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Robust Frequency Estimation Under Additive Symmetric α-Stable Gaussian Mixture Noise

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Abstract: Here the estimating problem of a single sinusoidal signal in the additive symmetric α-stable Gaussian (ASαSG) noise is investigated. The ASαSG noise here is expressed as the additive of a Gaussian noise and a symmetric α-stable distributed variable. As the probability density function (PDF) of the ASαSG is complicated, traditional estimators cannot provide optimum estimates. Based on the Metropolis-Hastings (M-H) sampling scheme, a robust frequency estimator is proposed for ASαSG noise. Moreover, to accelerate the convergence rate of the developed algorithm, a new criterion of reconstructing the proposal covariance is derived, whose main idea is updating the proposal variance using several previous samples drawn in each iteration. The approximation PDF of the ASαSG noise, which is referred to the weighted sum of a Voigt function and a Gaussian PDF, is also employed to reduce the computational complexity. The computer simulations show that the performance of our method is better than the maximum likelihood and the lp-norm estimators.

Keywords: Additive symmetric α-stable Gaussian mixture; metropolis-hastings algorithm; robust frequency estimation; probability density function approximation

1 Introduction

In the real-world applications, impulsive noise is commonly come across, especially in wireless communication or/and image processing [1–7]. Among these heavy-tailed noise models, α-stable [7], Student's t and Laplace distributions [8–13] are typical ones, whose probability density function (PDF) are usually described by a single known mathematical function. Furthermore, mixture noise models are proposed, which are Gaussian mixture and Cauchy Gaussian mixture [14–18]. However, all these noise models cannot represent the special noise type in some real-world applications like the astrophysical imaging processing [19] and multi-user communication network [20]. Take the astrophysical imaging processing as an example, the encountered noise is described as a variable following symmetric α-stable (SαS) distribution and a Gaussian distributed variable, known as additive symmetric α-stable [21] Gaussian (ASαSG) mixture noise. Here SαS is due to galactic radiation and the Gaussian noise is caused by the antenna of the satellite [22].

In the paper, the estimation problem is investigated for a single sinusoid signal embedded with the ASαSG mixture noise. As the PDF of the SαS noise cannot be written as an closed-form function, the PDF of ASαSG distribution, obtained by the convolution of the PDF of SαS distribution and the PDF of Gaussian distributions, cannot be expressed in an analytical form. Therefore, traditional estimators like maximum likelihood estimator (MLE), cannot provide the optimal and stable estimates. Therefore, to fix the estimation problem, we adopt a Markov chain Monte Carlo (MCMC) method, which can sample from a simple conditional distribution of a stable Markov chain [22], instead of a complicated target PDF. Since the conditional distribution is difficult to choose, the Metropolis-Hastings (M-H) method [23–27] is proposed, which draw samples from any simple distributions with a constraint of an acceptance ratio [28]. As only the conditional PDF of a stable Markov chain corresponds to the target PDF, the convergence of the chain influences the computational complexity of the proposed method. In order to improve computational cost, a proposal covariance reconstruction method is proposed, which iteratively updates the proposal variance with the residuals between adjacent samples. Here we consider an independent-parameter estimation problem, so the proposal covariance is defined as a diagonal matrix, with all non-zero elements being candidate proposal variances. To further reduce the complexity caused by the PDF of the ASαSG, the approximation of the SαS [22,29–33] is utilized, which is a weighted sum of a Cauchy PDF and a Gaussian PDF. And hence the PDF of ASαSG noise can be simplified as the sum of the Voigt profile [34,35] and a normal distribution. It is also worth to point out that our work is a generalization of [36,37], which consider the additive Gaussian and Cauchy noise ($\alpha = 1$).

The rest of this paper is organized as follows. Section 2 reviews the main idea of the M-H algorithm. The PDF approximation of the ASαSG is shown in Section 3. In Section 4, the proposed method is then given in detail, where development of the new proposed covariance updating criterion is also provided. Computer simulations are conducted in Section 5 to verify the robust of the proposed scheme. In Section 6, conclusions are drawn.

