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# Numerical Solving of a Boundary Value Problem for Fuzzy Differential Equations

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**Abstract:** In this work we solve numerically a boundary value problem for second order fuzzy differential equations under generalized differentiability in the form $y''(t) = p(t)y'(t) + q(t)y(t) + F(t)\ y(0) = \gamma,\ y(\ell) = \lambda$ where $t \in T = [0, \ell]$, $p(t) \geq 0$, $q(t) \geq 0$ are continuous functions on $[0, \ell]$ and $[\gamma]^\alpha = [\underline{\gamma}_\alpha, \overline{\gamma}_\alpha]$, $[\lambda]^\alpha = [\underline{\lambda}_\alpha, \overline{\lambda}_\alpha]$ are fuzzy numbers. There are four different solutions of the problem (0.1) when the fuzzy derivative is considered as generalization of the H-derivative. An algorithm is presented and the finite difference method is used for solving obtained problems. The applicability of presented algorithm is illustrated by solving an examples of boundary value problems for second order fuzzy differential equations.

**Keywords:** Boundary value problem, Second order fuzzy differential equations, Generalized differentiability, Finite difference method

## 1 Introduction

If a process modeled by ordinary differential equations has the input data with some uncertainties, then it is naturally modeled by a fuzzy differential equations (FDEs). In general, two different approaches are used for solving the fuzzy differential equations. In the first approach, it is assumed that the boundary or/and initial conditions are fuzzy, one seeks the solution by applying Zadeh's extension principle [Zadeh (1975)] to the solution of crisp problem (Solution via the Extension Principle [Buckley and Feuring (2000); Jowers, Buckley and Reilly (2007); Misukoshi, Chalco-Cano, Román-Flores and Bassanezi (2007)]) or the problem is solved by writing in the form of a family of differential inclusion (Solution via differential inclusion [Hüllermeier (1997); Chen, Fu, Xue and Wu (2008); Chen, Li and Xue (2011); Diamond (1999); Diamond (2000); Diamond (2002); Diamond

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and Watson (2000); O'Regan, Lakshmikantham and Nieto (2003)]). In the second approach, in addition to fuzzy boundary and initial conditions it is assumed that the derivatives in the equation are generalized in the H-derivative form or in strongly generalized H-derivative form.

The H-derivative of a fuzzy function was introduced in [Puri and Ralescu (1983)]. The existence and uniqueness of the solution of a FDEs were studied in [Buckley and Feuring (2000); Kaleva (1987)] under this setting. However, this approach has the drawback that it leads to solution which have increasing length of their support [Diamond and Kloeden (1994); Stefanini and Bede (2009)]. In [Bede (2006)] authors have demonstrated that for this reason a large class of boundary value problems have not a solution under this approach. To resolve these difficulties authors in [Bede and Gal (2004)] introduced the concept of generalized differentiability. FDEs have been investigated using this concept in [Stefanini and Bede (2009); Hüllermeier (1997); Kaleva (1987); Kaleva (2006); Khastan, Bahrami and Ivaz (2009); Lan and Nieto (2009); Nieto, Khastan and Ivaz (2009); Bede, Rudas and Bencsik (2007) ]. First order FDEs under strongly generalized derivatives are considered in [Bede, Rudas and Bencsik (2007)]. In [Nieto, Khastan and Ivaz (2009)] a linear fuzzy nuclear decay equation under generalized differentiability is studied and numerical solutions are found. In [Ma, Friedman and Kandel (1999)] the Euler method was applied for solving initial value problem for FDEs. The authors in [Abbasbandy and Allahviranloo (2004); Palligkinis, Papageorgiou and Famelis (2009)] develop four-stage order Runge-Kutta methods for FDEs. Numerical methods such as Adams and Nystörm methods and predictor-corrector methods for solving FDEs presented in [Allahviranloo, Ahmadi and Ahmadi (2007); Friedman, Ma and Kandel (1999); Khastan and Ivaz (2009)]. Numerical method for a boundary value problem for a linear second order FDEs was considered in [Allahviranloo and Khalilpour (2011)]. In [Khastan and Nieto (2010)] a boundary value problem for FDEs by using a generalized differentiability was considered and a new concept of solutions was presented. In this paper, we propose a numerical algorithm for finding such of solutions for boundary value problem for second order FDEs. The paper is organized as follow. In section 2, we present the basic definition and useful theoretical information. Boundary value problem for second-order FDEs under generalized differentiability, we study in section 3. Numerical algorithm for solving considered problem is introduced in section 4 and in section 5 we present some examples of numerical solutions to illustrate our method.

