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On Axisymmetric Longitudinal Wave Propagation in Double-Walled Carbon Nanotubes

S.D. Akbarov[1,2]

Abstract: An attempt is made into the investigation of longitudinal axisymmetric wave propagation in the DWCNT with the use of the exact equations of motion of the linear theory of elastodynamics. The DWCNT is modeled as concentrically-nested two circular hollow cylinders between which there is free space. The difference in the radial displacements of these cylinders is coupled with the van der Waals forces and it is assumed that full slipping conditions occur on the inner surface of the outer tube and on the outer surface of the inner tube. Numerical results on the influence of the problem parameters such as the thickness/radius ratio, the distance between the tubes of the DWCNT and the van der Waals forces on the dispersion curves are presented and discussed. In particular, it is established that new types of modes arise under propagation of axisymmetric longitudinal waves as a result of the van der Waals interaction between the tubes of the DWCNT. The limit values of the wave propagation velocity are also analyzed.

Keywords: axisymmetric longitudinal waves, double-walled CNT, wave dispersion, van der Waals forces

1 Introduction

During the last 20 years, after the discovery of carbon nanotubes (CNT) by Iijima (1991), a lot of investigations were focused on studying the dynamics of the single walled and multi-walled CNT (SWCNT and MWCNT) as structural elements (see Natsuki et al (2008), Wang (2005), Wang et al (2006), Mahdavi et al (2011) and references listed therein). Note that in these works the continuum approach, i.e. the continuum mechanics of deformable solid bodies were employed in studying the corresponding problems. Applicability areas of the continuum approach for the

[1] Yildiz Technical University, Faculty of Mechanical Engineering, Department of Mechanical Engineering, Yildiz Campus, 34349, Besiktas, Istanbul-TURKEY. *E-mail address:* akbarov@yildiz.edu.tr

[2] Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan, 37041, Baku, Azerbaijan.

study of the mechanical behaviors of the CNT and other types of nanostructures were discussed by Harik (2001), Guz and Rushchidsky (2003, 2012) and others.

Here we consider a brief review of the aforementioned works which relate to the subject of the present paper. We begin this review with the paper by Wang (2005) in which transverse wave propagation in a SWCNT is investigated by the use of the nonlocal elasticity theory. The SWCNT is modeled as Euler-Bernoulli and Timoshenko beams with infinite length. The effect of a small scale coefficient on wave dispersion is studied. Note that this small scale coefficient characterizes granular distance. It is established that after a certain critical value of the wave number, this coefficient begins to affect the character of the dispersion curves. Moreover, it is established that an increase in the values of this coefficient causes the wave propagation velocity to decrease.

In a paper by Wang and Varadan (2006), within the scope of the same assumptions and theories used in the paper by Wang (2005), transverse vibration of the SWCNT and DWCNT is studied. In analyzing the DWCNT, the double-beam theory is used and the van der Waals interaction effect at the interface of the DWCNTs inner and outer tubes is taken into account. The van der Waals interaction pressure at any point between two adjacent tubes is modeled as a linear function of the deflection jump at that point. Numerical results on the effect of the small scale coefficient on the natural frequency for the simply supported SWCNT and DWCNT are presented and discussed.

In a paper by Wang et al (2006), the Timoshenko beam model is used for free vibration analyses of the MWCNT. The deflections of the adjacent tubes are coupled due to the presence mentioned above of the van der Waals forces. For a solution to the corresponding eigenvalue problems the differential quadrature method is employed. Numerical results on the natural frequencies for various end conditions such as simply supported, clamped-clamped and clamped-simply supported conditions are presented, mainly for the DWCNT.

Transverse wave propagation in a DWCNT conveying fluid and embedded within an elastic medium was studied in a paper by Natsuki et al (2008). The DWCNT is modeled as a Euler-Bernoulli beam-pipe, for which the equation of motion is obtained from the force and moment balances, taking into account the fluid moving velocity, fluid mass density and the mass density of the per unit axial length. The deflection of nested tubes is considered to be coupled together through the van der Waals forces between the inner and outer nanotubes. The action of the surrounding elastic medium to the motion of the DWCNT is described by the Winkler spring model. In particular, it is established that the DWCNTs' conveying fluid has a lower wave speed than the DWCNTs without fluid.

The nonlinear vibration behavior of an embedded DWCNT is studied in a paper by Mahdavi et al (2011) by considering the nonlinear van der Waals interactions between the outer tube and surrounding medium, and between adjacent tubes. Using the Euler-Bernoulli and Timoshenko beam models, the relation between the deflection amplitude and the resonant frequency was derived. Moreover, in a paper by Mahdavi et al (2011) the effects of axial load and the CNT size on the nonlinear vibration of the embedded DWCNT was also examined.

It follows from the foregoing review that all the investigations related to the dynamics of the CNT as well as to the dynamics of the MWCNT were made within the scope of the approximate beam or shell theories. Consequently, the results of these works are acceptable for the thin-walled CNT, i.e. for the cases where $h/R \ll 1$ (where h is the thickness of the CNT and R is the radius of the middle surface of the CNT). However, according to Guz et al (2005), Liew et al (2004) and many other references listed in these papers, there are many cases for which the relation $0.1 \leq h/R \leq 0.25$ takes place and in these cases the CNT cannot be considered as thin-walled. Namely, in such cases, for investigation of the dynamics of the CNTs and MWCNTs it is necessary to use the exact three-dimensional equations of the theory of elastodynamics without any restrictions or simplifications assumed for displacements and stresses.

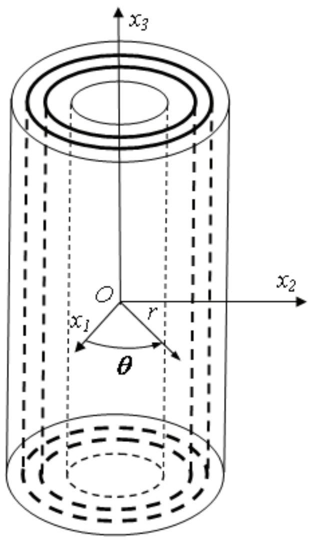
Moreover, the aforementioned approximate theories describe only a few wave modes and therefore these theories cannot be taken as sufficient for describing the dynamics of thick-walled CNTs and MWCNTs nor for thin-walled CNTs and MWCNTs.

Taking the foregoing discussions into account, in the present paper an attempt is made into the investigation of longitudinal axisymmetric wave propagation in the DWCNT with the use of the exact equations of motion of the linear theory of elastodynamics. The DWCNT is modeled as concentrically-nested two circular hollow cylinders between which there is free space. The difference in the radial displacements of these cylinders is coupled with the van der Waals forces.

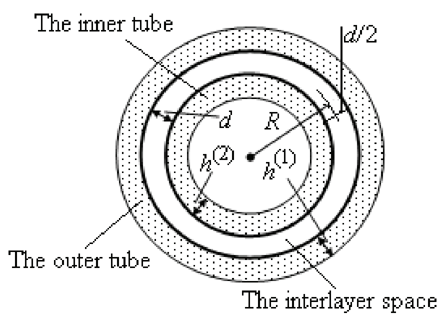
At the same time, the investigations carried out in the current paper can also be considered as developments of the studies by the author and his students (see Akbarov and Guliev (2009, 2010), Akbarov and Ipek (2010, 2012), Akbarov et al (2010, 2011) and Ozturk and Akbarov (2008, 2009a, 2009b)) related to axisymmetric wave propagation in compound cylinders for the case described above.