2 The M-H Sampling Method

A Markov chain [38–41] can be defined by a series of random variables $\{x_k\}$, which is

$$x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, \tag{1}$$

where x_{k+1} relies only on x_k, and the conditional PDF is expressed as $p(x_{k+1}|x_k)$.

Denote the PDF of x_{k+1} as $f(x_{k+1})$. The Markov chain is assumed to be stationary when

$$f(x) = \lim_{k\to\infty} p(x_{k+1}|x_k) f(x_k), \tag{2}$$

is satisfied with $f(x)$ being defined as $\lim_{k\to\infty} f(x_k)$. That is to say, for a stable Markov chain, with stationary PDF $f(x)$, the variables produced by $p(x_{k+1}|x_k)$ will be eventually tend to be sampled from $f(x)$. Therefore, to obtain a proper Markov chain, the choice of $p(x_{k+1}|x_k)$ is important and difficult in the real-world applications.

However, in some scenarios of complicated target PDF, the proper conditional PDF $p(x_{k+1}|x_k)$ of the chain is difficult to be selected. To avoid the choose of the conditional PDF, the M-H algorithm [42,43] is developed, which is to draw samples from a proposal distribution with a constraint of a rejection criterion. Denote the sample from the proposal distribution $q(x^*|x_k)$ as x^*, which is the candidate of the Markov chain. The rejection criterion is also required to determine whether this candidate is accepted as the member of the chain or not. The acceptance probability [28], referred to as $\mathcal{A}(x_k, x^*)$ is utilized to describe the rejection criteria, with the definition of

$$\mathcal{A}(x_k, x^*) = \min\left\{1, \frac{q(x_k|x^*)f(x^*)}{q(x^*|x_k)f(x_k)}\right\}. \quad (3)$$

Usually, the $q(x^*|x_k)$ are chosen as some simple distributions, such as uniform or/and Gaussian. Then the steps of the M-H method are described in Tab. 1.

Table 1: The M-H algorithm

(1) Initialize the sample x_1
(2) Draw sample u from $U(0, 1)$
(3) Sample a candidate x^* from the distribution $q(x^*\|x_k)$
(4) Calculate the acceptance ratio $\mathcal{A}(x_k, x^*)$ using the definition in (3)
(5) Rejection Criterion:
If $u < \mathcal{A}(x_k, x^*)$
$x_{k+1} = x^*$
else
$x_{k+1} = x_k$
(vi) Repeat steps (2)–(5) until a large number of iterations
(vii) Discard some samples before convergence

3 The PDF Approximation of Mixture Noise

The ASαSG noise q can be modelled as:

$$q = e + g, \quad (4)$$

where e denotes the SαS noise with unknown dispersion γ and g follows the normal distribution with unknown variance σ^2 [28].

Since the mixture noise is the additive of two random variables with different PDF, the PDF of mixed noise q, is usually calculated according to the convolution of SαS and Gaussian PDFs, which is

$$\begin{aligned} f(q|\sigma^2, \gamma) &= f_G(q|\sigma^2) * f_\alpha(q|\gamma) \\ &= \int_{-\infty}^{\infty} f_\alpha(q - \tau; \gamma) f_G(q|\sigma^2) d\tau, \end{aligned} \quad (5)$$

where $f_G(q|\sigma^2)$ and $f_\alpha(q|\gamma)$ denote the PDFs of Gaussian and SαS distributions, respectively.

As the SαS process has no closed-from PDF expression, it is usually expressed using characteristic function (CF) [44], which is

$$\varphi(t) = \exp(j\delta t - \gamma|t|^\alpha), 0 < \alpha \le 2 \quad (6)$$

where α is the characteristic parameter [7] reflecting the impulsiveness of the distribution, δ denotes the location parameter setting to 0 in our assumption and γ is the dispersion parameter describing the diffuseness of the process. Noticed that in the case of $\alpha = 2$, the process is the normal distribution with γ corresponding to the variance. While $\alpha = 1$, it corresponds to the Cauchy distribution.

