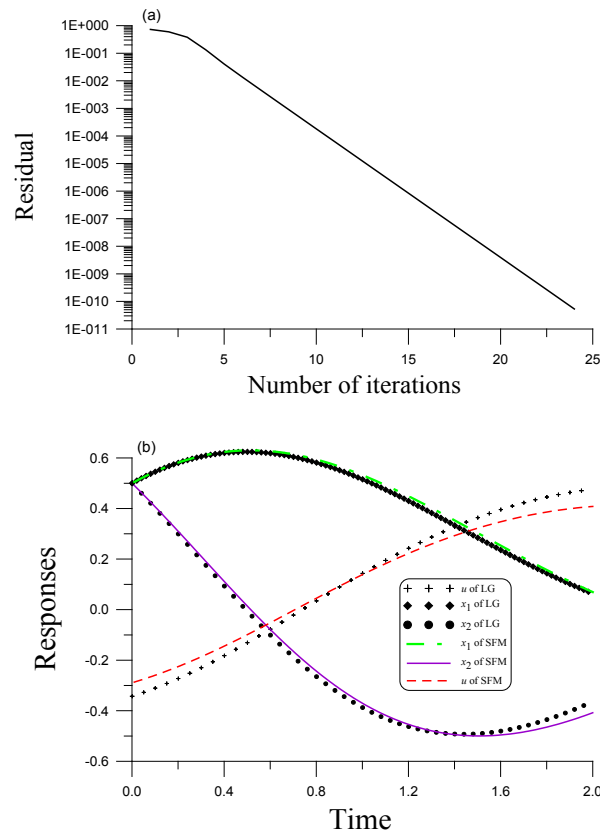

Figure 1: For the optimal control of an undamped Duffing oscillator in example 1, (a) convergence rate, and (b) comparing the solutions obtained by the present SFM and the Lie-group (LG) method

In order to investigate the robustness of the new algorithm SFM, we insert the following noisy data:

$\hat{A}_0 = A_0 + sR(i), \hat{B}_0 = B_0 + sR(i)$

as the inputs in the iteration process, where *s* is the level of noise and *R(i)* are random numbers in [-1,1]. Under a large noise *s*=0.1, and keeping other parameters unchanged, we find that the SFM does not converge within 500 iterations. However, the following results $\lambda_1(t_0)$=0.26862563, $\lambda_2(t_0)$=0.28515652 and *J*=0.15349061 are still close to $\lambda_1(t_0)$=0.27213155, $\lambda_2(t_0)$=0.28945859 and *J*=0.14780793 without considering noise. It confirms that the new algorithm SFM is stable and robust against large noise.

5.2 Example 2

Then, we solve the optimal control problem of a damped Duffing oscillator under a more complicated performance index:

$$J = \frac{1}{2}\int_{t_0}^{t_f}[x^2(t) + \dot{x}^2(t) + u^2(t)]\, dt, \tag{40}$$

which is subjected to the initial conditions with $x(t_0) = A_0 = 0.5$, $\dot{x}(t_0) = B_0 = -0.5$ and the end values $x(t_f)$ and $\dot{x}(t_f)$ are free.

In the Hamiltonian formulation, we can derive the following Euler-Lagrange equations:

$\dot{x}_1 = x_2, x_1(t_0) = A_0,$

$\dot{x}_2 = u - \gamma x_2 - (\alpha + \beta x_1^2)x_1, x_2(t_0) = B_0,$

$\dot{\lambda}_1 = (\alpha + 3\beta x_1^2)\lambda_2 - x_1, \lambda_1(t_f) = 0,$

$$\dot{\lambda}_2 = \gamma \lambda_2 - x_2 - \lambda_1, \lambda_2(t_f) = 0, \tag{41}$$

where $u = -\lambda_2$.