## 2 Basic Concepts

**Definition 2.1** *A fuzzy subset of $R$ is defined in terms of membership functions $u : R \to [0,1]$ which assigns to each $x \in R$ a grade of membership in the fuzzy* *set. Such a membership function is used to denote the corresponding fuzzy set. Denote by $\mathscr{F}$ the set of all fuzzy sets of R, and by E the class of fuzzy sets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:*

*1) u is normal, that is there exists an $x_0 \in R$ such that $u(x_0) = 1$.*

*2) u is fuzzy convex, that is for $x, y \in R$ and $0 < \lambda \leq 1$:*

$$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}. \tag{2.1}$$

*3) u is upper semi-continuous on R.*

*4) the closure of $\{x \in R | u(x) > 0\}$ is compact.*

*E is called the space of fuzzy numbers.*

**Definition 2.2** *For each $\alpha \in (0,1]$ the $\alpha$-level set $[u]^\alpha$ of a fuzzy set u is the subset of points $x \in R$ with*

$$[u]^\alpha = \{x \in R : u(x) \geq \alpha\}. \tag{2.2}$$

*The support $[u]^0$ of a fuzzy set is defined as the closure of the union of all its level sets, that is $[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha}$. It is clear that $\alpha$-level set of u is an $[\underline{u}^\alpha, \overline{u}^\alpha]$, where $\underline{u}$ and $\overline{u}$ are called lower and upper branches of u respectively. For $u \in E$ we define the length of u as*

$$len(u) = \sup_\alpha (\overline{u}^\alpha - \underline{u}^\alpha) \tag{2.3}$$

**Definition 2.3** *A fuzzy number in parametric form is presented by an ordered pair of functions $(\underline{u}_\alpha, \overline{u}_\alpha)$, $0 \leq \alpha \leq 1$, satisfying the following properties:*

*1) $\underline{u}_\alpha$ is a bounded nondecreasing left-continuous function of $\alpha$ over $(0,1]$ and right continuous for $\alpha = 0$.*

*2) $\overline{u}_\alpha$ is a bounded nonincreasing left-continuous function on $(0,1]$ and right continuous for $\alpha = 0$.*

*3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.*

**Definition 2.4** *The metric on E is defined by the equation*

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha),$$

*where $d_H([u]^\alpha, [v]^\alpha) = max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$ is a Hausdorff distance of two interval $[u]^\alpha$ and $[v]^\alpha$.*

**Definition 2.5** *(Triangular fuzzy number) If u is symmetric number with support* $[\underline{u},\overline{u}]$*, such that the* $\alpha$ *level set of* $[u]^\alpha$ *is* $[u]^\alpha = [\underline{u} + (\frac{\overline{u}-\underline{u}}{2})\alpha, \overline{u} + (\frac{\overline{u}-\underline{u}}{2})\alpha]$*, then u is called as triangular fuzzy number.*

**Definition 2.6** *Let u and v be two fuzzy sets. If there exists a fuzzy set w such that* $u = v + w$*, then w is called the H-difference of u and v and denoted by* $u \ominus v$*.*

**Definition 2.7** *(H differentiability or Hukuhara differentiability) Let* $I = (0,l)$ *and* $f : I \to \mathscr{F}$ *is a fuzzy function. We say that f is differentiable at* $t_0 \in I$ *if there exists an element* $f'(t_0) \in \mathscr{F}$ *such that the limits*

$$\lim_{h\to 0^+} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\ominus f(t_0-h)}{h} \tag{2.4}$$

