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Article

A New Modified EWMA Control Chart for Monitoring Processes Involving Autocorrelated Data

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Abstract: Control charts are one of the tools in statistical process control widely used for monitoring, measuring, controlling, improving the quality, and detecting problems in processes in various fields. The average run length (ARL) can be used to determine the efficacy of a control chart. In this study, we develop a new modified exponentially weighted moving average (EWMA) control chart and derive explicit formulas for both one and the two-sided ARLs for a p-order autoregressive (AR(p)) process with exponential white noise on the new modified EWMA control chart. The accuracy of the explicit formulas was compared to that of the well-known numerical integral equation (NIE) method. Although both methods were highly consistent with an absolute percentage difference of less than 0.00001%, the ARL using the explicit formulas method could be computed much more quickly. Moreover, the performance of the explicit formulas for the ARL on the new modified EWMA control chart was better than on the modified and standard EWMA control charts based on the relative mean index (RMI). In addition, to illustrate the applicability of using the proposed explicit formulas for the ARL on the new modified EWMA control chart in practice, the explicit formulas for the ARL were also applied to a process with real data from the energy and agricultural fields.

Keywords: Autoregressive process; new modified EWMA; average run length (ARL); numerical integral equation (NIE)

1 Introduction

Quality control of products or services plays a very important role in the business and manufacturing industries. Statistical process control (SPC) is a powerful set of tools that are used to inspect, control, and improve the quality of processes [1], and control charts used for monitoring processes and detecting shifts in the process mean comprise a key tool for SPC. Shewhart [2] introduced the first control chart that is still widely used for monitoring and detecting large shifts in the process mean but is unsuitable for detecting small changes. Later, several researchers derived control charts for detecting small and large changes in process mean. The cumulative sum (CUSUM) control chart proposed by Page [3] is better than the Shewhart control chart for detecting small shifts in the process mean (see also [4,5]). Furthermore, Roberts [6] presented the exponentially weighted moving average (EWMA) control chart as another option for detecting small shifts in the process mean (see also [7,8]). Khan et al. [9] developed a new EWMA control chart statistic based on the modified EWMA statistic [10] that considers the past and current behavior of the process by introducing an extra constant in the modified EWMA statistic proposed in [9]. They compared its efficacy with the modified and standard control charts and found that the proposed control chart was more efficient in terms of the average run length (ARL) (a popular measure for control chart performance) and could detect shifts more quickly. Anwar et al. [11] proposed the modified mxEWMA control chart for a process in the presence of auxiliary information, while Aslam et al. [12] proposed the Bayesian-modified EWMA control chart for the process mean involving various loss functions.

The ARL is the average number of observations before a control chart signals that a process is out-of-control. There are two components: ARL_0 and ARL_1. ARL_0 is the average number of observations for the process to remain in-control and should be as large as possible while ARL_1 is the average number of observations until the process is signaled as out-of-control and should be as small as possible. Various methods to estimate the ARL have been reported, such as Monte Carlo simulation, Markov chain, Martingale, and numerical integration equations (NIEs) based on several quadrature rules (midpoint, trapezoidal, Simson's rule, and Gauss-Legendre) [13]. Explicit formulas comprise a method for evaluating the ARL that requires solving integral equations. Crowder [7] used an integral equation approach to develop an approximation for the ARL of a Gaussian process on an EWMA control chart by using a Fredholm integral equation of the second kind. Champ et al. [14] also used this approach to evaluate the ARL on CUSUM and EWMA control charts and compared the results with those obtained by using the Markov chain approach. Moreover, the Fredholm integral equation of the second kind has been used to evaluate the ARL for many control charts [13]. Several researchers have focused on approximating the ARL to measure the efficacy of control charts by using many methods. Roberts [6] proposed using Monte Carlo simulation to estimate the ARL on the standard EWMA control chart. Harris et al. [15] studied serially correlated observations on a CUSUM control chart via Monte Carlo simulation. Vanbrackle et al. [16] investigated the NIE and Markov chain approaches to evaluate the ARL when the observations are from a first-order autoregressive (AR(1)) process with additional random error on EWMA and CUSUM control charts.

The modified EWMA statistic with an extra constant in the model that equally prioritizes historical and current information may degrade the performance of the control chart. Hence, we added one more constant to place more emphasis on current information over historical information. We hypothesized that the proposed control chart would provide very interesting properties (i.e., it would be more efficient at detecting small shifts in the process mean and would obtain the smallest ARL). Moreover, present a new modified EWMA control chart based on the modified EWMA statistic developed by Khan et al. [9] that prioritizes current information over historical information. In addition, we derive explicit formulas for the ARL for detecting changes in the process mean of a p-order autoregressive (AR(p)) process with exponential white noise running on the new modified EWMA control chart by using the Fredholm integral equation of the second kind and compared its efficiency with the ARL based on the well-know NIE method using the Gauss-Legendre rule.

2 The Properties of the Various EWMA Control Chart

The properties of the standard, modified, and new modified EWMA control charts are provided in the following subsections.

2.1 The Standard EWMA Control Chart

The standard EWMA control chart used for detecting small shifts in the process mean is defined as

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t;\ t = 1,\ 2,\ 3,\ \ldots, \tag{1}$$

where Z_t is the EWMA statistic, Y_t is the sequence of the AR(p) process with exponential white noise, and λ is an exponential smoothing parameter $(0 < \lambda \leq 1)$.

The stopping time occurs when an out-of-control observation is firstly detected, which is sufficient to decide that the process is out-of-control. The stopping time τ_b for the standard EWMA control chart can be written as

$$\tau_b = \inf\{t > 0;\ Z_t < a \text{ or } Z_t > b\}, \tag{2}$$

where a is a constant parameter known as the lower control limit (LCL) and b is a constant parameter known as the upper control limit (UCL). The upper side of the ARL for the AR(p) process on the standard EWMA control chart with an initial value $(Z_0 = u)$ can be found. Now, function $L(u)$is defined as

$$L(u) = \text{ARL} = E_\infty(\tau_b) \geq T,\ Z_0 = u, \tag{3}$$

where T is a fixed number (should be large) and $E_\infty(.)$ is the expectation under the assumption that observations ε_t follow an $F(y_t,\ \alpha)$ distribution.

The mean and the variance of the standard EWMA control chart can respectively be written as

$$E(Z_t) = \mu \tag{4}$$

and
$$Var(Z_t) = \left(\frac{\lambda}{2-\lambda}\right)\sigma^2. \tag{5}$$

For the control limit ($\text{CL} = \mu_0$), the UCL and LCL of the standard EWMA control chart are respectively defined as follows:

$$\text{LCL} = \mu_0 - L_1\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{6a}$$

and
$$\text{UCL} = \mu_0 + L_1\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{6b}$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is an appropriate control width limit $(L_1 > 0)$.