This problem is simple with *a*=0 and *b*=0, due to $\lambda_1(t_f) = 0$ and $\lambda_2(t_f) = 0$. Under the following parameters γ=0.02, α=1, β=0.15, t_0=0 and t_f=2, we apply the SFM with $s_{10} = -1$, $s_{10} = -0.2$, $N = 200$, $c_0 = d_0 = 0$, to solve this problem, which is convergent with 49 iterations as shown in Fig. 2(a), and the responses of x_1, x_2 and the control force $u = -\lambda_2$ are plotted in Fig. 2(b). The computed results by Eqs. (37), (38) and (40) are $\lambda_1(t_0)$=0.69182762, $\lambda_2(t_0)$=-0.423268357, and *J*=0.27674272. The results computed from the Lie-group approach [Tsai and Lee (2018)] were also plotted in Fig. 2(b), where $\lambda_1(t_0)$=0.605366, $\lambda_2(t_0)$=-0.4952084, and *J*=0.28393986. The differences of *u* and x_2 are apparent, and the new value *J*=0.27674272 is better than *J*=0.28393986.

5.3 Example 3

In this example we solve an optimal control problem of the undamped Duffing oscillator [Van Dooren and Vlassenbroeck (1982); El-Gindy, El-Hawary, Salim et al. (1995); El-Kady and Elbarbary (2002); Lakestani, Razzaghi and Dehghan (2006); Liu (2012)], where the optimal control problem for Eq. (11) is under the following performance index and boundary conditions:

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\, dt,$$

$$x(t_0) = A_0 = 0.5, \dot{x}(t_0) = B_0 = -0.5, x(t_f) = C_0 = 0, \dot{x}(t_f) = D_0 = 0. \quad (42)$$

We employ the second SFM algorithm based-on the result in Theorem 2 to solve this optimal orbit control problem, under the following parameters $\gamma=0$, $\alpha=1$, $\beta=0.15$, $t_0= -2$, $t_f =0$, $\lambda_1(t_0) =0.271$ and $\lambda_2(t_0) =-0.4858$, and with $N=200$, $a_0 = b_0 = 0$, which is convergent with one iteration under the convergence criterion $\epsilon = 10^{-2}$. The responses of x_1, x_2 and the control force $u = -\lambda_2$ are plotted in Fig. 3. The results computed from the Lie-group approach [Tsai and Lee (2018)] were also plotted in Fig. 3, and they are almost coincident. The new value $J=0.1858681$ is slightly better than $J=0.1858713$ obtained by the Lie-group approach [Tsai and Lee (2018)]. We note that the new value of J is slightly smaller than 0.1874, which was obtained by other methods [Van Dooren and Vlassenbroeck (1982); Razzaghi and Elnagar (1994); Lakestani, Razzaghi and Dehghan (2006)]. Note that the present method can achieve a better control strategy than other methods.
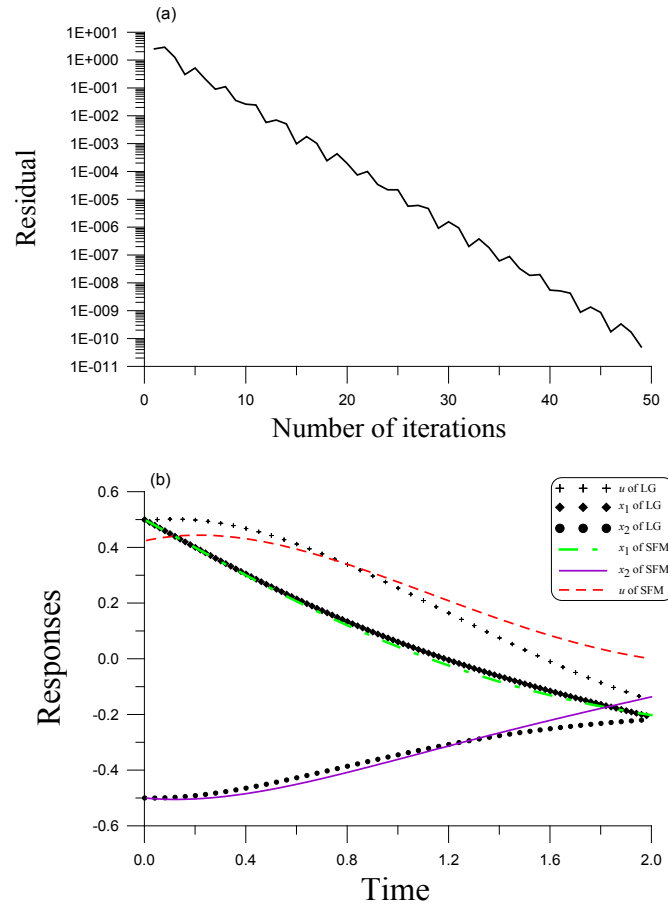