2 Formulation of the problem

We consider a DWCNT which is modeled as concentrically-nested two circular hollow cylinders with an infinite length between which there is free space (Fig.1). We assume that on the inner surface of the outer tube (cylinder) and on the outer surface of the inner tube (cylinder) of the DWCNT, full slipping conditions occur. At the same time, we assume that the difference between the radial displacements of the adjacent surfaces of the tubes is resisted with the van der Waals forces. Suppose that the radius of the middle surface of the free space cylinder is R; the thickness of the outer and inner layers (cylinders) we denote through $h^{(1)}$ and $h^{(2)}$, respectively (Fig. 1b), and the thickness of the interlayer free space we denote by d. For the case under consideration, below we will use the subscriptions (1) and (2) to denote the quantities related to the outer and inner cylinders respectively. We associate the cylindrical $Or\theta z$ and Cartesian $Ox_1x_2x_3$ system of coordinates with the central line of the DWCNT. We will use the cylindrical coordinates below.



a



b

Figure 1: The geometry of the DWCNT (a) and its cross section (b)

The materials of the constituents of the system under consideration are taken as isotropic and homogeneous. According to Charlier and Michenaud (1993), we suppose that full slipping conditions on the inner surface of the outer tube (i.e. at $r = R + d/2$) and on the outer surface of the inner tube (i.e. at $r = R - d/2$) of the DWCNT (Fig. 1) are satisfied. At the same time, according to Kelly (1981), we assume that the van der Waals forces resist the interlayer radial displacements of these adjacent tubes (layers).

Thus, within the scope of the foregoing conditions we investigate the axisymmetric longitudinal wave propagation along the Oz axis (Fig. 1a) with the use of the exact field equations and relations of the linear theory of elastodynamics. These equations and relations are:

The equations of motion

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = \rho \frac{\partial^2 u_z}{\partial t^2}, \tag{1}$$

The elasticity relations

$$\sigma_{(ij)} = \lambda e \delta_i^j + 2\mu\varepsilon_{(ij)},\ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\ \mu = \frac{E}{2(1+\nu)}, \quad (ij) = rr, \theta\theta, zz, rz, \tag{2}$$

The geometrical relations

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{zr} = \frac{1}{2}(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}). \tag{3}$$

Note that equations (1)-(3) are satisfied within the framework of each constituent of the system considered separately and in writing these equations the conventional notation is used and will also be used below.

Now we consider formulation of the boundary and contact conditions. Thus, according to the foregoing discussions we can write the following boundary conditions on the outer surface of the outer tube (i.e. at $r = R + d/2 + h^{(1)}$) and on the inner surface of the inner tube (i.e. at $r = R - d/2 - h^{(2)}$).

$$\sigma_{rr}^{(1)}\Big|_{r=R+d/2+h^{(1)}} = 0, \quad \sigma_{rz}^{(1)}\Big|_{r=R+d/2+h^{(1)}} = 0, \tag{4}$$

$$\sigma_{rr}^{(2)}\Big|_{r=R-d/2-h^{(2)}} = 0, \quad \sigma_{rz}^{(2)}\Big|_{r=R-d/2-h^{(2)}} = 0. \tag{5}$$

Moreover, the contact conditions between the tubes are:

$$(R+d/2)\sigma_{rr}^{(1)}\Big|_{r=R+d/2} = (R-d/2)\sigma_{rr}^{(2)}\Big|_{r=R-d/2},$$

$$\sigma_{rz}^{(1)}\Big|_{r=R+d/2} = 0, \quad \sigma_{rz}^{(2)}\Big|_{r=R-d/2} = 0,$$

$$\delta_w \left(u_r^{(1)}\Big|_{r=R+d/2} - u_r^{(2)}\Big|_{r=R-d/2} \right) = (R+d/2)\ \sigma_{rr}^{(1)}\Big|_{r=R+d/2}. \tag{6}$$

Note that the second and third conditions in (6) are mathematical simulations of full slipping on the inner surface of the outer tube and on the outer surface of the inner tube respectively. But the last condition in (6), according to the Lennard-Joes model (see Kelly (1981)), is the mathematical simulation of the van der Waals interaction between the tubes and the constant δ_w which has a stress dimension, and is determined through the interaction energy potential per unit axial length between the tubes. Moreover the values of the constant δ_w also depend on the curvature (radius) of the interface surface.

This completes formulation of the problem on the axisymmetric wave propagation in the DWCNT.

3 Method of solution

Substituting (2) and (3) into equation (1) we obtain the following equation of motion in displacement terms:

$$(\frac{\lambda^{(m)}}{\mu^{(m)}}+2)\frac{\partial^2 u_r^{(m)}}{\partial r^2}+(\frac{\lambda^{(m)}}{\mu^{(m)}}+1)\frac{\partial^2 u_z^{(m)}}{\partial r\partial z}+\frac{\lambda^{(m)}}{\mu^{(m)}}\frac{\partial}{\partial r}(\frac{\partial u_r^{(m)}}{\partial r})+$$

$$\frac{\partial^2 u_r^{(m)}}{\partial z^2}+\frac{2}{r}(\frac{\partial u_r^{(m)}}{\partial r}-\frac{u_r^{(m)}}{r})=\rho^{(m)}\frac{\partial^2 u_r^{(m)}}{\partial t^2},$$

$$(\frac{\lambda^{(m)}}{\mu^{(m)}}+2)\frac{\partial^2 u_z^{(m)}}{\partial z^2}+(\frac{\lambda^{(m)}}{\mu^{(m)}}+1)\frac{\partial^2 u_r^{(m)}}{\partial z\partial r}+\frac{\partial^2 u_z^{(m)}}{\partial r^2}+$$

$$(\frac{\lambda^{(m)}}{\mu^{(m)}}+1)\frac{1}{r}\frac{\partial u_r^{(m)}}{\partial z}+\frac{1}{r}\frac{\partial u_z^{(m)}}{\partial r}=\rho^{(m)}\frac{\partial^2 u_z^{(m)}}{\partial t^2},\quad m=1,2. \tag{7}$$

According to the monograph by Guz (2004), we use the following representation for the displacement:

$$u_r^{(m)}=-\frac{\partial^2}{\partial r\partial z}\mathrm{X}^{(m)},$$

$$u_z^{(m)}=\frac{1}{\lambda^{(m)}+\mu^{(m)}}\left[(\lambda^{(m)}+2\mu^{(m)})\Delta_1+\mu^{(m)}\frac{\partial^2}{\partial z^2}-\rho^{(m)}\frac{\partial^2}{\partial t^2}\right]\mathrm{X}^{(m)}, \tag{8}$$

where $\mathrm{X}^{(m)}$ satisfies the following equation:

$$\left[(\Delta_1+\frac{\partial^2}{\partial z^2})(\Delta_1+\frac{\partial^2}{\partial z^2})-\rho^{(m)}\frac{\lambda^{(m)}+3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}(\Delta_1+\right.$$

$$\left.\frac{\partial^2}{\partial z^2})\frac{\partial^2}{\partial t^2}+\frac{(\rho^{(m)})^2}{(\lambda^{(m)}+2\mu^{(m)})\mu^{(m)}}\frac{\partial^4}{\partial t^4}\right]\mathrm{X}^{(m)}=0. \tag{9}$$

In (8) and (9) the following notation is used:

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{10}$$

We represent the function $\mathrm{X}^{(m)} = \mathrm{X}^{(m)}(r,z,t)$ as

$$\mathrm{X}^{(m)} = \mathrm{X}_1^{(m)} \cos(kz - \omega t). \tag{11}$$

Substituting (11) into (9) and doing some mathematical manipulations we obtain the following equation for $\mathrm{X}_1^{(m)}(r)$:

$$\left(\Delta_1 + (k\zeta_2^{(m)})^2\right)\left(\Delta_1 + (k\zeta_3^{(m)})^2\right)\mathrm{X}_1^{(m)}(r) = 0. \tag{12}$$

The constants $\varsigma_2^{(m)}$ and $\varsigma_3^{(m)}$ are determined from the following equation:

$$(\zeta^{(m)})^4 - (\zeta^{(m)})^2\left[\frac{c^2-(c_1^{(m)})^2}{(c_2^{(m)})^2} + \frac{c^2-(c_2^{(m)})^2}{(c_1^{(m)})^2} + \frac{(\lambda^{(m)}+\mu^{(m)})^2}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}\right] +$$

$$\frac{(c^2-(c_1^{(m)})^2)}{(c_2^{(m)})^2}\frac{(c^2-(c_2^{(m)})^2}{(c_2^{(m)})^2} = 0, \tag{13}$$

where $c = \omega/k$ is the phase velocity; $c_1^{(m)} = \sqrt{(\lambda^{(m)}+2\mu^{(m)})/\rho^{(m)}}$ and $c_1^{(m)} = \sqrt{\mu^{(m)}/\rho^{(m)}}$.