*exists and are equal to* $f'(t_0)$*. Here the limits are taken in the metric space* $(\mathscr{F}, D)$*.*

It is obviously that Hukuhara differentiable function has increasing length of support. If the function doesn't has this properties then this function is not $H$-differentiable. To avoid this difficulty the authors in [Bede and Gal (2004)] introduced a more general definition of derivative for fuzzy number valued function in the following form:

**Definition 2.8** *Let* $f : I \to \mathscr{F}$ *and* $t_0 \in I$*. We say that*
*1) f is (1)-differentiable at* $t_0$*, if there exists an element* $f'(t_0) \in \mathscr{F}$ *such that for all* $h > 0$ *sufficiently near to* $0$*, there exist* $f(t_0 + h \ominus f(t_0)$*,* $f(t_0) \ominus f(t_0 - h)$*, and the limits*

$$\lim_{h\to 0^+} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\ominus f(t_0-h)}{h} \tag{2.5}$$

*exist and are equal to* $f'(t_0)$ *at* $t_0$*.*
*2) f is (2)-differentiable at* $t_0$*, if there exists an element* $f'(t_0) \in \mathscr{F}$ *such that for all* $h < 0$ *sufficiently near to* $0$*, there exist* $f(t_0+h) \ominus f(t_0)$*,* $f(t_0) \ominus f(t_0-h)$ *and the limits*

$$\lim_{h\to 0^-} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0)\ominus f(t_0-h)}{h} \tag{2.6}$$

*exist and are equal to* $f'(t_0)$*.*

We denote by $D_n^1 f(t_0)$ the first derivatives of $f$, if it is $(n)$-differentiable at $t_0$. $(n = 1,2)$. In [Chalco-Cano and Román-Flores (2008)] Chalco-Cano and Román-Flores for fuzzy-value functions got the following results.

**Theorem 2.9** *Let $f : I \to \mathscr{F}$ be fuzzy function, where $[f(t)]^\alpha = [\underline{f}_\alpha, \overline{f}_\alpha]$, for each $\alpha \in [0,1]$. Then,*

*1) If $f$ is (1) differentiable in the first form, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[D_1^1 f(t)]^\alpha = [\underline{f}'_\alpha, \overline{f}'_\alpha]$.*

*2) If $f$ is (2)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable and $[D_2^1 f(t)]^\alpha = [\overline{f}'_\alpha, \underline{f}'_\alpha]$.*

Now let fuzzy function $f$ is (1) or (2) differentiable, then the first derivative $D_1^1$ for $D_2^1 f$ might be $(n)$-differentiable $(n = 1,2)$ and there are four possibilities $D_1^1(D_1^1 f(t))$, $D_2^1(D_1^1 f(t))$, $D_1^1(D_2^1 f(t))$ and $D_2^1(D_2^1 f(t))$. The second derivatives $D_n^1(D_m^1 f(t))$ are denoted by $D_{n,m}^2 f(t)$ for $n,m = 1,2$. Similar to Theorem 2.9, in [Khastan, Bahrami and Ivaz (2009)] authors get following results for the second derivatives.

**Theorem 2.10** *([Khastan, Bahrami and Ivaz (2009)]) Let $D_1^1 f : I \to \mathscr{F}$ or $D_2^1 f : I \to \mathscr{F}$ be fuzzy functions, where $[f(t)]^\alpha = [\underline{f}^\alpha(t), \overline{f}^\alpha(t)]$ for $\forall \alpha \in [0,1]$. Then,*

*1) If $D_1^1 f$ is (1) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{1,1}^2 f(t)]^\alpha = [\underline{f}''_\alpha, \overline{f}''_\alpha]$.*

*2) If $D_1^1 f$ is (2) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{1,2}^2 f(t)]^\alpha = [\overline{f}''_\alpha, \underline{f}''_\alpha]$.*

*3) If $D_2^1 f$ is (1) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{2,1}^2 f(t)]^\alpha = [\overline{f}''_\alpha, \underline{f}''_\alpha]$.*