2.2 The Modified EWMA Control Chart

Khan et al. [9] developed a new EWMA control chart based upon the modified EWMA statistic of Patel et al. [10] that considers the past and current behavior of the process. This modified EWMA control chart is defined as

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1});\ t = 1,\ 2,\ 3,\ \ldots, \tag{7}$$

where M_t is the modified EWMA statistic, Y_t is the sequence of the AR(p) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \leq 1)$, and k is a constant $(k > 0)$. The stopping time τ_h for the modified EWMA control chart can be written as

$$\tau_h = \inf\{t > 0;\ M_t < g \text{ or } M_t > h\}, \tag{8}$$

where g is the LCL and h is the UCL. The upper side of the ARL for the AR(p) process on the modified EWMA control chart with an initial value ($M_0 = u$) can be found. Now, we define function $G(u)$as

$$\text{ARL} = G(u) = E_\infty(\tau_h) \geq T,\ M_0 = u. \tag{9}$$

The mean and the variance of the modified EWMA control chart are respectively defined as

$$E(M_t) = \mu \tag{10}$$

$$\text{and } Var(M_t) = \frac{(\lambda + 2\lambda k + 2k^2)\sigma^2}{(2-\lambda)}. \tag{11}$$

For the control limit (CL $= \mu_0$), the UCL and LCL of the modified EWMA control chart can respectively be expressed as

$$\text{LCL} = \mu_0 - L_2\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}} \tag{12a}$$

$$\text{and} \quad \text{UCL} = \mu_0 + L_2\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{12b}$$

where $L_2$is an appropriate control width limit $(L_2 > 0)$.

2.3 The Proposed New Modified EWMA Control Chart

The new modified EWMA control chart based on the modified EWMA control chart proposed by Khan et al. [9] is enhanced by adding one more constant to the model, which bestows more importance on current information than on historical information. The new modified EWMA control chart contains three constants: λ, k_1, and k_2 in its derivation. Roberts [6] used $k_1 = k_2 = 0$ in the original EWMA control chart whereas Khan et al. [9] suggested a modified EWMA control chart by assuming that $k_1 = k_2$, and similarly, Patel et al. [10] modified it by applying $k_1 = k_2 = 1$. The new modified EWMA control chart is derived as

$$N_t = (1-\lambda)N_{t-1} + \lambda Y_t + k_1 Y_t - k_2 Y_{t-1};\ t = 1,\ 2,\ 3,\ \ldots, \tag{13}$$

where N_t is the new modified EWMA statistic, Y_t is the sequence of the AR(p) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \leq 1)$, and k_1 and k_2 are constants $(k_1 > k_2 > 0)$.

The stopping time τ_r for the modified EWMA control chart can be written as

$$\tau_r = \inf\{t > 0;\ N_t < l \text{ or } N_t > r\}, \tag{14}$$

where l is the LCL and r is the UCL.

Now, the upper side of the *ARL* for the AR(p) process on the modified EWMA control chart with initial value $N_0 = u$ can be found. First, we define function $H(u)$as

$$\text{ARL} = H(u) = E_\infty(\tau_r) \geq T,\ N_0 = u. \tag{15}$$

The mean and the variance of the new modified EWMA control chart are respectively defined as

$$E(N_t) = (\lambda + k_1 - k_2)\frac{\mu_0}{\lambda} \tag{16}$$

$$\text{and } Var(N_t) = \left[\frac{(\lambda + k_1)^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}\right]\sigma^2. \tag{17}$$

Meanwhile, for control limit CL $= (\lambda + k_1 - k_2)\frac{\mu_0}{\lambda}$, the UCL and LCL of the modified EWMA control chart can respectively be expressed as

$$\text{LCL} = (\lambda + k_1 - k_2)\frac{\mu_0}{\lambda} - L_3\sigma\sqrt{\frac{(\lambda + k_1)^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}} \tag{18a}$$

$$\text{and} \quad \text{UCL} = (\lambda + k_1 - k_2)\frac{\mu_0}{\lambda} + L_3\sigma\sqrt{\frac{(\lambda + k_1)^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}}, \tag{18b}$$

where L_3 is an appropriate control width limit ($L_3 > 0$).

3 Explicit Formulas for the ARL of an AR(p) Process on the New Modified EWMA Control Chart

The AR(p) process is defined as

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t; \ t = 1,\ 2,\ 3,\ \ldots, \tag{19}$$

where δ is a constant ($\delta \geq 0$), ϕ_i is an autoregressive coefficient for $i = 1,\ 2,\ \ldots,\ p$ ($|\phi_p| < 1$), and ε_t is an independent and identically distributed (iid) sequence ($\varepsilon_t \sim Exp(\alpha)$). The initial value for the AR(p) process mean is $Y_{t-1},\ Y_{t-2},\ \ldots,\ Y_{t-p} = 1$.

3.1 The Explicit Formulas

Explicit formulas for the *ARL* of the new modified EWMA control chart for an AR(p) process are derived as follows:

$$N_t = (1-\lambda)Z_{t-1} + (\lambda + k_1)\delta + (\lambda + k_1)\phi_1 Y_{t-1} + \ldots + (\lambda + k_1)\phi_p Y_{t-p} + (\lambda + k_1)\varepsilon_t - k_2 Y_{t-1}.$$

If Y_1 signals the out-of-control state for N_1, $N_0 = u$, then

$$N_1 = (1-\lambda)u + (\lambda + k_1)\delta + (\lambda + k_1)\phi_1 Y_{t-1} + \ldots + (\lambda + k_1)\phi_p Y_{t-p} + (\lambda + k_1)\varepsilon_1 - k_2 v$$

If ε_1 is the in-control limit for N_1, then $l \leq N_1 \leq r$. Consider function $H(u)$

$$H(u) = 1 + \int H(N_1)f(\varepsilon_1)d(\varepsilon_1). \tag{20}$$

Eq. (20) is a Fredholm integral equation of the second kind [17], and thus $H(u)$can be rewritten as

$$H(u) = 1 + \int_l^r L\{(1-\lambda)u + (\lambda + k_1)\delta + (\lambda + k_1)\phi_1 Y_{t-1} + \ldots + (\lambda + k_1)\phi_p Y_{t-p} - k_2 Y_{t-1} + (\lambda + k)y\}f(y)dy.$$

Let $w = (1-\lambda)u + (\lambda + k_1)\delta + (\lambda + k_1)\phi_1 Y_{t-1} + \ldots + (\lambda + k_1)\phi_p Y_{t-p} - k_2 Y_{t-1} + (\lambda + k)y$.