Figure 2: For the optimal control of a damped Duffing oscillator in example 2, (a) convergence rate, and (b) comparing the solutions obtained by the present SFM and the Lie-group (LG) method
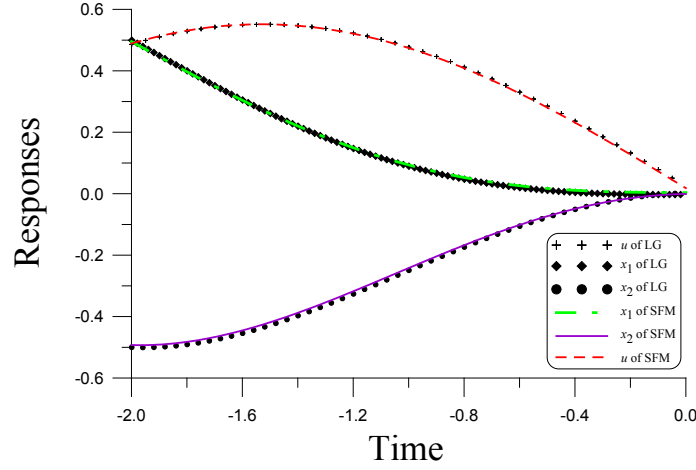
Figure 3: For the optimal control of the orbit of an undamped Duffing oscillator in example 3, comparing the solutions obtained by the present SFM and the Lie-group (LG) method

5.4 Example 4

We consider the following performance index for the undamped Duffing oscillator [Elgohary, Dong, Junkins et al. (2014)]:

$$J = \frac{1}{2} \left\| \mathbf{x}(t_f) - \mathbf{q} \right\|^2 + \frac{1}{2} \int_{t_0}^{t_f} u^2(t) \, dt, \tag{43}$$

where $\mathbf{q}$ is the desired final state at a specified final time. Here we fix $t_0=0$, $t_f=2$, $x(t_0) = \dot{x}(t_0) = 0.1$ and $q_1 = q_2 = 1$.

From Eqs. (6) and (43) we have $\boldsymbol{\lambda}^f = \mathbf{x}^f - \mathbf{q}$, which yields

$$\lambda_1(t_f) = x_1^f - q_1 = a - q_1, \lambda_2(t_f) = x_2^f - q_2 = b - q_2. \tag{44}$$