Thus, we determine the following expression for $\mathrm{X}_1^{(m)}(r)$ from the equations (12) and (13):

$$\mathrm{X}_1^{(m)}(r) = A_2^{(m)}E_0^{(m)}(kr\zeta_2^{(m)}) + A_3^{(m)}E_0^{(m)}(kr\zeta_3^{(m)}) +$$

$$B_2^{(m)}G_0^{(m)}(kr\zeta_2^{(m)}) + B_3^{(m)}G_0^{(m)}(kr\zeta_3^{(m)}), \tag{14}$$

where

$$E_0^{(m)}(kr\zeta_n^{(m)}) = \begin{cases} J_0(kr\zeta_n^{(m)}) & if\ (\zeta_n^{(m)})^2 > 0 \\ I_0(kr\left|\zeta_n^{(m)}\right|) & if\ (\zeta_n^{(m)})^2 < 0 \end{cases},$$

$$G_0^{(m)}(kr\zeta_n^{(m)}) = \begin{cases} Y_0(kr\zeta_n^{(m)}) & if\ (\zeta_n^{(m)})^2 > 0 \\ K_0(kr\left|\zeta_n^{(m)}\right|) & if\ (\zeta_n^{(m)})^2 < 0 \end{cases}. \tag{15}$$

In (15) $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind of the zeroth order, $I_0(x)$ and $K_0(x)$ are Bessel functions of a purely imaginary argument and Macdonald function of the zeroth order, respectively.

Thus, using (15), (14), (11), (8), (2) and (3) we obtain the following dispersion equation from the conditions (4), (5) and (6):

$$\left\|\beta_{ij}\right\| = 0, \tag{16}$$

The expressions of β_{ij} are given in Appendix A by formulae (A1) and (A2).

Note that the numerical results, which will be discussed below, on the dispersion of the considered wave propagation are obtained from the numerical solution to equation (16) and this solution is obtained by utilizing the well-known "bisection method". In this case, for fixed values of the problem parameters, for each value of kR, the roots of the dispersion equation with respect to the wave propagation velocity c, are found. In the present paper the main purpose of the numerical investigations is the study of the influence of the problem parameters, such as the thickness of the tubes, the van der Waals forces and the thickness of the free space cylinder on the lowest fundamental modes. However, for construction of the dispersion curves corresponding to these modes, it is necessary to use the certain N number roots of equation (16). In this case, the graphs of the dependencies among $(c)_1$, $(c)_2$,..., $(c)_N$ and kR create the net on the plane $\{kR, c\}$. Note that, in general, the graph corresponding to the dependence between $(c)_n$ and kR is discontinuous and contains parts related to the various dispersion modes. Consequently, under construction of the dispersion curves, at first, we separate these parts from each other and then group the parts which are continuations of each other and thus we obtain the lowest fundamental modes' dispersion curves.

4 Numerical results and discussions

Before analyzing the numerical results, we consider the possible changes in the range of the problem parameters. As noted above, we assume that the material of the hollow cylinders of which the DWCNT consists, is a CNT. According to works by Harik (2001), Ru (2000, 2001), Ruoff and Lorents (1995), Guz and Rushchitsky (2012), Wang et al (2006), Xiao et al (2005) and others, the changing range of modulus of elasticity E, the Poisson's coefficient ν and the thickness of tubes $h^{(1)}/R (= h^{(2)}/R)$ is

$$1TPa \leq E \leq 1.2TPa, \quad 0.25 \leq \nu \leq 0.35, \quad 0.01 \leq h^{(1)}/R \leq 0.25. \tag{17}$$

Moreover it follows from the foregoing references that the change range of the constant δ_w which characterizes the van der Waals forces, is

$$0 < \delta_w \leq 9.92TPa. \tag{18}$$

At the same time, the foregoing references show that the value of the interlayer spacing under which the van der Waals forces are equal to zero is $d \approx 0.34nm$. It was established that $0.066nm \leq h^{(1)} = h^{(2)} \leq 0.34$, therefore we can suppose that

$$1 \leq d/h^{(1)} \leq 5. \tag{19}$$

Assume that the materials of the tubes are the same, i.e. $E^{(1)} = E^{(2)}$, $\lambda^{(1)} = \lambda^{(2)}$, $\mu^{(1)} = \mu^{(2)}$ and $\nu^{(1)} = \nu^{(2)} = 0.25$.

Introduce the parameter

$$F = \frac{\mu}{\delta_w}\left(1 - \frac{h^{(1)}}{2R}\frac{d}{h^{(1)}}\right) \tag{20}$$

through which we characterize the influence of the van der Waals forces on the wave propagation velocity. Note that the value $F = 0$ corresponds to the case where the van der Waals forces between the tubes of the DWCNT are equal to zero, and the radial displacements of the tubes on the cylindrical surfaces which bound the interlayer free space are equal to each other. But the value $F = \infty$ corresponds to the case where the radial forces (stresses) acting on the cylindrical surfaces of the tubes are equal to zero. Consequently, in the case where $F = \infty$ there is no contact between the tubes of the DWCNT. According to the aforementioned references, we assume that

$$0 \leq F \leq 15, \tag{21}$$

because the numerical results show that in the cases where $F > 15$ an increase in the values of the parameter Fcauses an insignificant change in the values of the wave propagation velocity.

Before analyzing the numerical results we note that the corresponding algorithm and PC programs were tested on the problems related to the longitudinal axisymmetric wave propagation in circular solid and hollow compound cylinders considered in works by Rose (2004), Guz (2004), Akbarov and Ipek (2010, 2012) and others.

Now we consider the numerical results obtained for the dispersion curves, i.e. the curves illustrating the dependence between c/c_2 and kR. These curves are given in Figs. 2, 3 and 4 for the cases where $h^{(1)}/R(=h^{(2)}/R) = 0.1$, 0.15 and 0.2 respectively. The figures indicated by the letters$a$, b and c relate to the cases where $d/h^{(1)} = 1.0$, 2.5 and 5 respectively. In each of these figures the results are given for various values of the parameter F(20). It follows from observation of the graphs that the lowest dispersion curves under consideration are separated into three groups or modes. The first is similar to the first mode of the wave propagation in the corresponding hollow or compound hollow cylinder. However, the