By changing the integral variable, we obtain the following integral equation:

$$H(u) = 1 + \frac{1}{\lambda + k_1} \int_l^r H(w) f\left\{\frac{w - (1-\lambda)u}{(\lambda + k_1)} + \frac{k_2 Y_{t-1}}{(\lambda + k_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}\right\} dw. \tag{21}$$

If $Y_t \sim Exp(\alpha)$ the $f(y) = \frac{1}{\alpha} e^{\frac{-y}{\alpha}}$; $y \geq 0$, then

$$H(u) = 1 + \frac{1}{\lambda + k_1} \int_l^r H(w) \frac{1}{\alpha} e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k_1)} + \frac{k_2 Y_{t-1}}{(\lambda+k_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}\right\}} dw. \tag{22}$$

Let function $C(u) = e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)} - \frac{k_2 Y_{-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}}$, then we have

$$H(u) = 1 + \frac{C(u)}{\alpha(\lambda + k_1)} \int_l^r H(w) e^{\frac{-w}{\alpha(\lambda+k_1)}} dw; \;\; l \leq u \leq r.$$

Let $B = \int_l^r H(w) e^{\frac{-w}{\alpha(\lambda+k_1)}} dw$, then $H(u) = 1 + \frac{C(u)}{\alpha(\lambda+k_1)} \cdot B$. Consequently, we obtain

$$H(u) = 1 + \frac{1}{\alpha(\lambda + k_1)} e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)} - \frac{k_2 Y_{t-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}} \cdot B. \tag{23}$$

By solving for constant B, we obtain

$$B = \int_l^r H(w) e^{\frac{-w}{\alpha(\lambda+k_1)}} dw = \frac{-\alpha(\lambda + k_1)\left(e^{\frac{-r}{\alpha(\lambda+k_1)}} - e^{\frac{-l}{\alpha(\lambda+k_1)}}\right)}{1 + \frac{e^{\frac{-k_2 Y_{t-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda r}{\alpha(\lambda+k_1)}} - e^{\frac{-\lambda l}{\alpha(\lambda+k_1)}}\right)}.$$

By substituting constant B into Eq. (23), we arrive at

$$H(u) = 1 + \frac{e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)} - \frac{k_2 Y_{t-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}}}{\alpha(\lambda + k_1)} \left(\frac{-\alpha(\lambda + k_1)\left[e^{\frac{-r}{\alpha(\lambda+k_1)}} - e^{\frac{-l}{\alpha(\lambda+k_1)}}\right]}{1 + \frac{e^{\frac{-k_2 Y_{t-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}}}{\lambda}\left[e^{\frac{-\lambda r}{\alpha(\lambda+k_1)}} - e^{\frac{-\lambda l}{\alpha(\lambda+k_1)}}\right]}\right). \tag{24}$$

Therefore, the explicit two-sided formulas for the ARL of an AR(p) process running on the new modified EWMA control chart by using the Fredholm integral equation of the second kind can be defined as

$$ARL_{2-sided} = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)}}\left[e^{\frac{-r}{\alpha(\lambda+k_1)}} - e^{\frac{-l}{\alpha(\lambda+k_1)}}\right]}{\lambda e^{\frac{k_2 Y_{t-1}}{\alpha(\lambda+k_1)} - \frac{\delta}{\alpha} - \frac{\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}}{\alpha}} + e^{\frac{-\lambda r}{\alpha(\lambda+k_1)}} - e^{\frac{-\lambda l}{\alpha(\lambda+k_1)}}}. \tag{25}$$

when $l = 0$, the explicit one-sided formulas for the ARL on the new modified EWMA control chart can be written as follows:

$$ARL_{1-sided} = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)}}\left[e^{\frac{-r}{\alpha(\lambda+k_1)}} - 1\right]}{\lambda e^{\frac{k_2 Y_{t-1}}{\alpha(\lambda+k_1)} - \frac{\delta}{\alpha} - \frac{\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}}{\alpha}} + e^{\frac{-\lambda r}{\alpha(\lambda+k_1)}} - 1} \tag{26}$$

3.2 The Existence and Uniqueness of Explicit Formulas

Here, we show the existence and uniqueness of the solution to the integral equation in Eq. (22). First, we define

$$T(H(u)) = 1 + \frac{1}{\lambda + k_1} \int_l^r H(w) \frac{1}{\alpha} e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k_1)} + \frac{k_2 Y_{t-1}}{(\lambda+k_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}\right\}} dw \tag{27}$$

Theorem 1. (Banach's fixed-point theorem [18])

Let $C[l,\ r]$ be a set of all of the continuous functions on complete metric $(X,\ d)$, and assume that $T{:}X \rightarrow X$ is a contraction mapping with contraction constant $0 \le s < 1$; i.e., $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\| \forall H_1,\ H_2 \in X$. Subsequently, $H(.) \in X$ is unique at $T(H(u)) = H(u)$; i.e., it has a unique fixed point in X.

Proof: To show that T defined in Eq. (27) is a contraction mapping for $H_1,\ H_2 \in C[l,\ r]$, we use the inequality $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\| \forall H_1,\ H_2 \in C(l,\ r)$ with $0 \le s < 1$. Consider Eqs. (22) and (27), then

$$\begin{aligned}
\|T(H_1) - T(H_2)\|_\infty &= \sup_{u\in[l,r]} \left| \frac{C(u)}{\alpha(\lambda + k_1)} \int_l^r (H_1(w) - H_2(w)) e^{\frac{-w}{\alpha(\lambda+k_1)}} dw \right| \\
&\le \sup_{u\in[l,r]} \left| \|H_1 - H_2\|_\infty C(u) \left(e^{\frac{-l}{\alpha(\lambda+k_1)}} - e^{\frac{-r}{\alpha(\lambda+k_1)}} \right) \right| \\
&= \|L_1 - L_2\|_\infty \left| e^{\frac{-l}{\alpha(\lambda+k_1)}} - e^{\frac{-r}{\alpha(\lambda+k_1)}} \right| \sup_{u\in[l,r]} |C(u)| \\
&\le s\|L_1 - L_2\|_\infty,
\end{aligned}$$

where $s = \left| e^{\frac{-l}{\alpha(\lambda+k_1)}} - e^{\frac{-r}{\alpha(\lambda+k_1)}} \right| \sup_{u\in[l,r]} |C(u)|$ and $C(u) = e^{\frac{(1-\lambda)u}{\alpha(\lambda+k_1)} - \frac{k_2 Y_{t-1}}{\alpha(\lambda+k_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p}}{\alpha}}$; $0 \le s < 1$.

Therefore, as confirmed by applying Banach's fixed-point theorem, the solution exists and is unique.