Due to the complicated relationship between the CF and PDF, the PDF of ASαSG in (8) cannot be expressed with an analytic form due to the convolution and integral operations. Therefore, to obtain the closed-form PDF expression, we use the approximated PDF of the SαS noise. Because for a SαS variable, $\alpha = 1$ corresponds to the Cauchy distribution, and $\alpha = 2$ is the Gaussian process, its PDF is rewritten as the sum of a Cauchy ($\alpha = 1$) PDF and a Gaussian ($\alpha = 2$) PDF [38]:

$$f_\alpha(e|\gamma) = \xi(\alpha)f_1(e|\eta) + (1 - \xi(\alpha))f_2(e|\lambda^2), \tag{7}$$

where $0 \leq \xi(\alpha) \leq 1$ is the mixed coefficient, $f_1(e|\eta)$ and $f_2(e|\lambda)$ denote the unnormalized Cauchy and Gaussian processes, with the dispersion η and variance λ^2 [38], respectively. For the sake of the analytical form of $\xi(\alpha), f_1(e|\eta)$ and $f_2(e|\lambda^2)$, previous works [45–47] are developed, among which the most accurate expression is

$$\xi(\alpha) = \frac{2\Gamma(-p/\alpha) - \alpha\Gamma(-p/2)}{2\alpha\Gamma(-p) - \alpha\Gamma(-p/2)}, \tag{8}$$

$$f_1(e|\gamma) = \frac{\gamma}{\pi(e^2 + \gamma^2)}, \tag{9}$$

$$f_2(e|\gamma^2) = \frac{1}{2\sqrt{\pi}\gamma}\exp\left(-\frac{e^2}{4\gamma^2}\right), \tag{10}$$

where p denotes the fractional moment. According to the investigation in [48], the value of p is usually chosen as −1/4.

Then we express the PDF of the Gaussian variable g as

$$f_G\left(g|\sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{g_n^2}{2\sigma^2}\right). \tag{11}$$

With the use of (10)–(14), the PDF of ASαSG distribution in (8) is simplified as

$$f(q|\sigma^2, \gamma) = \xi(\alpha)f_3(q|\gamma, \sigma^2) + (1 - \xi(\alpha))f_4(q|\gamma^2, \sigma^2), \tag{12}$$

where

$$f_3(q|\gamma, \sigma^2) = f_1(q|\gamma) * f_G(q|\sigma^2), \tag{13}$$

$$f_4(q|\gamma^2, \sigma^2) = f_2(q|\gamma^2) * f_G(q|\sigma^2). \tag{14}$$

According to [32], $f_3(q|\gamma, \sigma^2)$ and $f_4(q|\gamma^2, \sigma^2)$ are in fact the Voigt profile and Gaussian distribution's PDF, whose expression are

$$f_3(q|\gamma, \sigma^2) = \frac{\text{Re}\{\omega\}}{\sigma\sqrt{2\pi}}, \tag{15}$$

$$f_4(q|\gamma^2, \sigma^2) = \frac{1}{\sqrt{2\pi(\sigma^2 + 2\gamma^2)}}\exp\left(-\frac{q^2}{2\sigma^2 + 4\gamma^2}\right), \tag{16}$$

where $\omega = \exp\left(-\left(\frac{q + i\gamma}{\sigma\sqrt{2}}\right)^2\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^{q+i\gamma}\exp(t^2)dt\right)\right)$.

4 Proposed Method

In general, the observations have the form of:

$$y_n = s_n + q_n, \tag{17}$$

where q_n denotes the independent and identically distributed (i.i.d.) ASαSG noise term, and

$$s_n = A\cos(\omega n + \phi) = a_1\cos(\omega n) + a_2\sin(\omega n), \tag{18}$$

with $a_1 = A\cos(\phi), a_2 = -A\sin(\phi)$. Here A, ω and φ are amplitude, frequency and phase, respectively. The task of the estimation is to find ω from observations $\{y_n\}_{n=1}^{N}$.