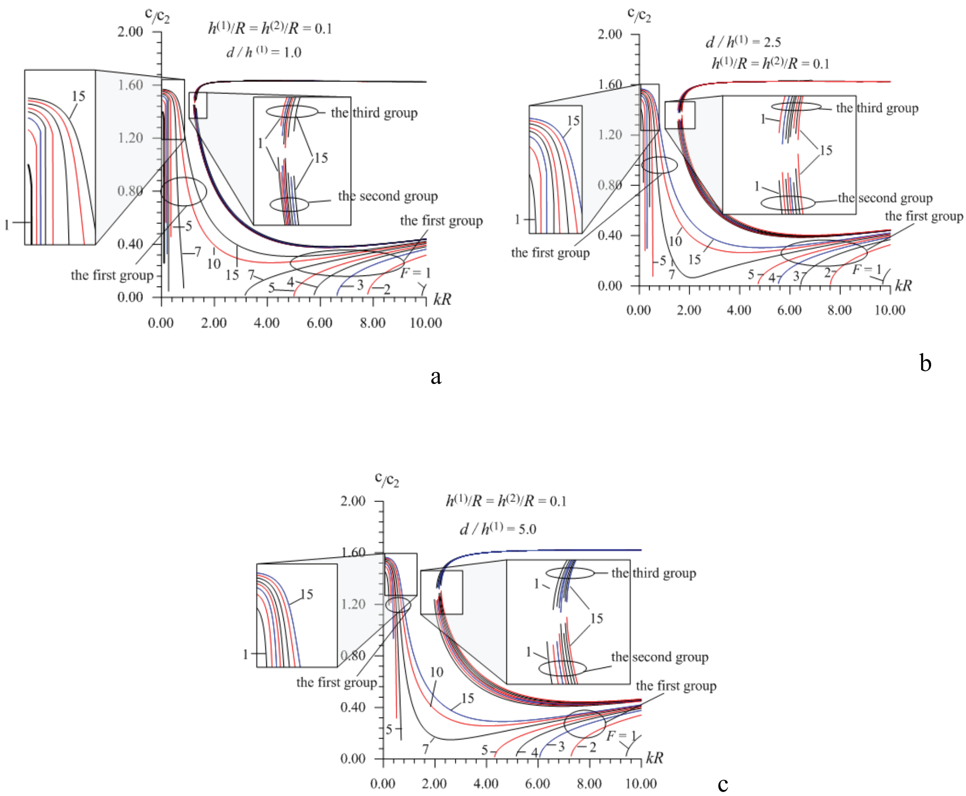


Figure 2: Dispersion curves constructed for the case where $h^{(1)}/R = h^{(2)}/R = 0.1$ under $d/h^{(1)} = 1$ (a), 2.5 (b) and 5 (c)

character of the dispersion mode depends significantly on the values of the parameter $F(20)$ which characterizes the van der Waals forces acting between the tubes of the DWCNT. The mentioned dependence can be formulated as follows:

There exists such a value of the parameter F(denoted by F') before which (i.e. under $F < F'$) a certain zone stopband, determined as

$$(kR)' < kR < (kR)'' \tag{22}$$

appears for the first group of modes. Note that the values of F', $(kR)'$ and $(kR)''$ depend on $h^{(1)}/R(= h^{(2)}/R)$ and on $d/h^{(1)}$. Moreover, note that the values of $(kR)'$ and $(kR)''$ depend also on the parameter F and

$$((kR)'' - (kR)') \to 0 \text{ as } F \to F' - 0. \tag{23}$$

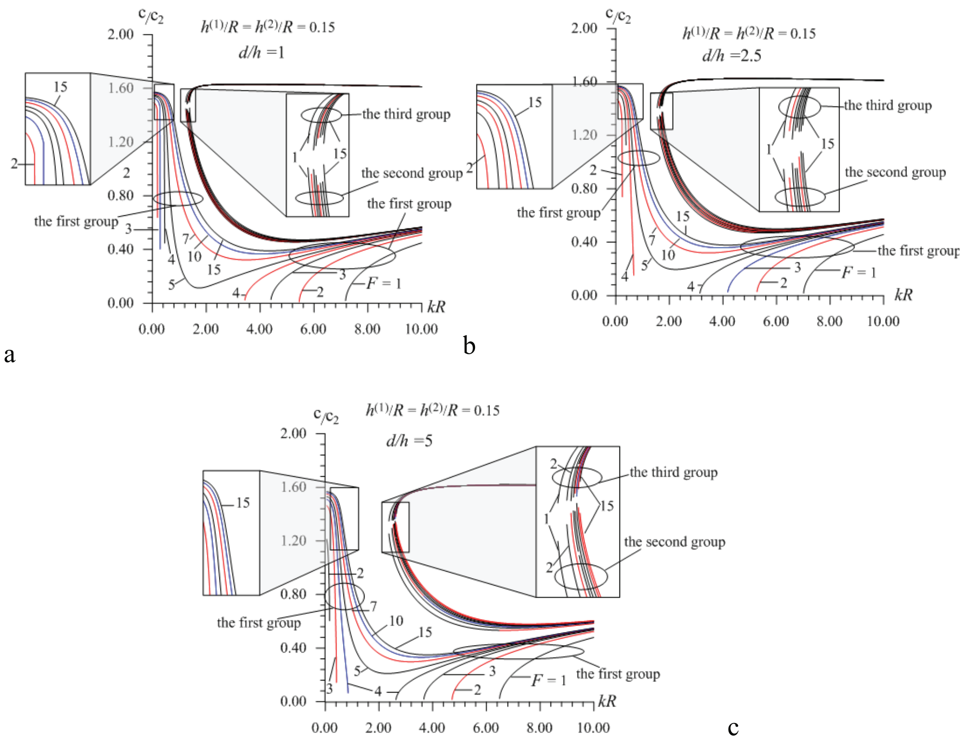


Figure 3: Dispersion curves constructed for the case where $h^{(1)}/R = h^{(2)}/R = 0.15$ under $d/h^{(1)} = 1$(a), 2.5 (b) and 5 (c)

The results given in Figs. 2, 3 and 4 show that the wave propagation velocity of the first group of modes increases (decreases) monotonically with kR under $kR > (kR)''$(under $kR < (kR)'$). Moreover, these results show that the values of $(kR)''$ and $(kR)'$ decrease with $h^{(1)}/R$.

To estimate the influence of the parameter $d/h^{(1)}$(i.e. the distance between the tubes of the DWCNT) on the wave propagation velocity we consider the dispersion curves constructed in the case where $h^{(1)}/R = 0.2$ which are given in Fig. 5. Note that in this figure the dispersion curves related to the corresponding hollow cylinder (i.e. related to the case where the free space between the tubes is ignored and full contact conditions are satisfied) and the dispersion curve related to the case where $d/h^{(1)} = 0$ and $F = 0$(full slipping) are also given. It follows from Fig. 5 and Figs. 2, 3 and 4 that the wave propagation velocity related to the first group of modes decreases (increases) with $d/h^{(1)}$ (with F).

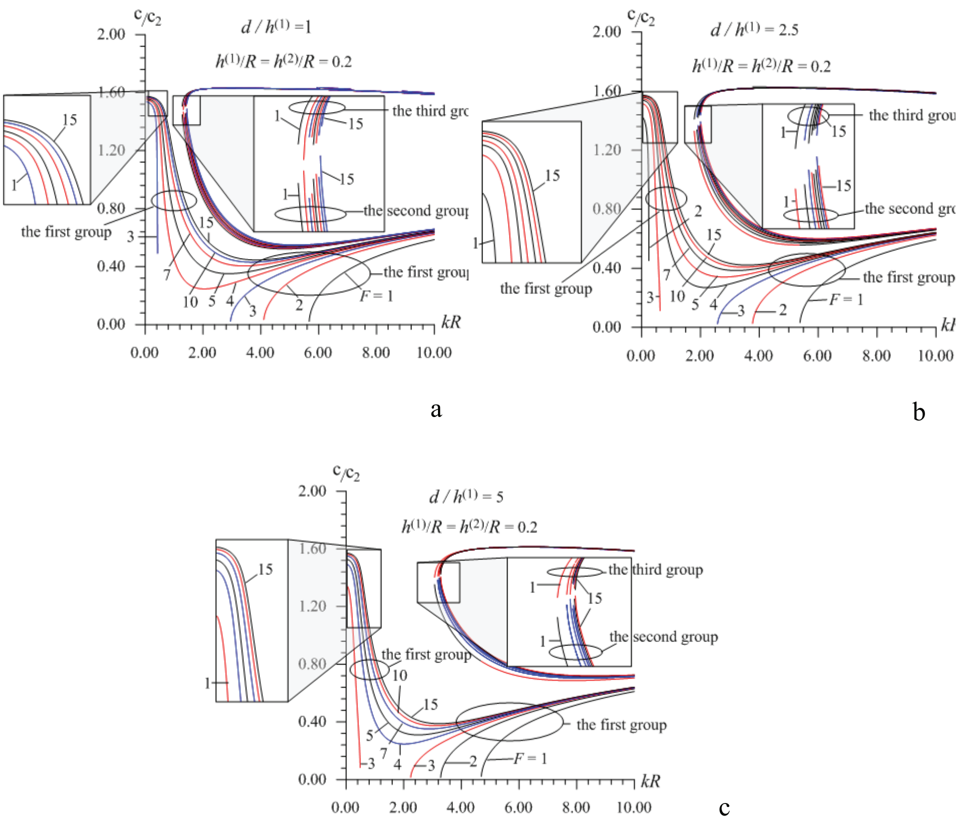


Figure 4: Dispersion curves constructed for the case where $h^{(1)}/R = h^{(2)}/R = 0.2$ under $d/h^{(1)} = 1$(a), 2.5 (b) and 5 (c)

Now we consider the second and third groups of the wave propagation modes. Note that these modes appear after certain cut off values of kR(denoted by $(kR)_{cf}$) and as if there is some disconnection between the beginning points (i.e. at $kR = (kR)_{cf}$) of the second and the third order dispersion curves. The length of these disconnection parts is not large, but these parts mean that there are some intervals for c/c_2 under which the wave propagation under consideration does not exist for the second or third group of curves or modes.