4 The NIE for the ARL of an AR(p) Process on the New Modified EWMA Control Chart

The NIE approach is widely used for evaluating the ARL. It can be based on several quadrature rules (midpoint, trapezoidal, Simson's rule, and Gauss-Legendre), all of which give ARLs that are very close to each other [19]. When considering the problem of integrating function *f(w)* over [*l, r*], the interval of integration [*l, r*] is finite when using the midpoint, trapezoidal, and Simpson's rules whereas it is infinite for the Gauss-Legendre rule [13]. Therefore, in this study, we used the Gauss-Legendre rule to evaluate the ARL. An integral equation of the second kind for the ARL on the new modified EWMA control chart for the AR(p) process in Eq. (24) can be approximated by using the quadrature formula. The Gauss-Legendre quadrature rule is applied as follows:

$$\text{Given } f(a_j) = f\left\{ \frac{a_j - (1-\lambda)a_i}{(\lambda + k_1)} + \frac{k_2 Y_{t-1}}{(\lambda + k_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} \right\}. \tag{28}$$

The approximation for the integral is in the form

$$\int_{l}^{r} H(w)f(w)dw \approx \sum_{j=1}^{m} w_j f(a_j), \tag{29}$$

where $a_j = \frac{r-l}{m}\left(j - \frac{1}{2}\right) + l$ and $w_j = \frac{r-l}{m};\ j = 1,\ 2,\ \ldots,\ m$

Using the Gauss-Legendre quadrature formula, numerical approximation $\tilde{H}(u)$ for the integral equation can be found as the solution for the following linear equations:

$$\tilde{H}(u) = 1 + \frac{1}{\lambda + k_1}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left\{\frac{a_j - (1-\lambda)u}{(\lambda + k_1)} + \frac{k_2 Y_{t-1}}{(\lambda + k_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p}\right\}. \tag{30}$$

5 Comparison of the Efficacies of the NIE Method and the Explicit Formulas

Here, the details of a simulation study to compare the efficacies of the NIE method $(\tilde{H}(u))$ and the explicit formulas $(H(u))$ for the ARL of an AR(p) process on the new modified EWMA control chart are provided. The parameter values were set as $ARL_0 = 370$; λ =0.05 or 0.1; in-control parameter $\alpha_0 = 1$; and a shift size of 0.001, 0.005, 0.01, 0.03, 0.05, 0.07, 0.1, 0.2, or 0.3. The absolute percentage difference between the ARL methods is defined as

$$Diff(\%) = \frac{|H(u) - \tilde{H}(u)|}{H(u)} \times 100. \tag{31}$$

Eqs. (24) and (30) were used to evaluate the ARL of the AR(p) process with exponential white noise on the new modified EWMA control chart. The number of nodes equal to 1000 iterations was used to obtain the ARL results from the NIE method. The results are reported in Tabs. 1 and 2.

Table 1: One-sided comparison of the ARL derived using explicit formulas and the NIE method for an AR(1) process on the new modified EWMA control chart with $\delta = 2,\ k_1 = 1,$ and $k_2 = 0.5$

λ	ϕ	r	Shift	Explicit	Time[a]	NIE	Time	Diff%
			0.00	370.0016295659	<0.01	370.0016288807	9.969	0.00000019
			0.001	256.2864996053	<0.01	256.2864991740	10.016	0.00000017
			0.005	114.9134566837	<0.01	114.9134565160	9.718	0.00000015
			0.01	67.9834203601	<0.01	67.9834202668	10.031	0.00000014
	0.2	0.18698742	0.03	25.7862742391	<0.01	25.7862742074	10.391	0.00000012
			0.05	15.9140681363	<0.01	15.9140681181	10.296	0.00000011
			0.07	11.5230080802	<0.01	11.5230080679	10.204	0.00000011
			0.10	8.1762920907	<0.01	8.1762920828	10.188	0.00000010
			0.20	4.2605039280	<0.01	4.2605039250	10.172	0.00000007
0.05			0.30	2.9901442894	<0.01	2.9901442878	10.079	0.00000005
			0.00	370.0021173682	<0.01	370.0021158170	10.109	0.00000042
			0.001	265.5101680556	<0.01	265.5101670391	9.563	0.00000038
			0.005	124.6480713761	<0.01	124.6480709630	9.906	0.00000033
			0.01	74.9296968644	<0.01	74.9296966319	10.125	0.00000031

(Continued)

Table 1 (continued)

λ	ϕ	r	Shift	Explicit	Time[a]	NIE	Time	Diff%
	−0.2	0.27963495	0.03	28.8629221195	<0.01	28.8629220393	10.046	0.00000028
			0.05	17.8886497252	<0.01	17.8886496790	10.297	0.00000026
			0.07	12.9802917477	<0.01	12.9802917163	10.250	0.00000024
			0.10	9.2254646270	<0.01	9.2254646067	10.046	0.00000022
			0.20	4.8072341584	<0.01	4.8072341505	10.125	0.00000016
			0.30	3.3605755174	<0.01	3.3605755132	10.297	0.00000012

Note: [a]The computations for the explicit and NIE methods were carried out on a Windows 10 Professional with RAM of 8 GB and an Intel Core i5 CPU.

Table 2: Two-sided comparison of the ARL derived using explicit formulas and the NIE method for an AR(3) process on the new modified EWMA control chart with $\delta = 2,\ k_1 = 3,\ k_2 = 2,\ \phi_1 = 0.4,$ and $\phi_2 = -0.2$

λ	ϕ_3	l	r	Shift	Explicit	Time	NIE	Time	Diff%
0.10	0.3	0.1	0.58889287	0.00	370.0044182916	<0.01	370.0044167443	9.734	0.00000042
				0.001	198.2939468455	<0.01	198.2939463089	10.640	0.00000027
				0.005	69.7755200152	<0.01	69.7755199045	11.531	0.00000016
				0.01	38.7890925696	<0.01	38.7890925192	11.000	0.00000013
				0.03	14.3211977680	<0.01	14.3211977535	9.781	0.00000010
				0.05	8.9927022553	<0.01	8.9927022473	10.781	0.00000009
				0.07	6.6635363431	<0.01	6.6635363378	10.844	0.00000008
				0.10	4.9013616591	<0.01	4.9013616557	10.562	0.00000007
				0.20	2.8408720199	<0.01	2.8408720186	10.344	0.00000005
				0.30	2.1630741647	<0.01	2.1630741640	10.297	0.00000003
	−0.3	0.1	0.99684163	0.00	370.0042221823	<0.01	370.0042167448	10.141	0.00000147
				0.001	214.9980933489	<0.01	214.9980912107	10.359	0.00000099
				0.005	80.7227527927	<0.01	80.7227523255	9.922	0.00000058
				0.01	45.5958782733	<0.01	45.5958780613	11.266	0.00000046
				0.03	17.0250185306	<0.01	17.0250184705	10.828	0.00000035
				0.05	10.6997523123	<0.01	10.6997522790	11.125	0.00000031
				0.07	7.9214725775	<0.01	7.9214725551	10.703	0.00000028
				0.10	5.8124775312	<0.01	5.8124775167	10.125	0.00000025
				0.20	3.3324603669	<0.01	3.3324603611	10.188	0.00000017
				0.30	2.5081216551	<0.01	2.5081216520	10.109	0.00000012

From the results in Tabs. 1 and 2, we can see that the ARL values derived by using the explicit formulas were the same as those of the NIE method, with the numerical approximations having an absolute percentage difference of less than 0.00001%. However, the computational time for the NIE method was 9.563 s–11.531 s whereas that for the explicit formulas was less than 1 s.