Under the following parameters $\gamma=0$, $\alpha=1$, $\beta=0.9$, $t_0=0$ and $t_f=2$, we apply the SFM with $s_{10}=-1$, $s_{20}=-0.2$, $N=200$, $a_0 = b_0 = c_0 = d_0 = 0$, to solve this problem, which is convergent with 62 iterations as shown in Fig. 4(a), and the responses of x_1, x_2 and the control force $u = -\lambda_2$ are plotted in Fig. 4(b). The computed results by Eqs. (37), (38) and (43) are $\lambda_1(t_0) =0.703491$, $\lambda_2(t_0) =0.085527$, and $J=0.49385273$. The results computed from the Lie-group approach [Tsai and Lee (2018)] were also plotted in Fig. 4(b) for the purpose of comparison, and in addition to x_1, they are not close. However, the results obtained by the LG are $\lambda_1(t_0)=0.762934$, $\lambda_2(t_0)=0.085703$, and $J=0.5191889$, which is not better than the new result $J=0.49385273$.
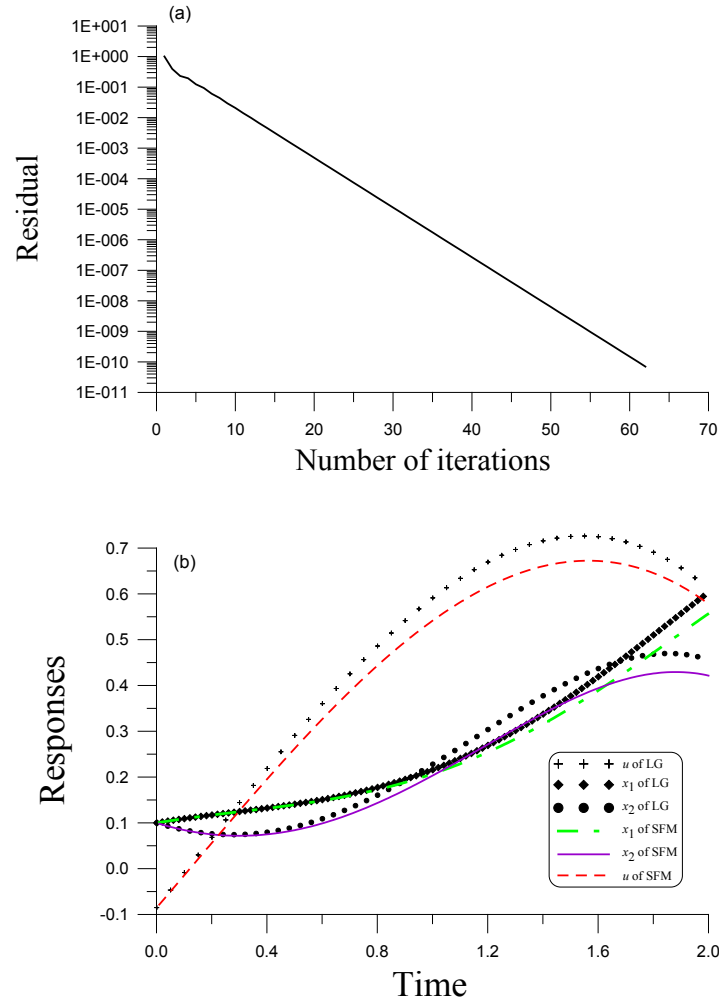
Figure 4: For the optimal control of undamped Duffing oscillator in example 4, (a) convergence rate, and (b) comparing the solutions obtained by the present SFM and the Lie-group (LG) method

5.5 Example 5

Finally, we consider an optimal control problem of a damped Duffing oscillator under a complicated performance index:

$$J = \frac{1}{2} \int_{t_0}^{t_f} \left[x^2(t) + \dot{x}^2(t) + \exp\left(u^2(t)\right) \right] dt, \tag{45}$$

which is subjected to the initial conditions $x(t_0) = A_0 = 0.5$, $\dot{x}(t_0) = B_0 = -0.5$ and the end values $x(t_f)$ and $\dot{x}(t_f)$ are free.

We can derive

$$\dot{x}_1 = x_2, x_1(t_0) = A_0,$$

$$\dot{x}_2 = u - \gamma x_2 - (\alpha + \beta x_1^2) x_1, x_2(t_0) = B_0,$$

$$\dot{\lambda}_1 = (\alpha + 3\beta x_1^2)\lambda_2 - x_1, \lambda_1(t_f) = 0,$$
$$\dot{\lambda}_2 = \gamma\lambda_2 - x_2 - \lambda_1, \lambda_2(t_f) = 0, \tag{46}$$

where u is solved from

$$\frac{\partial H}{\partial u} = u\exp(u^2) + \lambda_2 = 0. \tag{47}$$

It is difficult to express u as a function of λ_2. However, by using Eq. (47), we can obtain another ODEs system:

$$\dot{x}_1 = x_2, x_1(t_0) = A_0,$$
$$\dot{x}_2 = u - \gamma x_2 - (\alpha + \beta x_1^2)x_1, x_2(t_0) = B_0,$$
$$\dot{\lambda}_1 = -(\alpha + 3\beta x_1^2)u\exp(u^2) - x_1, \lambda_1(t_f) = 0,$$
$$\dot{u} = \frac{1}{\exp(u^2) + 2u^2\exp(u^2)}[\gamma u\exp(u^2) + x_2 + \lambda_1], u(t_f) = 0. \tag{48}$$