*4) If $D_2^1 f$ is (2) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{2,2}^2 f(t)]^\alpha = [\underline{f}''_\alpha, \overline{f}''_\alpha]$.*

**Proof.** (see [Khastan, Bahrami and Ivaz (2009)]) ■

## 3 Boundary Value Problem for Second-Order Fuzzy Differential Equations

Consider fuzzy boundary value problem for a second-order fuzzy differential equations

$$y''(t) = p(t)y'(t) + q(t)y(t) + F(t) \tag{3.1}$$

$$y(0) = \gamma, y(\ell) = \lambda \tag{3.2}$$

where $\gamma$, $\lambda \in \mathscr{F}$ and $F : [0,\ell] \to \mathscr{F}$ is a fuzzy function. According to [Khastan and Nieto (2010)], we define the concept of solution of this problem as:

**Definition 3.1** *Let $y : [0,\ell] \to \mathscr{F}$ be a fuzzy function and $n,m \in \{1,2\}$, we say that $y$ is a $(n,m)$-solution for problem (3.1)-(3.2) on $[0,\ell]$, if $D_n^1 y$, $D_{n,m}^2 y$ exist on $[0,\ell]$, $D_{n,m}^2 y(t) = p(t)D_n^1 y(t) + q(t)y(t) + F(t)$ and $y(0) = \gamma$, $y(\ell) = \lambda$.*

**Definition 3.2** *Let $y : [0,\ell] \to \mathscr{F}$ be a fuzzy function and $n,m \in \{1,2\}$, we say that $y$ is a $(n,m)$-solution for problem (3.1) on an interval $J \subset [0,\ell]$, if $D_n^1 y$, $D_{n,m}^2 y$ exist on $[0,\ell]$, $D_{n,m}^2 y(t) = p(t)D_n^1 y(t) + q(t)y(t) + F(t)$ on $J$.*

Let $y$ be an $(n,m)$ solution for (3.1)-(3.2). To find it take into account the Theorem 2.9, 2.10, we can reformulate problem (3.1)-(3.2) as a system of boundary value problems that we call corresponding $(n,m)$-system for problem (3.1)-(3.2). Four boundary value problems system are possible for problem (3.1)-(3.2), as follows [Khastan, Bahrami and Ivaz (2009)].
$(1,1)$-system

$$\begin{cases} \underline{y}''(t;\alpha) = p(t)\underline{y}'(t;\alpha) + q(t)\underline{y}(t;\alpha) + \underline{F}(t;\alpha) \\ \overline{y}''(t;\alpha) = p(t)\overline{y}'(t;\alpha) + q(t)\overline{y}(t;\alpha) + \overline{F}(t;\alpha) \\ \underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ \underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{cases} \tag{3.3}$$

$(1,2)$-system

$$\begin{cases} \overline{y}''(t;\alpha) = p(t)\underline{y}'(t;\alpha) + q(t)\underline{y}(t;\alpha) + \overline{F}(t;\alpha) \\ \underline{y}''(t;\alpha) = p(t)\overline{y}'(t;\alpha) + q(t)\overline{y}(t;\alpha) + \underline{F}(t;\alpha) \\ \underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ \underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{cases} \tag{3.4}$$

$(2,1)$-system

$$\begin{cases} \overline{y}''(t;\alpha) = p(t)\overline{y}'(t;\alpha) + q(t)\underline{y}(t;\alpha) + \overline{F}(t;\alpha) \\ \underline{y}''(t;\alpha) = p(t)\underline{y}'(t;\alpha) + q(t)\overline{y}(t;\alpha) + \underline{F}(t;\alpha) \\ \underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ \underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{cases} \tag{3.5}$$

$(2,2)$-system

$$\begin{cases} \underline{y}''(t;\alpha) = p(t)\overline{y}'(t;\alpha) + q(t)\underline{y}(t;\alpha) + \underline{F}(t;\alpha) \\ \overline{y}''(t;\alpha) = p(t)\underline{y}'(t;\alpha) + q(t)\overline{y}(t;\alpha) + \overline{F}(t;\alpha) \\ \underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ \underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{cases} \tag{3.6}$$