6 Comparison of the ARL Derived Using Explicit Formulas

After verifying the accuracy of the explicit formulas, we used simulated data and the relative mean index (*RMI*) to compare the performances of the ARL derived using explicit formulas for an AR(p) process on standard, modified, and new modified EWMA control charts. The RMI is defined as

$$RMI(r) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_i(r) - Min[ARL_i(s)]}{Min[ARL_i(s)]}\right), \tag{32}$$

where $ARL_i(r)$ is the ARL of the control chart for the shift size in row *i* and $Min[ARL_i(s)]$ denotes the smallest ARL of the three control charts in comparison to the shift size in row *i*, for $i = 1,\ 2,\ \ldots,\ n$. The control chart with the smallest RMI is the best at detecting changes in the process mean for a particular set of criteria.

For the one-sided comparison of the ARL for an AR(1) process on the standard, modified, and new modified EWMA control charts, the parameter values were set as $ARL_0 = 370$; $\lambda = 0.05$ or 0.1; in-control parameter $\alpha_0 = 1$; and a shift size of 0.001, 0.005, 0.01, 0.03, 0.05, 0.07, 0.1, 0.2, or 0.3. The results are reported in Tab. 3.

Table 3: One-sided comparison of the ARL for the AR(1) process on standard, modified, and new modified EWMA control charts with $\delta = 2,$ and $\phi = 0.2$

λ	Shift	EWMA	Modified EWMA	New modified EWMA ($k_1 = 2$)			
			$k = 2$	$k_2 = 1.6$	$k_2 = 1.2$	$k_2 = 0.8$	$k_2 = 0.4$
		$b = 1.1454 \times 10^{-8}$	$h = 0.604752918$	$r = 0.49685332$	$r = 0.408306731$	$r = 0.335609146$	$r = 0.27590156$
	0.00	370	370	370	370	370	370
	0.001	361.9102	232.6537	227.3313	222.1699	217.1575	212.2882
	0.005	331.4143	93.8920	89.6714	85.7573	82.1143	78.7156
	0.01	297.1766	53.9851	51.2257	48.6989	46.3746	44.2296
0.05	0.03	194.2090	20.2956	19.1553	18.1228	17.1829	16.3239
	0.05	129.0593	12.6838	11.9586	11.3040	10.7100	10.1686
	0.07	87.1623	9.3238	8.7882	8.3057	7.8686	7.4710
	0.10	49.8241	6.7677	6.3796	6.0307	5.7153	5.4290
	0.20	9.9814	3.7609	3.5524	3.3659	3.1982	3.0467
	0.30	3.1300	2.7652	2.6196	2.4899	2.3739	2.2695
RMI		**5.9692**	**0.2164**	**0.1544**	**0.0981**	**0.0469**	**0.0000**
		$b = 0.000483728$	$h = 0.609657639$	$r = 0.502604301$	$r = 0.414540081$	$r = 0.342035616$	$r = 0.28230037$
	0.00	370	370	370	370	370	370
	0.001	365.4813	229.8330	224.4713	219.2918	214.2793	209.4254
	0.005	348.0222	91.6510	87.5021	83.6682	80.1106	76.8000
	0.01	327.5441	52.5314	49.8378	47.3793	45.1238	43.0470
0.1	0.03	258.5037	19.7180	18.6118	17.6130	16.7059	15.8783
	0.05	205.8593	12.3305	11.6282	10.9960	10.4234	9.9024
	0.07	165.3390	9.0725	8.5543	8.0886	7.6676	7.2851
	0.10	120.8131	6.5950	6.2199	5.8834	5.5797	5.3044
	0.20	47.6711	3.6813	3.4801	3.3004	3.1390	2.9934
	0.30	21.9089	2.7162	2.5757	2.4509	2.3392	2.2388
RMI		**12.5708**	**0.2144**	**0.1528**	**0.0971**	**0.0463**	**0.0000**

For the two-sided comparison of the ARL for an AR(2) process on the three control charts, the parameter values and the shift sizes were the same as for the one-sided comparison. The results are reported in Tab. 4.

Table 4: Two-sided comparison of the ARL for an AR(2) process on standard, modified, and new modified EWMA control charts with $\delta = 2,\ \phi_1 = 0.2,$ and $\phi_2 = -0.1$

λ	Shift	EWMA	Modified EWMA	New modified EWMA ($k_1 = 3,\ l = 0.1$)			
		$a = 0.1$	$k = 3,\ g = 0.1$	$k_2 = 2$	$k_2 = 1.5$	$k_2 = 1$	$k_2 = 0.5$
		$b = 0.1000000935342$	$h = 1.10555778$	$r = 0.82269292$	$r = 0.71283641$	$r = 0.619754113$	$r = 0.54086378$
	0.00	370	370	370	370	370	370
	0.001	362.6678	219.8254	210.4749	206.0100	201.6722	197.4560
	0.005	334.8841	84.1605	77.6066	74.6460	71.8670	69.2534
	0.01	303.3988	47.7752	43.6287	41.7827	40.0651	38.4630
0.05	0.03	206.3952	17.8986	16.2260	15.4907	14.8119	14.1832
	0.05	142.5270	11.2492	10.1883	9.7236	9.2955	8.9000
	0.07	99.8509	8.3243	7.5401	7.1975	6.8823	6.5914
	0.10	60.0940	6.1019	5.5321	5.2837	5.0555	4.8454
	0.20	13.7451	3.4853	3.1752	3.0410	2.9182	2.8056
	0.30	4.4582	2.6138	2.3947	2.3004	2.2144	2.1359
RMI		**7.8516**	**0.2316**	**0.1276**	**0.0817**	**0.0393**	**0.0000**
		$b = 0.101371684$	$h = 1.115786514$	$r = 0.83228197$	$r = 0.72207165$	$r = 0.62859342$	$r = 0.54926663$
	0.00	370	370	370	370	370	370
	0.001	365.8141	218.6787	209.2031	204.7010	200.3398	196.1112
	0.005	349.6039	83.3369	76.7611	73.8064	71.0411	68.4469
	0.01	330.5169	47.2552	43.1063	41.2691	39.5645	37.9783
0.1	0.03	265.5012	17.6958	16.0271	15.2973	14.6253	14.0043
	0.05	215.0886	11.1253	10.0681	9.6073	9.1840	8.7936
	0.07	175.6468	8.2360	7.4553	7.1157	6.8042	6.5172
	0.10	131.4553	6.0410	5.4742	5.2283	5.0029	4.7957
	0.20	55.7018	3.4567	3.1492	3.0164	2.8953	2.7844
	0.30	27.1233	2.5960	2.3790	2.2858	2.2011	2.1237
RMI		**15.2481**	**0.2325**	**0.1277**	**0.0817**	**0.0393**	**0.0000**

From the results in Tabs. 3 and 4, it is evident that the ARL values derived by using the explicit formulas for the new modified EWMA control chart are smaller than those for the standard and modified EWMA control charts for all shift sizes and λ for $k_1 > k_2$, and thus the RMI values of the ARL on the new modified EWMA control chart were smaller than those for the standard and modified EWMA control charts for all λ.