We introduce the notation c_I, c_{II} and c_{III} to indicate the wave propagation velocity related to the first, second and third groups of modes. According to the results given in Figs. 2, 3 and 4 we can write that

$$c_I < c_{II} < c_{III} \text{ for } kR > (kR)_{cf}. \tag{24}$$

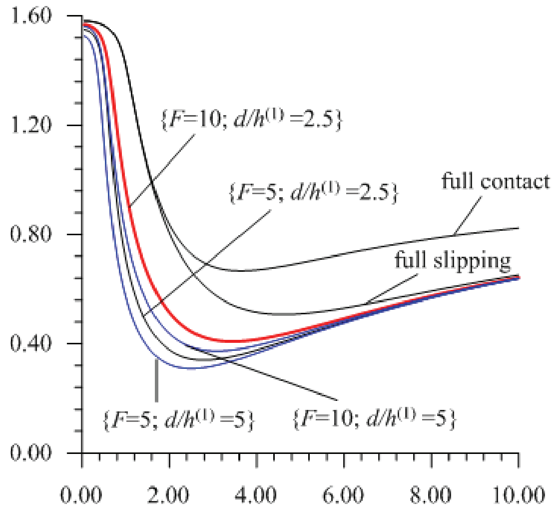


Figure 5: Graphs illustrating the influence of the parameters F and $d/h^{(1)}$on the wave propagation velocity

Also, the results show that the dependence between c_{II} and kR, as well as between c_{III} and kR are non-monotonic. This conclusion is observed clearly from the graphs given in Fig. 6 which are constructed for the case where $0 < kR \leq 30$ under $h^{(1)}/R = 0.15$ and $d/h^{(1)} = 1.0$. Note that the graphs given in Fig. 6 are the same ones given in Fig. 3a and constructed for $0 < kR \leq 10$.

Thus, it follows from the foregoing results that $c_{II}(c_{III})$ decreases (increases) monotonically before a certain value of kR and $c_{II}(c_{III})$ increases (decreases) monotonically after this value of kR with kR.

Analyses of the results show that the values of $(kR)_{cf}$increase with $d/h^{(1)}$, but the influence of the $h^{(1)}/R$ on the values of $(kR)_{cf}$ is insignificant. Moreover, the analyses show that the values of $c_{II}(c_{III})$ increase (decrease) with the parameter F.

Consider the limit values of c_I, c_{II} and c_{III}. As the velocities c_{II} and c_{III} have a meaning in the cases where $kR > (kR)_{cf}$, therefore there are no limit values for the velocities c_{II} and c_{III} under $kR \to 0$. But the values of c_I have finite limit values as $kR \to 0$ and their magnitude depends on the parameters F, $d/h^{(1)}$ and $h^{(1)}/R$. Note that with F (i.e. as $F \to 15$) the magnitude of the influence of the parameters $d/h^{(1)}$ and $h^{(1)}/R$ on the limit values becomes insignificant.

According to the results given in Fig. 6 and similar results obtained for the cases where $h^{(1)}/R = 0.1$ and 0.2, we can write that

$$c_I; \quad c_{II} \to c_R, \quad c_{III} \to c_2 \text{ as } kR \to \infty. \tag{25}$$

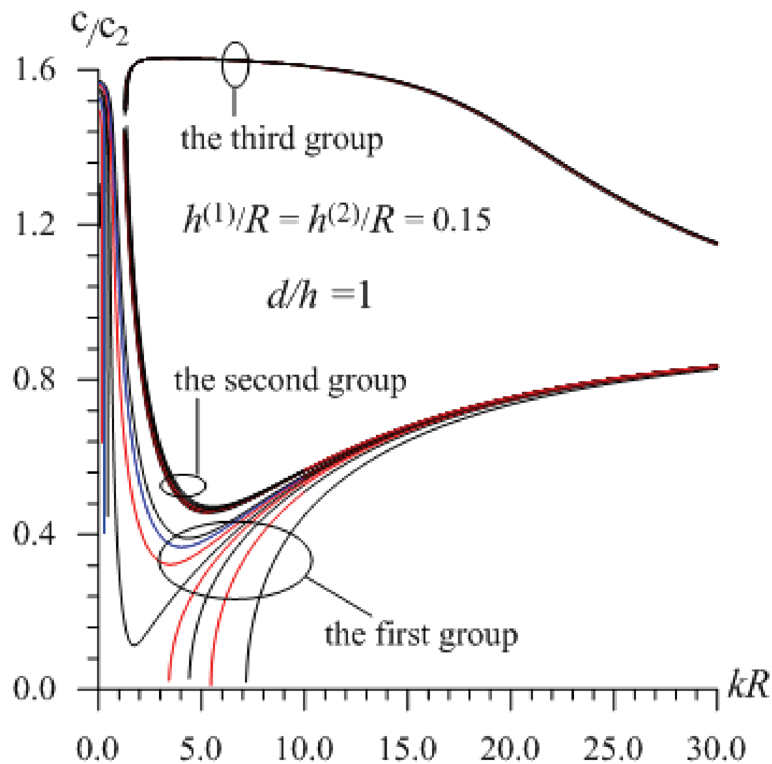


Figure 6: Illustration of the high wave number limit values of the wave propagation velocity for the case where $h^{(1)}/R = h^{(2)}/R = 0.15$ and $d/h^{(1)} = 1$

In (25) c_R is the velocity of the Rayleigh wave.

It follows from the foregoing results that the axisymmetric longitudinal waves with an arbitrary velocity which is greater than zero can propagate in the DWCNT. However such propagation can occur after certain cut off values of the wave number parameter. It should be noted that this particularity of the wave propagation is typical for the DWCNT only and that it does not appear for the SWCNT, or for compound cylinders, the constituents of which are ideally contacted. This is because it is known that the wave propagation in the SWCNT and in the compound cylinders occurs after a certain minimal velocity. Thus, it follows from the foregoing discussions that the aforementioned and other particularities indicated under consideration of the numerical results and related to the wave propagation in the DWCNT, are caused by the character of the contact relation between the tubes of the DWCNT. Namely, these particularities are caused with the van der Waals forces resisting the jump of the radial displacements of the tubes.

Moreover, it means that the DWCNTs can be used as sensors for indicating the waves with very low frequencies. In other words, the DWCNTs can be used as a wave guide for the waves with arbitrary frequencies or propagation velocities.