Summarize, it is seen that $(m,n)$-system can be written as follows:
$(m,n)$-system

$$\begin{cases} \overline{y}''_\alpha(t) = |3-(m+n)|q(t)\overline{y}(t;\alpha) + |2-(m+n)||4- \\ (m+n)|q(t)\underline{y}(t;\alpha) + |2-n|p(t)\overline{y}'(t;\alpha) + |1-n|p(t)\underline{y}'(t;\alpha) + \overline{F}(t;\alpha) \\ \\ \underline{y}''_\alpha(t) = |3-(m+n)|q(t)\underline{y}(t;\alpha) + |2-(m+n)||4- \\ (m+n)|q(t)\overline{y}(t;\alpha) + |2-n|p(t)\underline{y}'(t;\alpha) + |1-n|p(t)\overline{y}'(t;\alpha) + \underline{F}(t;\alpha) \\ \\ \underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ \\ \underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{cases} \tag{3.7}$$

Now problem is to solve the system (3.7) for $\forall(m,n)$, $m,n \in \{1,2\}$. We first choose the pair $(m_0,n_0)$, then we solve the problem (3.7) for this pair, after that we find such a domain in which the solution and its derivatives have valid level sets according to the type of differentiability. For example, for finding $(2,1)$ solution, we solve system (3.7) for $(m,n) = (2,1)$ and then we look for a domain where the solution is $(2,1)$-differentiable. (3.7) is solved numerically, the procedure that we present in the next section.

## 4 Numerical Solution.

Denote the right hand side in the equations in (3.7) by $G$ we can rewrite (3.7) as follows:

$$\begin{aligned} &\underline{y}''(t;\alpha) = G(t,\overline{y}(t;\alpha),\underline{y}(t;\alpha),\overline{y}'(t;\alpha),\underline{y}'_\alpha(t;\alpha)) \\ &\overline{y}''(t;\alpha) = G(t,\underline{y}(t;\alpha),\overline{y}(t;\alpha),\underline{y}'(t;\alpha),\overline{y}'(t;\alpha)) \\ &\underline{y}(0;\alpha) = \underline{\gamma}_\alpha,\ \overline{y}(0;\alpha) = \overline{\gamma}_\alpha \\ &\underline{y}(\ell;\alpha) = \underline{\lambda}_\alpha,\ \overline{y}(\ell;\alpha) = \overline{\lambda}_\alpha \end{aligned} \tag{4.1}$$

Let we write again eq.(4.1) as taking $x = \underline{y}(t;\alpha)$, $z = \overline{y}(t;\alpha)$ :

$$\begin{aligned} &x'' = G(t,z,x,z',x') \\ &z'' = G(t,x,z,x',z') \\ &x(0) = \underline{\gamma}_\alpha,\ z(0) = \overline{\gamma}_\alpha \end{aligned} \tag{4.2}$$

$x(\ell) = \underline{\lambda}_\alpha,\ z(\ell) = \overline{\lambda}_\alpha$

We solve this system numerically by applying the finite-difference method. Let $t_0 = 0$, $t_n = 1$, $t_i = ih$ $(i = 1, 2, ..., n-1)$ be a system of equally spaced grid points with $h = 1/n$ and $x_i = x(t_i)$, $z_i = z(t_i)$. The finite difference approximation of (4.2) can be written as follows:

$$
\begin{aligned}
&\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} = G(t_i, z_i, x_i, \frac{z_{i+1} - z_{i-1}}{2h}, \frac{x_{i+1} - x_{i-1}}{2h}) \\
&\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} = G(t_i, x_i, z_i, \frac{x_{i+1} - x_{i-1}}{2h}, \frac{z_{i+1} - z_{i-1}}{2h}) \\
&z_0 = \overline{\gamma}_\alpha, z_n = \overline{\lambda}_\alpha \\
&x_0 = \underline{\gamma}_\alpha, x_n = \underline{\lambda}_\alpha
\end{aligned}
\tag{4.3}
$$