The property of the new modified EWMA control chart ensured that the ARL decreased as k_1 was increased for $k_1 > 1$, and so the ARL value obtained by using the explicit formulas for the new modified EWMA control chart was lower than those for the standard and modified EWMA control charts under each set of conditions. For $k_2 < k_1$, the ARL value decreased as k_2 became smaller (i.e., the importance of the historical information was reduced), which made the new modified EWMA control chart more efficient at detecting changes than the standard and modified EWMA control charts. Finally, the ARL was reduced as λ was increased.

7 Practical Applications

To confirm the results of the simulation study, we applied the explicit formulas for the ARL of an AR(1) process involving 72 real data observations of the price of crude oil (Unit: US Dollars per barrel) from January 2015 to December 2020 (data from the West Texas Intermediate [20]) on the standard, modified, and new modified EWMA control charts. The parameters were set as λ =0.05 or 0.1; $\alpha_0 = 3.2028$; $\delta = 50.6834$; $\phi_1 = 0.8750$; and a shift size of 0.0005, 0.001, 0.003, 0.005, 0.007, 0.01, 0.05, 0.1, 0.2, or 0.3. The results are summarized in Tab. 5.

Table 5: Two-sided comparison of the ARL for the AR(1) process for the price of crude oil on standard, modified, and new modified EWMA control charts for $ARL_0 = 370$

λ	Shift	EWMA	Modified EWMA	New modified EWMA ($k_1 = 2,\ l = 0.1$)			
		$a = 0.1$	$k = 2,\ g = 0.1$	$k_2 = 1.6$	$k_2 = 1.2$	$k_2 = 0.8$	$k_2 = 0.4$
		b = 0.10000000000390551	h = 0.100002254057	r = 0.1000003624251	r = 0.1000000582736	r = 0.100000009369687	r = 0.10000000150653
	0.0000	370	370	370	370	370	370
	0.0005	365.8525	205.2767	195.2779	186.2230	177.9507	170.4133
	0.001	361.6463	142.1241	132.7348	124.5198	117.2545	110.8132
	0.003	345.6715	63.9040	58.3806	53.7426	49.7917	46.3764
	0.005	330.7717	41.3439	37.5461	34.3940	31.7350	29.4609
0.05	0.007	316.8048	30.6249	27.7398	25.3585	23.3590	21.6565
	0.010	297.5174	22.1160	19.9976	18.2567	16.8004	15.5645
	0.050	148.9292	5.0514	4.5852	4.2058	3.8912	3.6260
	0.100	76.8896	2.8206	2.5856	2.3953	2.2382	2.1065
	0.200	27.2089	1.7266	1.6125	1.5214	1.4472	1.3861
	0.300	11.6765	1.3852	1.3138	1.2580	1.2135	1.1777
RMI		**15.4952**	**0.3257**	**0.2214**	**0.1349**	**0.0621**	**0.0000**
		b = 0.1000000000118177	h = 0.100001858937	r = 0.1000003121884	r = 0.10000005242868	r = 0.100000008804831	r = 0.100000001478671
	0.0000	370	370	370	370	370	370
	0.0005	330.6441	203.1446	193.4803	184.7025	176.6885	169.3114
	0.001	298.7322	140.0937	131.0839	123.1683	116.1538	109.8300
	0.003	214.9672	62.6915	57.4384	53.0032	49.2083	45.9224
	0.005	167.4812	40.5081	36.9051	33.8968	31.3467	29.1541
0.1	0.007	136.7188	29.9899	27.2560	24.9854	23.0693	21.4283
	0.010	106.7896	21.6504	19.6450	17.9863	16.5915	15.4018
	0.050	24.2697	4.9529	4.5124	4.1514	3.8503	3.5954
	0.100	10.6448	2.7731	2.5512	2.3702	2.2199	2.0933
	0.200	4.0887	1.7051	1.5975	1.5109	1.4400	1.3813
	0.300	2.2907	1.3724	1.3052	1.2522	1.2098	1.1754
RMI		**3.5157**	**0.3140**	**0.2141**	**0.1309**	**0.0605**	**0.0000**

Table 6: Two-sided comparison of the ARL of the AR(2) process for the price of rubber running on standard, modified, and new modified EWMA control charts for $ARL_0 = 370$

λ	Shift	EWMA	Modified EWMA	New modified EWMA ($k_1 = 3,\ l = 0.1$)			
		$a = 0.1$	$k = 3,\ g = 0.1$	$k_2 = 2$	$k_2 = 1.5$	$k_2 = 1$	$k_2 = 0.5$
		b = 0.100000000000139998	h = 0.100000073991	r = 0.10000000268798	r = 0.100000000512329	r = 0.1000000000976499	r = 0.1000000000186121
	0.00000	370	370	370	370	370	370
	0.00001	362.1586	236.1255	219.9381	212.6027	205.8036	199.8819
	0.00003	346.1650	137.0466	121.4735	114.9273	109.0444	104.0794
	0.00005	331.9876	96.6312	84.0090	78.8578	74.3121	70.2980
	0.00007	318.1125	74.6795	64.2644	60.0751	56.4003	53.1342
0.05	0.00010	300.0840	55.7570	47.5727	44.3238	41.5003	39.0026
	0.00050	165.3761	13.0791	11.0028	10.2018	9.5141	8.9178
	0.00100	100.9062	6.9224	5.8486	5.4362	5.0831	4.7774
	0.00200	51.5182	3.7743	3.2283	3.0195	2.8413	2.6873
	0.00300	31.5322	2.7219	2.3574	2.2186	2.1005	1.9988
RMI		**9.9060**	**0.3766**	**0.1923**	**0.1191**	**0.0554**	**0.0000**
		b = 0.100000000000310172	h = 0.1000000650949	r = 0.100000002494682	r =.0.100000000488371	r = 0.1000000000956057	r = 0.10000000001871652
	0.00000	370	370	370	370	370	370
	0.00001	321.5076	234.9812	219.0408	211.9030	205.0963	199.5842
	0.00003	252.2377	135.9119	120.7211	114.3536	108.5909	103.4988
	0.00005	208.4280	95.6981	83.4219	78.4083	73.9497	70.0573
	0.00007	177.2149	73.9035	63.7891	59.7133	56.1250	53.0099
0.1	0.00010	144.4376	55.1433	47.2053	44.0444	41.2793	38.8710
	0.00050	40.7542	12.9227	10.9143	10.1354	9.4648	8.8826
	0.00100	20.6675	6.8424	5.8041	5.4031	5.0587	4.7601
	0.00200	9.8227	3.7346	3.2067	3.0037	2.8299	2.6795
	0.00300	6.1586	2.6959	2.3436	2.2087	2.0935	1.9941
RMI		**2.3074**	**0.3675**	**0.1881**	**0.1167**	**0.0542**	**0.0000**

We also carried out another comparison for the ARL of an AR(2) process using 72 real data observations of the price of rubber (Unit: US Dollars per kilogram) from January 2015 to December 2020 (Singapore Exchange Ltd. (SGX) [21]) on the standard, modified, and new modified EWMA control charts. The parameters were set as λ =0.05 or 0.1; $\alpha_0 = 0.0989$; $\delta = 1.6660$; $\phi_1 = 1.2821$; $\phi_2 = -0.4578$; and a shift size of 0.00001, 0.00003, 0.00005, 0.00007, 0.0001, 0.0005, 0.001, 0.002, or 0.003. The results are summarized in Tab. 6.