4.1 Posterior of Unknown Parameters

Let $\boldsymbol{\theta} = [a_1, a_2, \omega, \gamma, \sigma^2]^T$, which is the unknown parameter vector. According to [49], the priors of noise parameter γ and σ^2, are usually considered following the conjugate inverse-gamma distribution. Assuming that the priors for $\boldsymbol{\theta}$ and the observed data y_n are statistically independent, they can be expressed as

$$f(y_n|\boldsymbol{\theta}) = f_G(y_n - s_n|\gamma, \sigma^2), \tag{19}$$

$$f(a_1, a_2) = \frac{1}{\sqrt{2\pi}\delta}\exp\left(-\frac{a_1^2 + a_2^2}{2\delta^2}\right), \tag{20}$$

$$f(\omega) = \frac{1}{\pi}, \omega \in [0, \pi], \tag{21}$$

$$f(\gamma) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\exp\left(-\frac{\beta_1}{\gamma}\right), \tag{22}$$

$$f(\sigma^2) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\exp\left(-\frac{\beta_2}{\sigma^2}\right), \tag{23}$$

where $\beta_1 = \beta_2 = 0.01$ and $\alpha_1 = \alpha_2 = 10^{-10}$ according to [50].

By employing Bayes' theorem [19], we have

$$\begin{aligned} &f(a_1, a_2, \omega, \gamma, \sigma^2|\boldsymbol{y}) \\ &= f(\boldsymbol{y}|a_1, a_2, \omega, \gamma, \sigma^2)f(a_1, a_2)f(\omega)f(\gamma) \\ &= C^N\prod_{n=2}^{N}\left\{\xi(\alpha)\frac{\mathrm{Re}\{\omega_n\}}{\sigma\sqrt{2\pi}} + (1 - \xi(\alpha))\frac{\exp\left(-\frac{e_n^2}{2\sigma^2 + 4\gamma^2}\right)}{\sqrt{2\pi(\sigma^2 + 2\gamma^2)}}\right\}, \end{aligned} \tag{24}$$

where $\mathbf{y} = [y_1\ y_2 \cdot\cdot\cdot y_N]^T$, $\mathrm{Re}\{\cdot\}$ denotes the real part and

$$C = \frac{\beta_1^{\alpha_1}\beta_2^{\alpha 2}}{\pi\sqrt{2\pi}\delta\Gamma(\alpha_1)\Gamma(\alpha_2)}\exp\left(-\frac{\beta_1}{\gamma} - \frac{\beta_2}{\sigma^2} - \frac{a_1^2 + a_2^2}{2\delta^2}\right), \tag{25}$$

$$\omega_n = \exp\left(-\left(\frac{e_n + i\gamma}{\sigma\sqrt{2}}\right)^2\right)\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^{\frac{e_n + i\gamma}{\sigma\sqrt{2}}}\exp(t^2)dt\right)\Bigg), \tag{26}$$

with $e_n = y_n - a_1\cos(\omega n) - a_2\sin(\omega n)$.

Furthermore, it can be easily seen in (27) that the expectations of posteriors of the unknown parameters are their true values. Therefore, the mean of unknown parameter samples drawn by M-H algorithm are the unbiased.

4.2 Proposed M-H Algorithm

Although we have known the PDF expression of the ASαSG noise, estimators like MLE and the l_p-norm methods [51], are not able to be utilized due to poor performance and convergence problems. Furthermore, since the posteriors of unknown parameters are complicated, directly sampling on them is difficult.