Instead of λ_2, we directly solve u by integrating the above ODEs.
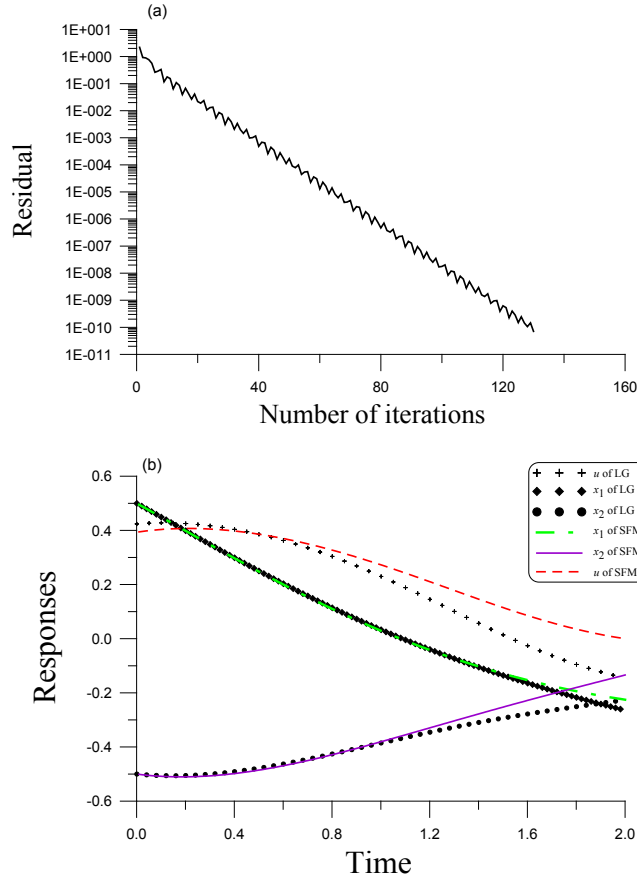

Figure 5: For the optimal control of a damped Duffing oscillator with a complicated performance index in example 5, (a) convergence rate, and (b) comparing the solutions obtained by the present SFM and the Lie-group (LG) method

Under the following parameters $\gamma=0.02$, $\alpha=1$, $\beta=0.15$, $t_0=0$ and $t_f=2$, we apply the SFM with $s_{10}=-1$, $s_{20}=-0.2$, $N=200$, $c_0=d_0=0$, to solve this problem, which is convergent with 130 iterations as shown in Fig. 5(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 5(b). The computed results are $\lambda_1(t_0)=0.7044912$, $\lambda_2(t_0)=-0.4586696$, and $J=1.282944$. The results computed from the Lie-group approach [Tsai and Lee (2018)] were also plotted in Fig. 5(b) for the purpose of comparison, and they are different in u and x_2, but close in x_1. The results computed by the LG are $\lambda_1(t_0)=0.610798$, $\lambda_2(t_0)=-0.506856$, and $J=1.29008$. The new $J=1.282944$ is better. When $\gamma=0$ we can obtain $J=1.28782$, which is better than $J=1.466$ obtained by Liu [Liu (2012)], and $J=1.29299$ obtained by Tsai et al. [Tsai and Lee (2018)].

Under a large noise $s=0.1$, and keeping other parameters unchanged, although we find that the SFM does not converge within 500 iterations, the following results $\lambda_1(t_0)=0.66021372$, $\lambda_2(t_0)=-0.4997156$ and $J=1.288407$ are close to $\lambda_1(t_0)=0.7044912$, $\lambda_2(t_0)=-0.4586696$ and $J=1.282944$ without considering noise. In Fig. 6, we compare the computed results of u, x_1 and x_2 by using the SFM with and without considering noise; even under a large noise $s=0.1$, they are close. They confirm again that the new algorithm SFM is stable and robust against large noise.
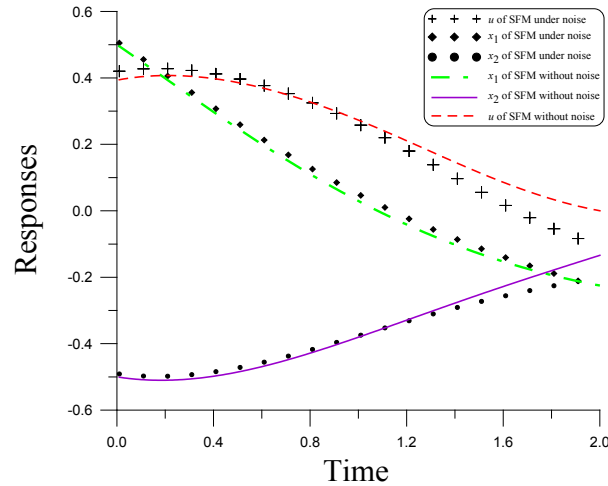