From the results using real data in Tabs. 5 and 6, it is evident that the ARL values derived by using the explicit formulas for the new modified EWMA control chart were less than those for the standard and modified EWMA control charts for all shift sizes and λ for $k_1 > k_2$. This corresponds to the RMI values for the new modified EWMA control chart being less than those for the standard and modified EWMA control charts for all k_2; $k_1 > k_2$. In addition, as k_2 decreased, the ARL_1 and the RMI decreased. Detection of shifts in the means of the AR(1) and AR(2) processes with real data on the three types of EWMA control charts are plotted in Figs. 1 and 2, respectively.

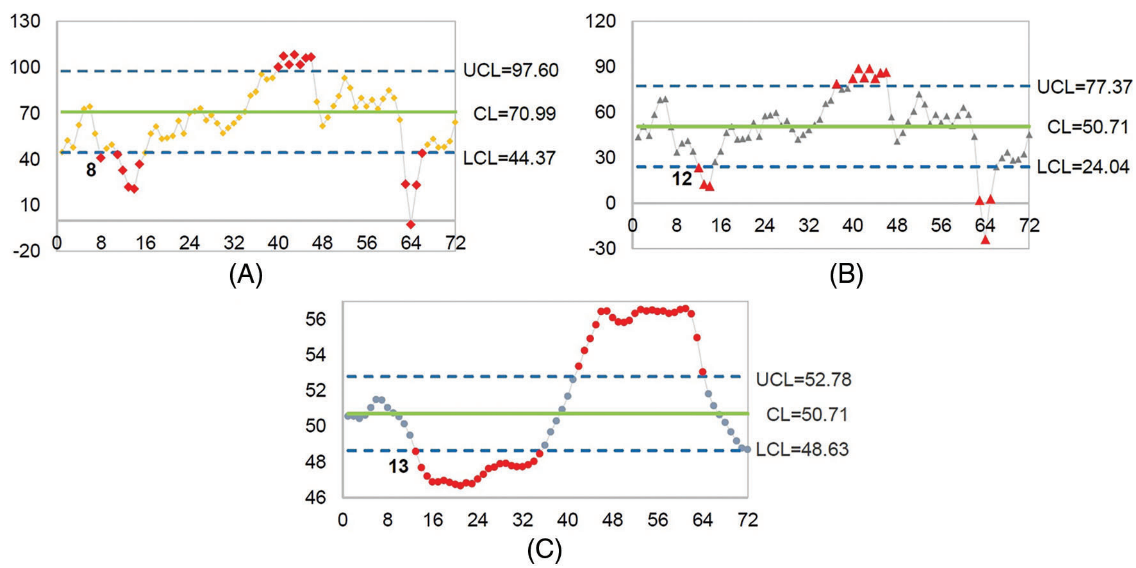

Figure 1: Mean shift detection for the AR(1) process for the price of crude oil. (A) The new modified EWMA control chart, (B) The modified EWMA control chart, and (C) The standard EWMA control chart

The results in Fig. 1 indicate that the new modified EWMA control chart could detect a change in the price of crude oil for the first time at the 8th observation, while the standard and modified EWMA control charts achieved this at the 13th and 12th observations, respectively.

The results in Fig. 2 show that the new modified EWMA control chart could detect the price of rubber at the 8th observation for the first time whereas the standard and the modified EWMA control charts could only do so at the 12th and 9th observations, respectively. Hence, in both cases, detecting a shift in the process mean by the new modified EWMA control chart was sooner than either the standard or modified EWMA control charts, and therefore, it performed better.

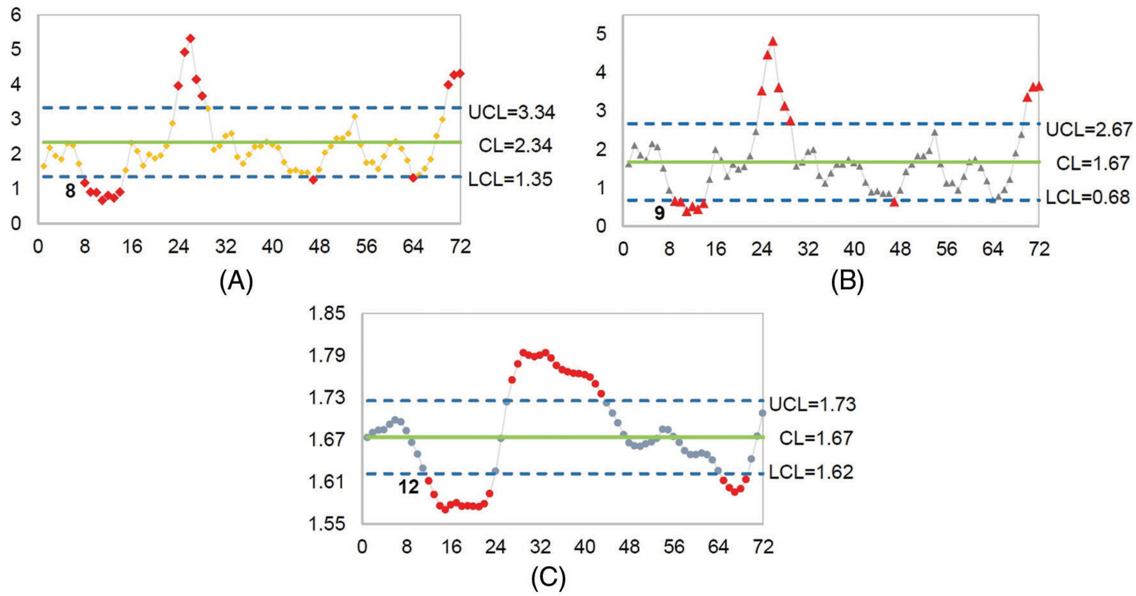

Figure 2: Mean shift detection of the AR(2) process for the price of rubber. (A) The new modified EWMA control chart, (B) The modified EWMA control chart, and (C) The standard EWMA control chart

8 Conclusions

A new modified EWMA control chart to detect a change in the process mean of an AR(p) process with exponential white noise was proposed. We derived explicit formulas for the ARL on the new modified EWMA control chart and checked its accuracy by comparing its absolute percentage difference with the widely used NIE method via a simulation study. The results show that although both methods were highly consistent with an absolute percentage difference of less than 0.00001%, the explicit formula method could be computed much more quickly. A comparison of the ARL derived by using explicit formulas on standard, modified, and new modified EWMA control charts shows that the proposed control chart was more efficacious than the others in terms of RMI. Application of the proposed control chart for AR(p) processes with exponential white noise using real data observations and a comparison of its performance with the standard and modified EWMA control charts show that the new modified EWMA control chart performed better than the others for a two-sided shift with all of the smoothing parameter values tested. In addition, as k_2 decreased, its ARL_1 and the RMI decreased. Based on the findings, the explicit formulas for the ARL of an AR(p) process with exponential white noise detected a change in the process mean more quickly on the new modified EWMA control chart than on the standard and modified EWMA control charts. Although the conclusions drawn from the results of this study are only applicable to AR(p) processes, it would be interesting to discover whether our approach is relevant for others, especially where autoregression is involved.