Therefore, in order to accurate estimate **θ**, the M-H algorithm is used to sample all unknown parameters. To draw samples easily, the multivariate Gaussian distribution is chosen as the M-H proposal distribution, whose PDF is

$$q(x|\mu) = \frac{1}{2\pi\sqrt{|\sum|}}\exp\left(-\frac{1}{2}\boldsymbol{x}^T{\sum}^{-1}\boldsymbol{x}\right), \tag{27}$$

where $\boldsymbol{x} = [x_1\ x_2\ x_3\ x_4\ x_5]^T$ with all elements corresponding to the candidates of the $a_1, a_2, \omega, \gamma, \sigma^2$, respectively and Σ denote covariance matrix of the proposal distribution. Since all elements in **θ** are assumed to be independent, Σ is a diagonal matrix, whose main diagonal entries, namely, are proposal variances. As a hyperparameter, the large value of the proposal variance makes the chain converge faster with a sharp fluctuation around the true value. On the other hand, the smaller value will cause a small-amplitude oscillation but a slower convergence rate [22]. Therefore, for a M-H algorithm, the choice of proposal variance is a difficult and meaningful task, due to its influence of the accuracy and the computational cost.

In this paper, we propose employing a batch-mode samples to update the values of the proposed variance in the proposal covariance matrix. The details of the proposal covariance matrix $\Sigma^{(k)}$ are shown in Fig. 1. With the use of the k-th estimate, denoted by $\boldsymbol{\theta}^{(k)}$, $\Sigma^{(k)}(m,m)$ is written as

$$\Sigma^{(k)}(m,m) = \sum_{l=0}^{L-1}\left(\boldsymbol{\theta}^{(k-l)} - \boldsymbol{\theta}^{(k-l-1)}\right)^2, m = 1, \cdots, 5, \tag{28}$$

where L is the length of the batch-mode window.

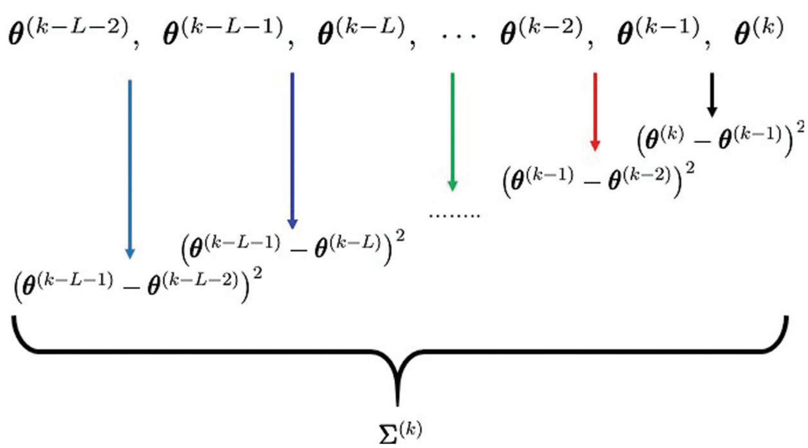

Figure 1: The batch-mode proposed covariance

According to the previous discussion, the initialization of the M-H method, denoted by $\theta^{(\boldsymbol{1})}$, can be chosen arbitrarily. Then to avoid initial bias [52], we threw away the first P samples before the stable of the Markov chain, which is named the burn-in period. With the using the batch-mode proposal covariance criterion, the k-th iteration $\theta^{(\boldsymbol{k})}$ can be obtained from the $\theta^{(\boldsymbol{k-1})}$ using the steps in Tab. 2.

Table 2: The details of the proposed algorithm

1. Set $\theta^{(1)}$ as all ones;
2. Draw P samples using the algorithm shown in Tab. 1, where the proposal covariance matrix is $\sum^{(1)} = \boldsymbol{I}_5 \times 5$;
3. For $k = P+1, \cdots, K+P$
3.1 calculate the proposal covariance matrix $\Sigma^{(k)}$ using batch-mode in (31);
3.2 generate $\boldsymbol{\theta}^{(\mathrm{k})}$ from M-H algorithm shown in Tab. 1 with the new proposal covariance $\Sigma^{(k)}$.