The numerical results discussed above are also typical in the following sense. According to Wilson et al (2002), the density of the CNT can be taken as $\rho = 1.4 \times 10^3 kg/m^3$. As an example, we consider the case where $E = 1.2TPa$ and $\nu = 0.25$ for which $c_2 = 1.85 \times 10^4 m/s$. For the considered case, the obtained numerical results with respect to c/c_2 must be estimated according to $c_2 = 1.85 \times 10^4 m/s$. However, for traditional materials, such as steel and aluminum similar results must be estimated according to $c_2 = 3210m/s$(for steel) and $c_2 = 3110m/s$(for aluminum). Consequently, the ratio c/c_2 for the CNT and for traditional materials must be distinguished according to the foregoing values of c_2 and according to the wavelengths, the values of which can be calculated from the values of kR, taking the changing range $R \sim 5nm - 35nm$ into account.

5 Conclusions

Thus in the present paper, longitudinal axisymmetric wave propagation in the DWCNT has been investigated by utilizing the exact equations of motion of the linear theory of elastodynamics. This investigation has been made within the scope of the following assumptions:

1. The DWCNT has been modeled as concentrically-nested two circular hollow cylinders between which there is free space;

2. The difference in the radial displacements of these cylinders has been coupled with the van der Waals forces;

3. Full slipping conditions are satisfied on the inner surface of the outer tube and on the outer surface of the inner tube.

The foregoing conditions follow from the characteristic features of the DWCNT. The numerical results presented and discussed above on the new type of dispersion curves with these features are found by utilizing the exact equations of motion of the linear theory of elastodynamics. It should be noted that these conditions were also assumed in the works carried out within the framework of the approximate beam and shell theories. However, these approximate theories cannot uncover these new types of dispersion curves either in the quantitative or the qualitative sense. Therefore, from the author's point of view, for more detailed study and correct understanding of the MWCNTs' dynamical behavior it is necessary to use the approach developed in the present paper for all related future investigations.

Appendix A

We write the expressions for calculation of the term β_{ij} which enter the dispersion equation (16)

$$\beta_{11}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$\begin{cases}
(\lambda^{(2)}+2\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\\
\frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\frac{s_2^{(2)}}{\gamma^{(2)}}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right),\\
(\lambda^{(2)}+2\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)-\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\\
\frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\frac{s_2^{(2)}}{\gamma^{(2)}}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right),
\end{cases}$$

$$\beta_{21}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$\begin{cases}
-\mu^{(2)}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3J_1\left(\chi_{2h^{(2)}}^{(2)}\right)-J_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\right.\right.\\
\left.\left.\frac{s_2^{(2)}}{(\gamma^{(2)})^2}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(J_2\left(\chi_{2h^{(2)}}^{(2)}\right)-J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right),\\
\mu^{(2)}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+I_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\right.\right.\\
\left.\left.\frac{s_2^{(2)}}{(\gamma^{(2)})^2}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\beta_2^{(2)}s_2^{(2)}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right),
\end{cases}$$

$$\beta_{13}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$\begin{cases}
(\lambda^{(2)}+2\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(1)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\\
\frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right)-Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)-\frac{s_2^{(2)}}{\gamma^{(2)}}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right),\\
(\lambda^{(2)}+2\mu^{(2)})\left(\left(s_2^{(2)}\right)^2\frac{1}{2}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right)+K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right)+\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(2)}K_1\left(\chi_{2h^{(2)}}^{(2)}\right)+\\
\frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right)+J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)+\frac{s_2^{(2)}}{\gamma^{(2)}}I_1\left(\chi_{2h^{(2)}}^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right),
\end{cases}$$

$$\beta_{23}\left(s_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$\begin{cases} -\mu^{(2)} s_2^{(2)} Y_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3Y_1\left(\chi_{2h^{(2)}}^{(2)}\right) - Y_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right) + \right.\right. \\ \left.\left. \frac{s_2^{(2)}}{(\gamma^{(2)})^2} Y_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right) - Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) + \beta_2^{(2)} s_2^{(2)} Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right), \\ -\mu^{(2)} s_2^{(2)} K_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3K_1\left(\chi_{2h^{(2)}}^{(2)}\right) + K_3\left(\chi_{2h^{(2)}}^{(2)}\right)\right) + \right.\right. \\ \left.\left. \frac{s_2^{(2)}}{(\gamma^{(2)})^2} K_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right) + K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) + \beta_2^{(2)} s_2^{(2)} K_1\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right), \end{cases}$$

$$\beta_{n2} = \beta_{n1}\left(s_3^{(2)}, \chi_{3h^{(2)}}^{(2)}\right), \quad \beta_{n4} = \beta_{n3}\left(s_3^{(2)}, \chi_{3h^{(2)}}^{(2)}\right),$$

$$\beta_{n5} = \beta_{n6} = \beta_{n7} = \beta_{n8} = 0, \quad n = 1, 2,$$

$$\beta_{31}\left(s_2^{(2)}, \chi_2^{(2)}\right) =$$

$$R^{(2)} \begin{cases} (\lambda^{(2)} + 2\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2 \frac{1}{2}\left(J_2\left(\chi_2^{(2)}\right) - J_0\left(\chi_2^{(2)}\right)\right)\right) + \frac{\lambda^{(2)}}{\gamma^{(2)}} s_2^{(2)} J_1\left(\chi_2^{(2)}\right) + \\ \frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(J_2\left(\chi_2^{(2)}\right) - J_0\left(\chi_2^{(2)}\right)\right) - \frac{s_2^{(2)}}{\gamma^{(2)}} J_1\left(\chi_2^{(2)}\right) - \beta_2^{(2)} J_0\left(\chi_2^{(2)}\right)\right), \\ (\lambda^{(2)} + 2\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2 \frac{1}{2}\left(I_2\left(\chi_2^{(2)}\right) + I_0\left(\chi_2^{(2)}\right)\right)\right) - \frac{\lambda^{(2)}}{\gamma^{(2)}} s_2^{(2)} I_1\left(\chi_2^{(2)}\right) + \\ \frac{\lambda^{(2)}}{2\gamma^{(2)}}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_2^{(2)}\right) + I_0\left(\chi_2^{(2)}\right)\right) + \frac{s_2^{(2)}}{\gamma^{(2)}} I_1\left(\chi_2^{(2)}\right) - \beta_2^{(2)} I_0\left(\chi_2^{(2)}\right)\right), \end{cases}$$

$$\beta_{41}\left(s_2^{(2)}, \chi_2^{(2)}\right) =$$

$$\begin{cases} -\mu^{(2)} s_2^{(2)} J_1\left(\chi_2^{(2)}\right) + \frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3J_1\left(\chi_2^{(2)}\right) - J_3\left(\chi_2^{(2)}\right)\right) + \right.\right. \\ \left.\left. \frac{s_2^{(2)}}{(\gamma^{(2)})^2} J_1\left(\chi_2^{(2)}\right) + \frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(J_2\left(\chi_2^{(2)}\right) - J_0\left(\chi_2^{(2)}\right)\right) + \beta_2^{(2)} s_2^{(2)} J_1\left(\chi_2^{(2)}\right)\right)\right), \\ \mu^{(2)} s_2^{(2)} I_1\left(\chi_2^{(2)}\right) + \frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3I_1\left(\chi_2^{(2)}\right) + I_3\left(\chi_2^{(2)}\right)\right) - \right.\right. \\ \left.\left. \frac{s_2^{(2)}}{(\gamma^{(2)})^2} I_1\left(\chi_2^{(2)}\right) + \frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(I_2\left(\chi_2^{(2)}\right) + I_0\left(\chi_2^{(2)}\right)\right) - \beta_2^{(2)} s_2^{(2)} I_1\left(\chi_2^{(2)}\right)\right)\right), \end{cases}$$

$$\beta_{51}\left(s_2^{(2)}, \chi_2^{(2)}\right) = 0, \quad \beta_{61}\left(s_2^{(2)}, \chi_2^{(2)}\right) = \begin{cases} -s_2^{(2)} J_1\left(\chi_2^{(2)}\right), \\ s_2^{(2)} I_1\left(\chi_2^{(2)}\right), \end{cases}$$