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Appendix A. The Mean and the variance of the new modified EWMA control chart

The new modified EWMA control chart based on the modified EWMA control chart proposed by Khan et al. [9] from Eq. (13) is defined as

$$N_t = (1-\lambda)N_{t-1} + \lambda Y_t + k_1 Y_t - k_2 Y_{t-1};\ t = 1,\ 2,\ 3,\ \ldots,$$

where k_1 and k_2 are constants ($k_1 > k_2 > 0$).

The mean of the new modified EWMA control statistic is $E(N_t) = [\lambda + k_1 - k_2]\frac{\mu_0}{\lambda}$. It may be shown that

$$\begin{aligned} N_t &= (1-\lambda)N_{t-1} + \lambda Y_t + k_1 Y_t - k_2 Y_{t-1} \\ &= (1-\lambda)^3 N_{t-3} + (1-\lambda)^2[\lambda Y_{t-2} + k_1 Y_{t-2} - k_2 Y_{t-3}] + (1-\lambda)[\lambda Y_{t-1} + k_1 Y_{t-1} - k_2 Y_{t-2}] + \lambda Y_t + k_1 Y_t - k_2 Y_{t-1} \\ &= (1-\lambda)^3 N_{t-3} + \lambda(1-\lambda)^2 Y_{t-2} + \lambda(1-\lambda)Y_{t-1} + \lambda Y_t + (1-\lambda)^2 k_1 Y_{t-2} + (1-\lambda)k_1 Y_{t-1} \\ &\quad + (1-\lambda)^0 k_1 Y_t - (1-\lambda)^2 k_2 Y_{t-3} - (1-\lambda)k_2 Y_{t-2} - (1-\lambda)^0 k_2 Y_{t-1}, \end{aligned}$$

and continuing like this recursively for $Y_{t-j};\ j = 1,\ 2,\ 3,\ \ldots,\ t$, we obtain

$$N_t = (1-\lambda)^t N_0 + \sum_{j=0}^{t-1}(1-\lambda)^j[(\lambda + k_1)Y_{t-j} - k_2 Y_{t-j-1}]$$

Hence, $\sum_{j=0}^{t-1}(1-\lambda)^j[(\lambda + k_1)Y_{t-j} - k_2 Y_{t-j-1}]$ accounts for sum of the past and latest change in the process.

The unaccounted current fluctuations accumulated to time t in new modified EWMA statistic.

Let $N_t = (1-\lambda)^t N_0 + \sum_{j=0}^{t-1}(1-\lambda)^j[(\lambda + k_1)Y_{t-j} - k_2 Y_{t-j-1}]$.

Take the expectation on both sides, we have

$$\begin{aligned} E(N_t) &= (1-\lambda)^t E(N_0) + \sum_{j=0}^{t-1}(1-\lambda)^j E[(\lambda + k_1)Y_{t-j} - k_2 Y_{t-j-1}] \\ &= (1-\lambda)^t \mu_0 + \left[\frac{1-(1-\lambda)^t}{1-(1-\lambda)}\right][(\lambda + k_1)\mu_0 - k_2\mu_0] = \mu_0 + \frac{1}{\lambda}[k_1\mu_0 - k_2\mu_0];\ t \to \infty \\ &= [\lambda + k_1 - k_2]\frac{\mu_0}{\lambda} \end{aligned}$$

And the variance is $Var(N_t) = \left[\frac{(\lambda+k_1)^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}\right]\sigma^2$. The derive of the variance of N_t is

$$N_t = (1-\lambda)^t N_0 + (\lambda + k_1)\sum_{j=0}^{t-1}(1-\lambda)^j Y_{t-j} - k_2\sum_{j=0}^{t-1}(1-\lambda)^j Y_{t-j-1}$$

$$Var(N_t) = (1-\lambda)^{2t} Var(N_0) + (\lambda + k_1)^2 \sum_{j=0}^{t-1} (1-\lambda)^{2j} Var(Y_{t-j}) + k_2^2 \sum_{j=0}^{t-1} (1-\lambda)^{2j} Var(Y_{t-j-1})$$

$$+ 2(\lambda + k_1)(-k_2) Cov\left[\sum_{j=0}^{t-1} (1-\lambda)^j Y_{t-j},\ \sum_{j=0}^{t-1} (1-\lambda)^j Y_{t-j-1}\right]$$

$$= (1-\lambda)^{2t}\sigma^2 + (\lambda + k_1)^2 \sum_{j=0}^{t-1} (1-\lambda)^{2j}\sigma^2$$

$$+ k_2^2 \sum_{j=0}^{t-1} (1-\lambda)^{2j}\sigma^2 - 2(\lambda + k_1)k_2 \sum_{j=0}^{t-1} (1-\lambda)^{2j+1} Cov(Y_{t-j},\ Y_{t-j-1})$$

$$= (1-\lambda)^{2t}\sigma^2 + (\lambda + k_1)^2\sigma^2\left[\frac{1-(1-\lambda)^{2t}}{\lambda(2-\lambda)}\right] + k_2^2\sigma^2\left[\frac{1-(1-\lambda)^{2t}}{\lambda(2-\lambda)}\right] - 2(\lambda + k_1)k_2 \sum_{j=0}^{t-1} (1-\lambda)^{2j+1}\rho\sigma\sigma$$

$$= \left[\frac{(\lambda + k_1)^2}{\lambda(2-\lambda)} + \frac{k_2^2}{\lambda(2-\lambda)} - \frac{2(\lambda + k_1)k_2(1-\lambda)}{\lambda(2-\lambda)}\right]\sigma^2$$

when $t \to \infty,\ \rho \to 1$

$$= \left[\frac{\lambda^2 + 2\lambda k_1 + k_1^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}\right]\sigma^2$$

Therefore, the mean and the variance of the new modified EWMA control chart are respectively defined as

$E(N_t) = (\lambda + k_1 - k_2)\frac{\mu_0}{\lambda}$, and $Var(N_t) = \left[\frac{(\lambda + k_1)^2 + k_2^2 - 2\lambda k_2 + 2\lambda^2 k_2 - 2k_1 k_2 + 2\lambda k_1 k_2}{\lambda(2-\lambda)}\right]\sigma^2.$