After the the steps in Tab. 2, the chain of a_1, a_2 and ω will be convergent and tend to their true values. Therefore, the estimates of signal parameters, referred to as $\widehat{a}_1, \widehat{a}_2$ and $\widehat{\omega}$, can be obtained by the mean of $\theta^{(k)}(1)$, $\theta^{(k)}(2)$ and $\theta^{(k)}(3)$ $(k = P+1, \cdots, K+P)$, respectively. With the definition of a_1 and a_2, the estimates of amplitude and phase, denoted by $\hat{A}$ and $\hat{\phi}$, are

$$\hat{A} = \sqrt{\hat{a}_1^2 + \hat{a}_2^2}, \tag{29}$$

$$\hat{\phi} = \operatorname{atan}\left(\frac{\hat{a}_2}{\hat{a}_1}\right), \tag{30}$$

where $\operatorname{atan}(\cdot)$ is the arctangent operator.

5 Cramér-Rao Lower Bound (CRLB)

Let $\boldsymbol{\psi} = [A\ \omega\ \phi\ \gamma\ \sigma^2]^T$. According to the definition in [22], the CRLBs of $\boldsymbol{\psi}$ can be obtained by the diagonal elements of $\boldsymbol{F}^{-1}$. Here $\boldsymbol{F}$ is called the Fisher information matrix with $^{-1}$ denoting the inverse operator. The (k,l) entry $(k,l = 1, \cdots, 5)$ of $\boldsymbol{F}$ is

$$\boldsymbol{F}(k,l) = -E\left\{\frac{\partial \log f(\boldsymbol{y}|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\left(\frac{\partial \log f(\boldsymbol{y}|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right)^T\right\} = -E\left\{\sum_{n=1}^{N}\frac{\partial \log f(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\left(\frac{\partial \log f(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right)^T\right\}, \tag{31}$$

where $E\{\cdot\}$ is the expectation operator and

$$\frac{\partial \log f(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}} = \xi(\alpha)\frac{\partial \log f_3(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}} + (1 - \xi(\alpha))\frac{\partial \log f_4(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}, \tag{32}$$

with

$$\frac{\partial \log f_3(y_n \boldsymbol{\psi})}{\partial \boldsymbol{\psi}} = \frac{1}{\sigma^2 \mathrm{Re}\{w_n\}}\begin{bmatrix} \cos(\omega n + \phi)\mathrm{Re}\{(y_n - s_n + i\gamma)w_n\} \\ An\sin(\omega n + \phi)\mathrm{Re}\{(y_n - s_n + i\gamma)w_n\} \\ A\sin(\omega n + \phi)\mathrm{Re}\{(y_n - s_n + i\gamma)w_n\} \\ -\mathrm{Re}\{i(y_n - s_n + i\gamma)w_n\} + \dfrac{2\sigma}{\sqrt{2\pi}} \\ \mathrm{Re}\left\{(y_n - s_n + i\gamma)^2 w_n\right\} + \dfrac{\gamma}{\sqrt{2\pi\sigma^2}} - \dfrac{\mathrm{Re}\{w_n\}}{2} \end{bmatrix}, \tag{33}$$

$$\frac{\partial \log f_4(y_n|\boldsymbol{\psi})}{\partial \boldsymbol{\psi}} = f_4(y_n|\boldsymbol{\psi}) \begin{bmatrix} \dfrac{(y_n - s_n)\cos(\omega n + \phi)}{\sigma^2 + 2\gamma^2} \\ -\dfrac{An(y_n - s_n)\sin(\omega n + \phi)}{\sigma^2 + 2\gamma^2} \\ -\dfrac{A(y_n - s_n)\sin(\omega n + \phi)}{\sigma^2 + 2\gamma^2} \\ \left(\dfrac{(y_n - s_n)^2}{(\sigma^2 + 2\gamma^2)^2} - \dfrac{1}{\sigma^2 + 2\gamma^2}\right)\sigma \\ \sqrt{2}\left(\dfrac{(y_n - s_n)^2}{(\sigma^2 + 2\gamma^2)^2} - \dfrac{1}{\sigma^2 + 2\gamma^2}\right)\gamma \end{bmatrix}, \tag{34}$$