$$\beta_{33}\left(s_2^{(2)}, \chi_2^{(2)}\right) =$$

$$
R^{(2)}\begin{cases}(\lambda^{(2)}+\mu^{(2)})\left(-\left(s_2^{(2)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)\right)+\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)+\\ \frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)-\frac{s_2^{(2)}}{\gamma^{(2)}}Y_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}Y_0\left(\chi_2^{(2)}\right)\right),\\ (\lambda^{(2)}+\mu^{(2)})\left(\left(s_2^{(2)}\right)^2\frac{1}{2}\left(K_2\left(\chi_2^{(2)}\right)+K_0\left(\chi_2^{(2)}\right)\right)\right)+\frac{\lambda^{(2)}}{\gamma^{(2)}}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)+\\ \frac{\lambda^{(2)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_2^{(2)}\right)+I_0\left(\chi_2^{(2)}\right)\right)+\frac{s_2^{(2)}}{\gamma^{(2)}}I_1\left(\chi_2^{(2)}\right)-\beta_2^{(2)}I_0\left(\chi_2^{(2)}\right)\right),\end{cases}
$$

$$
\beta_{43}\left(s_2^{(2)},\chi_2^{(2)}\right)=
$$

$$
\begin{cases}-\mu^{(2)}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)+\frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3Y_1\left(\chi_2^{(2)}\right)-Y_3\left(\chi_2^{(2)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(2)}}{(\gamma^{(2)})^2}Y_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(Y_2\left(\chi_2^{(2)}\right)-Y_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}Y_1\left(\chi_2^{(2)}\right)\right)\right),\\ -\mu^{(2)}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)+\frac{\mu^{(2)}}{4}\left(\beta_1^{(2)}\left(\left(s_2^{(2)}\right)^3\left(3K_1\left(\chi_2^{(2)}\right)+K_3\left(\chi_2^{(2)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(2)}}{(\gamma^{(2)})^2}K_1\left(\chi_2^{(2)}\right)+\frac{\left(s_2^{(2)}\right)^2}{2\gamma^{(2)}}\left(K_2\left(\chi_2^{(2)}\right)+K_0\left(\chi_2^{(2)}\right)\right)+\beta_2^{(2)}s_2^{(2)}K_1\left(\chi_2^{(2)}\right)\right)\right),\end{cases}
$$

$$
\beta_{53}\left(s_2^{(2)},\chi_2^{(2)}\right)=0,
$$

$$
\beta_{63}\left(s_2^{(2)},\chi_2^{(2)}\right)=\begin{cases}-s_2^{(2)}Y_1\left(\chi_2^{(2)}\right),\\ -s_2^{(2)}K_1\left(\chi_2^{(2)}\right),\end{cases}
$$

$$
\beta_{n2}=\beta_{n1}\left(s_3^{(2)},\chi_3^{(2)}\right),\quad \beta_{n4}=\beta_{n3}\left(s_3^{(2)},\chi_3^{(2)}\right),\quad n=3,4,5,6,
$$

$$
\beta_{35}\left(s_2^{(1)},\chi_2^{(1)}\right)=
$$

$$
R^{(1)}\begin{cases}(\lambda^{(1)}+2\mu^{(1)})\left(-\left(s_2^{(1)}\right)^2\frac{1}{2}\left(J_2\left(\chi_2^{(1)}\right)-J_0\left(\chi_2^{(1)}\right)\right)\right)+\frac{\lambda^{(1)}}{\gamma^{(1)}}s_2^{(1)}J_1\left(\chi_2^{(1)}\right)+\\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(J_2\left(\chi_2^{(1)}\right)-J_0\left(\chi_2^{(1)}\right)\right)-\frac{s_2^{(1)}}{\gamma^{(1)}}J_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}J_0\left(\chi_2^{(1)}\right)\right),\\ (\lambda^{(1)}+2\mu^{(1)})\left(-\left(s_2^{(1)}\right)^2\frac{1}{2}\left(I_2\left(\chi_2^{(1)}\right)+I_0\left(\chi_2^{(1)}\right)\right)\right)-\frac{\lambda^{(1)}}{\gamma^{(2)}}s_2^{(1)}I_1\left(\chi_2^{(1)}\right)+\\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_2^{(1)}\right)+I_0\left(\chi_2^{(1)}\right)\right)+\frac{s_2^{(1)}}{\gamma^{(1)}}I_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}I_0\left(\chi_2^{(1)}\right)\right),\end{cases}
$$

$$
\beta_{37}\left(s_2^{(1)},\chi_2^{(1)}\right)=
$$

$$R^{(1)}\begin{cases}(\lambda^{(1)}+2\mu^{(1)})\left(-\left(s_2^{(1)}\right)^2\frac{1}{2}\left(Y_2\left(\chi_2^{(1)}\right)-Y_0\left(\chi_2^{(1)}\right)\right)\right)+\frac{\lambda^{(1)}}{\gamma^{(1)}}s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(Y_2\left(\chi_2^{(1)}\right)-Y_0\left(\chi_2^{(1)}\right)\right)-\frac{s_2^{(1)}}{\gamma^{(1)}}Y_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}Y_0\left(\chi_2^{(1)}\right)\right),\\ (\lambda^{(1)}+2\mu^{(1)})\left(\left(s_2^{(1)}\right)^2\frac{1}{2}\left(K_2\left(\chi_2^{(1)}\right)+K_0\left(\chi_2^{(1)}\right)\right)\right)+\frac{\lambda^{(1)}}{\gamma^{(1)}}s_2^{(1)}K_1\left(\chi_2^{(1)}\right)+\\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_2^{(1)}\right)+I_0\left(\chi_2^{(1)}\right)\right)+\frac{s_2^{(1)}}{\gamma^{(1)}}I_1\left(\chi_2^{(1)}\right)-\beta_2^{(1)}I_0\left(\chi_2^{(1)}\right)\right),\end{cases}$$

$$\beta_{45}=\beta_{46}=\beta_{47}=\beta_{48}=0$$

$$\beta_{55}\left(s_2^{(1)},\chi_2^{(1)}\right)=$$

$$\begin{cases}-\mu^{(1)}s_2^{(1)}J_1\left(\chi_2^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3J_1\left(\chi_2^{(1)}\right)-J_3\left(\chi_2^{(1)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}J_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(J_2\left(\chi_2^{(1)}\right)-J_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}J_1\left(\chi_2^{(1)}\right)\right)\right),\\ \mu^{(1)}s_2^{(1)}I_1\left(\chi_2^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3I_1\left(\chi_2^{(1)}\right)+I_3\left(\chi_2^{(1)}\right)\right)-\right.\right.\\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}I_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(I_2\left(\chi_2^{(1)}\right)+I_0\left(\chi_2^{(1)}\right)\right)-\beta_2^{(1)}s_2^{(1)}I_1\left(\chi_2^{(1)}\right)\right)\right),,\end{cases}$$

$$\beta_{57}\left(s_2^{(1)},\chi_2^{(1)}\right)=$$

$$\begin{cases}-\mu^{(1)}s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3Y_1\left(\chi_2^{(1)}\right)-Y_3\left(\chi_2^{(1)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}Y_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(Y_2\left(\chi_2^{(1)}\right)-Y_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)\right)\right),\\ -\mu^{(1)}s_2^{(1)}K_1\left(\chi_2^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3K_1\left(\chi_2^{(1)}\right)+K_3\left(\chi_2^{(1)}\right)\right)+\right.\right.\\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}K_1\left(\chi_2^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(K_2\left(\chi_2^{(1)}\right)+K_0\left(\chi_2^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}K_1\left(\chi_2^{(1)}\right)\right)\right),\end{cases}$$

$$\beta_{36}=\beta_{35}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{38}=\beta_{37}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{56}=\beta_{55}\left(s_3^{(1)},\chi_3^{(1)}\right),\quad \beta_{58}=\beta_{57}\left(s_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{65}\left(s_2^{(1)},\chi_2^{(1)}\right)=-\frac{1}{\delta_w}\beta_{35}\left(s_2^{(1)},\chi_2^{(1)}\right)+\begin{cases}-s_2^{(1)}J_1\left(\chi_2^{(1)}\right),\\ s_2^{(1)}I_1\left(\chi_2^{(1)}\right),\end{cases},$$