Figure 6: For example 5, comparing the solutions obtained by the SFM with and without considering noise

6 Conclusions

For the optimally controlled problems of nonlinear Duffing oscillators to find the optimal control forces with different performance indexes, we have transformed the Euler-Lagrange equations into a two-point boundary value problem equipped with constraints. The paper is witnessed to derive the shape functions to exactly satisfy the given boundary conditions. The major contribution is the introduction of a new concept of shape functions method and then derive the new initial value problems for the new variables, which automatically and exactly satisfy all the specified boundary conditions. The initial values of the new variables can be given arbitrarily, for example the zero values, while

the terminal values are determined iteratively. The present method can handle the minimization problem with a complicated performance index, where the control force can be solved very fast with high accuracy. Numerical examples demonstrated that the new method can obtain a smaller value of the performance index than other methods, including the Lie-group method and the Lie-group adaptive method.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

Agrawal, A. K.; Yang, J. N.; Wu, J. C. (1998): Non-linear control strategies for Duffing systems. *International Journal of Non-Linear Mechanics*, vol. 33, pp. 829-841.

Cvetićanin, L. (2013): Ninety years of Duffing's equation. *Theoretical and Applied Mechanics*, vol. 2013, pp. 49-63.

El-Gindy, T. M.; El-Hawary, H. M.; Salim, M. S.; El-Kady, M. (1995): A Chebyshev approximation for solving optimal control problems. *Computers & Mathematics with Applications*, vol. 29, pp. 35-45.

El-Kady, M.; Elbarbary, E. M. E. (2002): A Chebyshev expansion method for solving nonlinear optimal control problems. *Applied Mathematics and Computation*, vol. 129, pp. 171-182.

Elgohary, T. A.; Dong, L.; Junkins, J. L.; Atluri, S. N. (2014): Time domain inverse problems in nonlinear systems using collocation & radial basis functions. *Computer Modeling in Engineering and Sciences*, vol. 100, pp. 59-84.

Garg, D.; Patterson, M.; Hager, W. W.; Rao, A. V.; Benson, D. A. et al. (2010): A unified framework for the numerical solution of optimal control problems using pseudospectral methods. *Automatica*, vol. 46, pp. 1843-1851.

Hu, N.; Wen, X. (2003): The application of duffing oscillator in characteristic signal detection of early fault. *Journal of Sound and Vibration*, vol. 268, pp. 917-931.

Lakestani, M.; Razzaghi, M.; Dehghan, M. (2006): Numerical solution of the controlled Duffing oscillator by semi-orthogonal spline wavelets. *Physica Scripta*, vol. 74, pp. 362-366.

Liu, C. S. (2012): The optimal control problem of nonlinear Duffing oscillator solved by the Lie-group adaptive method. *Computer Modeling in Engineering and Sciences*, vol. 86, pp. 171-197.

Liu, C. S. (2014): A new sliding control strategy for nonlinear system solved by the Lie-group differential algebraic equation method. *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, pp. 2012-2038.

Rad, J.; Kazem, S.; Parand, K. (2012): A numerical solution of the nonlinear controlled Duffing oscillator by radial basis functions. *Computers & Mathematics with Applications*, vol. 64, pp. 2049-2065.

Razzaghi, M.; Elnagar, G. (1994): Numerical solution of the controlled Duffing oscillator by the pseudospectral method. *Journal Computational and Applied*

Mathematics, vol. 56, pp. 253-261.

Suhardjo, J.; Spencer, B. F. J.; Sain, M. K. (1992): Non-linear optimal control of a Duffing system. *International Journal of Non-Linear Mechanics*, vol. 27, pp. 157-172.

Tsai, C. L.; Lee, H. C. (2018): Optimal control of nonlinear duffing oscillators: lie-group approaches. *Journal of Marine Science and Technology*, vol. 26, pp. 285-296.

van Dooren, R.; Vlassenbroeck, J. (1982): Chebyshev series solution of the controlled Duffing oscillator. *Journal of Computational Physics*, vol. 47, pp. 321-329.