and $s_n = A\cos(\omega n + \phi)$. According to the definition in (29), the Voigt profile is complicated. And hence, the closed-form expressions of CRLBs are not easy to be derived. Therefore, the calculation of the CRLBs in (34) adopts an approximate numerical method:

$$\widehat{F}(k, l) \approx \frac{1}{M}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{\partial \log f\left(y_n^m|\boldsymbol{\psi}\right)}{\partial \boldsymbol{\psi}}\left(\frac{\partial \log f\left(y_n^m|\boldsymbol{\psi}\right)}{\partial \boldsymbol{\psi}}\right)^T, \tag{35}$$

where y_n^m represents the observed signal in the m-th independent trial and M denotes the number of independent runs. It can be easily to prove that (38) can approach (34) with a large M being choosing.

6 Simulation Results

In this section, computer simulations are conducted to verify the effectiveness of our method. Then the mean square frequency error (MSFE), denoted by $E\{(\widehat{\omega} - \omega)^2\}$, was employed to represent the performance measure of the estimation. The sinusoid signal s_n is constructed according to (21), with all parameters being $A = 10.30$, $\omega = 2.14$ and $\varphi = 0.55$. While for of ASαSG noise, the shape parameter α is chosen as 1.2. The initialization of the proposed algorithm is set to all ones and the iteration number of the M-H chain is $K = 8000$ [28]. To verify the performance, the simulations of the MLE and *lp*-norm estimator ($p = 1.1$) [52] are included, because they are typical robust estimators for the heavy-tailed noise. Meanwhile, the CRLB is also provided as a benchmark. In our experiments, all results are based on 600 independent runs with a data length of $N = 100$. Furthermore, all results are obtained by using Matlab on Intel (R) Core (TM) i7-4790 CPU@3.60GHz [22].

First of all, to obtain the proper $\Sigma^{(k)}$ in (28), the value of L is investigated. The dispersion parameters of ASαSG noise are set to $\gamma = 0.05$ and $\sigma^2 = 0.5$ [28]. Figs. 2 and 3 show the MSFE in different values of L and the computational cost *vs.* L, respectively. Here the computational time is measured using the stopwatch timer in the simulator. It can be seen in Fig. 2 that the MSFE of our method can be aligned with CRLB when $L \geq 600$. While according to the result in Fig. 3, the computational cost of the proposed algorithm becomes higher for larger L. Take the higher accuracy and lower computational complexity into account, we choose L as 600 [28] in the following test.

Second, we study the convergence rate of the M-H chain and the value of the burn-in period P. In this test, the density parameter is the same values to the previous test and the proposal covariance matrix is calculated by (31) with $L = 600$. Figs. 4 and 5 show the samples of ω, A, φ, γ and σ^2 in different iteration k. In these figures, we can see that after the first 2000 samples, the chain of all unknown parameters approaches their true values. In this case, the corresponding burn-in period P in our simulations is 2000 [22].