This system can be solved by applying the following iteration schema

$$
\begin{aligned}
&\frac{x_{i+1}^{s+1} - 2x_i^{s+1} + x_{i-1}^{s+1}}{h^2} = G(t_i, z_i^s, x_i^{s+1}, \frac{z_{i+1}^s - z_{i-1}^s}{2h}, \frac{x_{i+1}^{s+1} - x_{i-1}^{s+1}}{2h}) \\
&\frac{z_{i+1}^{s+1} - 2z_i^{s+1} + z_{i-1}^{s+1}}{h^2} = G(t_i, x_i^{s+1}, z_i^{s+1}, \frac{x_{i+1}^{s+1} - x_{i-1}^{s+1}}{2h}, \frac{z_{i+1}^{s+1} - z_{i-1}^{s+1}}{2h}) \\
&z_0^{s+1} = \overline{\gamma}_\alpha, z_n^{s+1} = \overline{\lambda}_\alpha \\
&x_0^{s+1} = \underline{\gamma}_\alpha, x_n^{s+1} = \underline{\lambda}_\alpha
\end{aligned}
\tag{4.4}
$$

For the initial iteration values of $x$ and $z$, the linear functions connected the left and right boundary values of this functions are taken. Solution of system (4.4) at each iteration step is obtained by using TDMA (Three Diagonal Matrix Algorithm) method. The algorithm of presented method for problem (4.2) is as follows.

### *4.1 Algorithm.*

Step 1. Set the value of $(m_0, n_0)$

Step 2. Enter problem data: Value of accuracy, $\varepsilon$; set iteration counter $s = 0$.

Step 3. Form the initial conditions: $x^0$, $z^0$

Step 4. Form the boundary conditions: $x^{s+1}(0)$, $x^{s+1}(l)$, $z^{s+1}(0)$, $z^{s+1}(l)$.

Step 5. Solve the first equation of (4.4) CALL FDM (Finite Difference Method subprogram)

Step 6. Solve the second equation of (4.4) CALL FDM.

Step 7. Test convergence:

if $\|x^{s+1}-x^s\|<\varepsilon$ AND $\|z^{s+1}-z^s\|<\varepsilon$, then go to step 8

else $s=s+1$; go to 4.

Step 8. Find the domain where the solution is valid.

Step 9. Find the valid domain where the solution is $(m_0,n_0)$ solution.

## 5 Numerical Solution Example

**Example 5.1** *Let's consider the following fuzzy boundary-value problem:*

$$\begin{cases} y''(t)=2y(t)+y'(t)+F(t) \\ y(0)=\gamma,\ y(1)=\lambda \end{cases} \tag{5.1}$$

*where* $\gamma^\alpha=\lambda^\alpha=[\alpha-1,1-\alpha]\frac{6}{25}$ *and*
$[F(t)]^\alpha=[2(\alpha-1)-\frac{2}{25}(25t^2-25t+6)(\alpha-1)-(2t-1)(\alpha-1);$
$2(1-\alpha)-\frac{2}{25}(25t^2-25t+6)(1-\alpha)-(2t-1)(1-\alpha)]$
*The graph of F is shown in Fig. 5.1 for* $\alpha=0$. *It is easy to see that exact* $(1,1)$ *solution for the problem (5.1) is*

$$y(t;\alpha)=[\frac{1}{25}(25t^2-25t+6)(\alpha-1),\frac{1}{25}(25t^2-25t+6)(1-\alpha)]$$

*In Fig.5.2 the results of numerical solution and exact solution are presented. It is seen that there is a uniformly good approximation to exact solution. We see* $\overline{y}(t;\alpha)$ *and* $\underline{y}(t;\alpha)$ *represent a valid fuzzy number when* $25t^2-25t+6\geq 0$, *that is for* $t\leq\frac{2}{5}$ *and* $t\geq\frac{3}{5}$. *For* $t\leq\frac{2}{5}$ *we have* $(2,2)$ *solution and for* $t\geq\frac{3}{5}$ *we have* $(1,1)$ *solution.*

*For* $(1,2)$ *solution, by solving* $(1,2)$ *system applying the presented numerical algorithm we get the results that illustrated in Fig.5.3. For* $t\leq 0.434$, *we have* $(1,2)$ *solution and for* $t\geq 0.444$, *we have* $(2,1)$ *solution. Since we solve* $(1,2)$ *system then the solution for this problem is on* $[0,0.434]$.