$$\beta_{67}\left(s_2^{(1)},\chi_2^{(1)}\right)=-\frac{1}{\delta_w}\beta_{37}\left(s_2^{(1)},\chi_2^{(1)}\right)+\begin{cases}-s_2^{(1)}Y_1\left(\chi_2^{(1)}\right),\\ -s_2^{(1)}K_1\left(\chi_2^{(1)}\right),\end{cases},$$

$$\beta_{66} = \beta_{65}\left(s_3^{(1)}, \chi_3^{(1)}\right), \quad \beta_{68} = \beta_{67}\left(s_3^{(1)}, \chi_3^{(1)}\right),$$

$$\beta_{75}\left(s_2^{(1)}, \chi_{2h^{(1)}}^{(1)}\right) =$$

$$\begin{cases} (\lambda^{(1)} + 2\mu^{(1)})\left(-\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right) - J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right) + \frac{\lambda^{(1)}}{\gamma^{(1)}} s_2^{(1)} J_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right) - J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right) - \frac{s_2^{(1)}}{\gamma^{(1)}} J_1\left(\chi_{2h^{(1)}}^{(1)}\right) - \beta_2^{(1)} J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right), \\ (\lambda^{(1)} + 2\mu^{(1)})\left(-\left(s_2^{(1)}\right)^2 \frac{1}{2}\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right) + I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right) - \frac{\lambda^{(1)}}{\gamma^{(1)}} s_2^{(1)} I_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(1)}\left(s_2^{(1)}\right)^2\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right) + I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right) + \frac{s_2^{(1)}}{\gamma^{(1)}} I_1\left(\chi_{2h^{(1)}}^{(1)}\right) - \beta_2^{(2)} I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right), \end{cases}$$

$$\beta_{85}\left(s_2^{(1)}, \chi_{2h^{(1)}}^{(1)}\right) =$$

$$\begin{cases} -\mu^{(1)} s_2^{(1)} J_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3J_1\left(\chi_{2h^{(1)}}^{(1)}\right) - J_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right) + \right.\right. \\ \left.\left. \frac{s_2^{(1)}}{(\gamma^{(2)})^2} J_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(J_2\left(\chi_{2h^{(1)}}^{(1)}\right) - J_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right) + \beta_2^{(1)} s_2^{(1)} J_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right), \\ \mu^{(1)} s_2^{(1)} I_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3I_1\left(\chi_{2h^{(1)}}^{(1)}\right) + I_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right) - \right.\right. \\ \left.\left. \frac{s_2^{(1)}}{(\gamma^{(2)})^2} I_1\left(\chi_{2h^{(1)}}^{(1)}\right) + \frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(I_2\left(\chi_{2h^{(1)}}^{(1)}\right) + I_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right) - \beta_2^{(1)} s_2^{(1)} I_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right), \end{cases}$$

$$\beta_{77}\left(s_2^{(1)}, \chi_{2h^{(1)}}^{(1)}\right) =$$

$$\begin{cases} (\lambda^{(1)} + 2\mu^{(1)})\left(-\left(s_2^{(2)}\right)^2 \frac{1}{2}\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right) - Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right) + \frac{\lambda^{(1)}}{\gamma^{(1)}} s_2^{(1)} Y_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(Y_2\left(\chi_{2h^{(2)}}^{(2)}\right) - Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) - \frac{s_2^{(2)}}{\gamma^{(1)}} Y_1\left(\chi_{2h^{(2)}}^{(2)}\right) - \beta_2^{(2)} Y_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right), \\ (\lambda^{(1)} + 2\mu^{(1)})\left(\left(s_2^{(2)}\right)^2 \frac{1}{2}\left(K_2\left(\chi_{2h^{(2)}}^{(2)}\right) + K_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right)\right) + \frac{\lambda^{(1)}}{\gamma^{(1)}} s_2^{(2)} K_1\left(\chi_{2h^{(2)}}^{(2)}\right) + \\ \frac{\lambda^{(1)}}{2}\left(\beta_1^{(2)}\left(s_2^{(2)}\right)^2\left(I_2\left(\chi_{2h^{(2)}}^{(2)}\right) + J_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right) + \frac{s_2^{(2)}}{\gamma^{(1)}} I_1\left(\chi_{2h^{(2)}}^{(2)}\right) - \beta_2^{(2)} I_0\left(\chi_{2h^{(2)}}^{(2)}\right)\right), \end{cases}$$

$$\beta_{87}\left(s_2^{(1)}, \chi_{2h^{(1)}}^{(1)}\right) =$$

$$\begin{cases} -\mu^{(1)}s_2^{(1)}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)-Y_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(Y_2\left(\chi_{2h^{(1)}}^{(1)}\right)-Y_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right), \\ -\mu^{(1)}s_2^{(1)}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\mu^{(1)}}{4}\left(\beta_1^{(1)}\left(\left(s_2^{(1)}\right)^3\left(3K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+K_3\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\right.\right. \\ \left.\left.\frac{s_2^{(1)}}{(\gamma^{(2)})^2}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)+\frac{\left(s_2^{(1)}\right)^2}{2\gamma^{(2)}}\left(K_2\left(\chi_{2h^{(1)}}^{(1)}\right)+K_0\left(\chi_{2h^{(1)}}^{(1)}\right)\right)+\beta_2^{(1)}s_2^{(1)}K_1\left(\chi_{2h^{(1)}}^{(1)}\right)\right)\right), \end{cases}$$

$$\beta_{n6}=\beta_{n5}\left(s_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),\quad \beta_{n8}=\beta_{n7}\left(s_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),$$

$$\beta_{n1}=\beta_{n2}=\beta_{n3}=\beta_{n4}=0,\quad n=7,8. \tag{A1}$$

In relation to (A1) the following notation is used:

$$\chi_2^{(n)}=kR^{(n)}\left|\zeta_2^{(n)}\right|,\quad \chi_3^{(n)}=kR^{(n)}\left|\zeta_3^{(n)}\right|,\quad s_2^{(n)}=\left|\zeta_2^{(n)}\right|,\quad s_3^{(n)}=\left|\zeta_3^{(n)}\right|,\quad n=1,2,$$

$$R^{(1)}=R+\frac{d}{2},\quad R^{(2)}=R-\frac{d}{2},\quad \gamma^{(n)}=kR^{(n)}$$

$$\chi_{2h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}(1+\frac{d}{2h^{(2)}})\right)\left|\zeta_2^{(2)}\right|,\quad \chi_{3h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}(1+\frac{d}{2h^{(2)}})\right)\left|\zeta_3^{(2)}\right|,$$

$$\chi_{2h^{(1)}}^{(1)}=kR\left(1-\frac{h^{(1)}}{R}(1-\frac{d}{2h^{(2)}})\right)\left|\zeta_2^{(1)}\right|,\quad \chi_{3h^{(1)}}^{(1)}=kR\left(1-\frac{h^{(1)}}{R}(1-\frac{d}{2h^{(2)}})\right)\left|\zeta_3^{(1)}\right|,$$

$$\beta_1^{(n)}=\frac{\lambda^{(n)}+2\mu^{(n)}}{\lambda^{(n)}+\mu^{(n)}},\quad \beta_2^{(n)}=\frac{\mu^{(n)}}{\lambda^{(n)}+\mu^{(n)}}-\frac{c^2}{(c_1^{(n)})^2-(c_2^{(n)})^2}. \tag{A2}$$

Moreover, note that in (A1) the expressions given through the functions $J_n(x)$ and $Y_n(x)$(the functions $I_n(x)$ and $K_n(x)$), relate to the case where $\left(\zeta_m^{(1)}\right)^2>0$ (where $\left(\zeta_m^{(1)}\right)^2<0$).

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