*For* $(2,1)$ *solution, by solving* $(2,1)$ *system we get the results presented in Fig.5.4. For* $t\geq 0.525$, *we have* $(2,1)$ *solution and* $t\leq 0.515$, *we have* $(1,2)$ *solution. Because of solving* $(2,1)$ *system then the solution for this problem is on* $[0.525,1]$.

## 6 Conclusion

In this paper, a numerical procedure for a boundary value problem for fuzzy differential equations is proposed. The proposed method was tested on a test example, and has been effective. This method can also be used to solve nonlinear problems

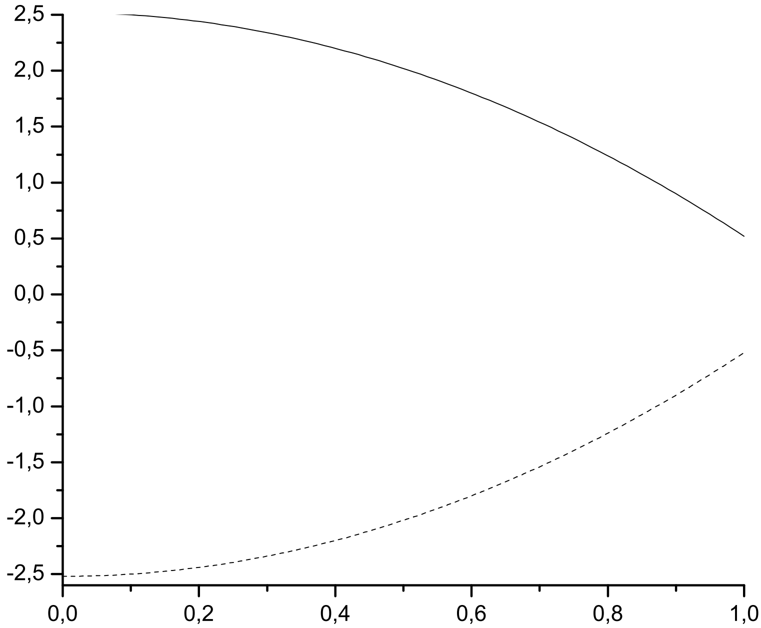


Figure 5.1: $\underline{F}(t;\alpha) = dash; \overline{F}(t;\alpha) = solid$

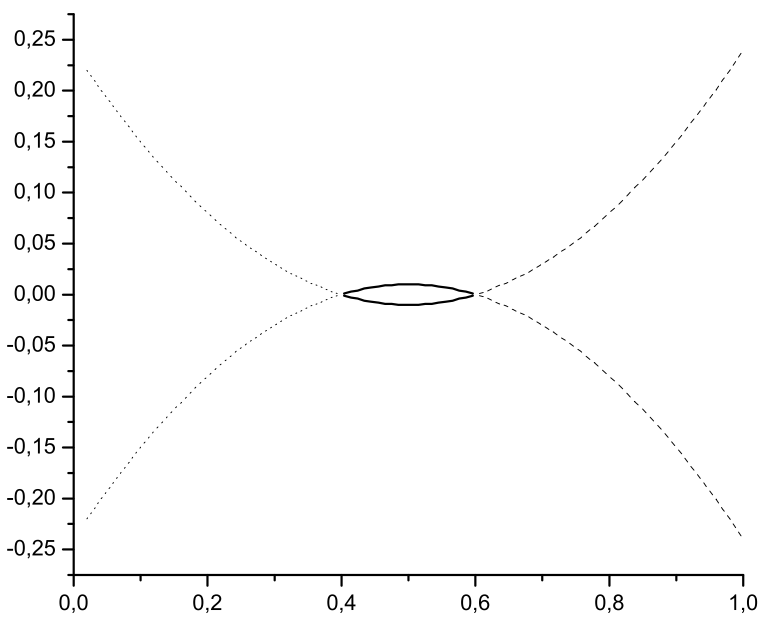


Figure 5.2: The graph of $(1,1)$-solution and $(2,2)$-solution: $(1,1)$-solution (dash), $(2,2)$-solution (dot), unvalid part (solid).

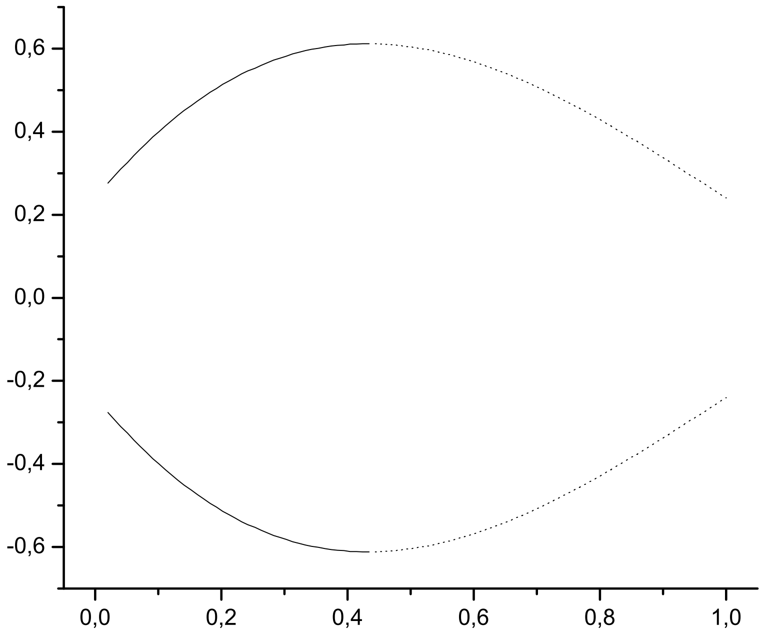


Figure 5.3: The graph of $(1,2)$-solution and $(2,1)$-solution: $(1,2)$-solution (solid), $(2,1)$-solution (dot).

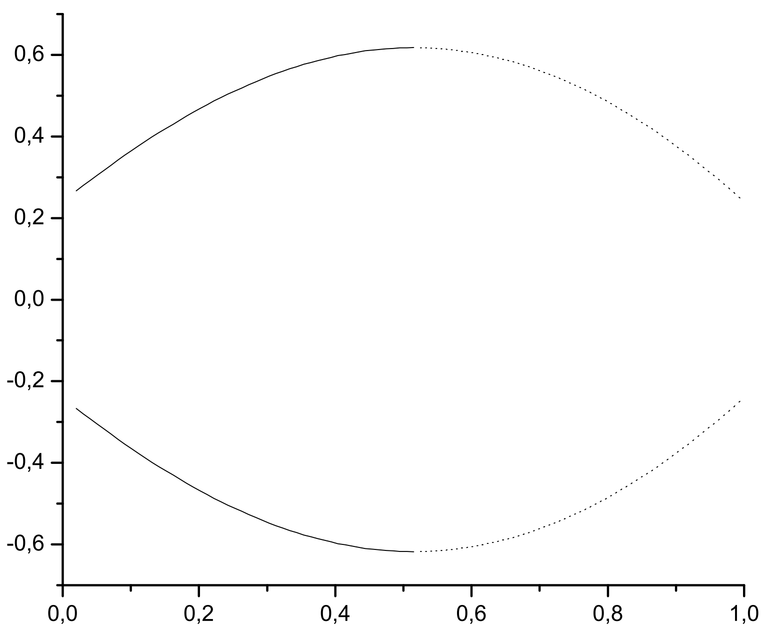


Figure 5.4: The graph of $(2,1)$-solution and $(1,2)$-solution: $(2,1)$-solution (dot), $(1,2)$-solution (solid).

with known results on the existence and uniqueness of solutions. This will be the subject of our future work.

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