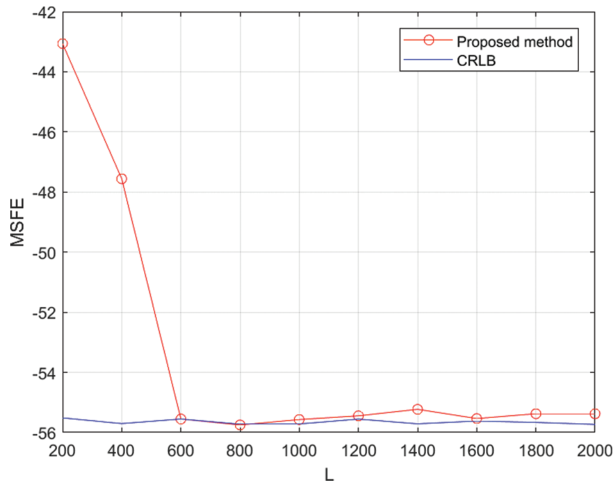

Figure 2: MSFE *vs.* L

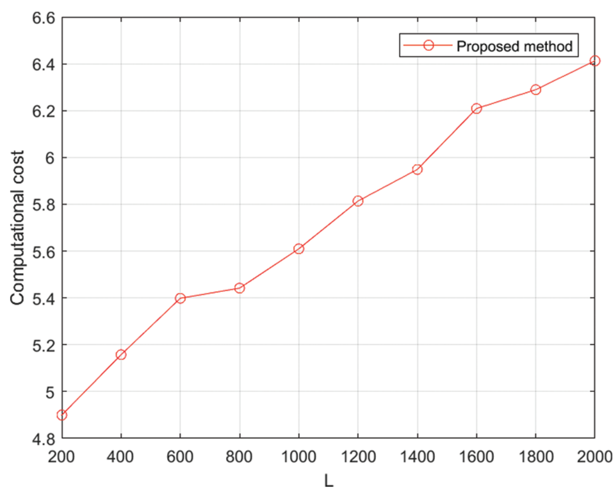

Figure 3: The computational cost of the proposed method *vs.* L

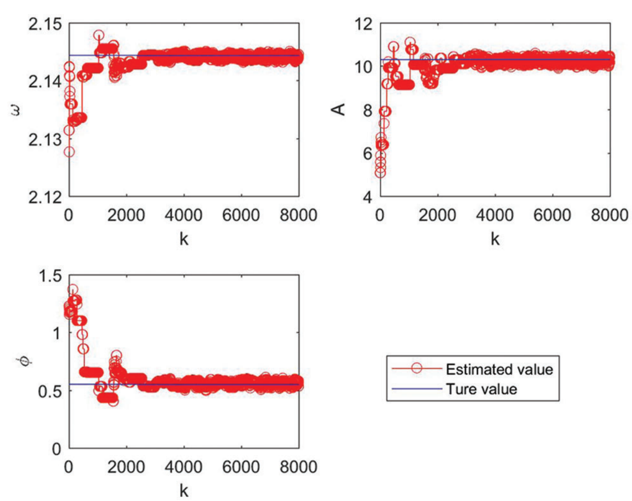

Figure 4: Estimates of unknown parameters *vs.* iteration number k

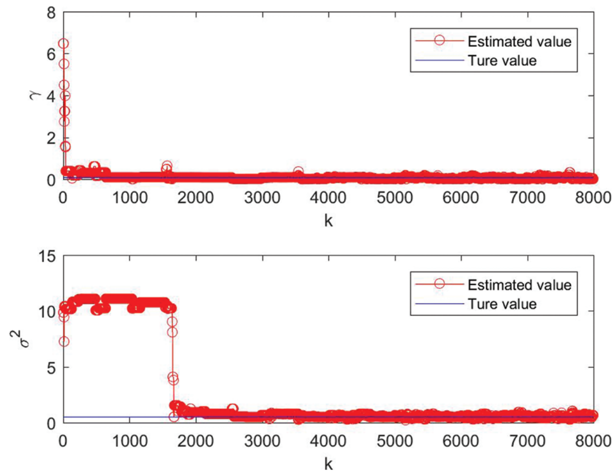

Figure 5: Estimates of density parameters *vs*. iteration number *k*

In the following, the MSFE performance of our estimator, MLE and *lp*-norm estimator are considered. Since there is no signal-to-noise ratio in ASαSG noise, in the proposed method, γ is scaled to generate different noise conditions. With the use of previous tests, we throw away the first 2000 samples to guarantee the stationary of the chain. It is indicated in Fig. 6 that the MSFE of our proposed method can attain the CRLB for the noise conditions $\gamma \in [-30, 5]$ dB [22]. Furthermore, the proposed method performs better than the *lp*-norm estimator and MLE, because it is much closer to CRLB.

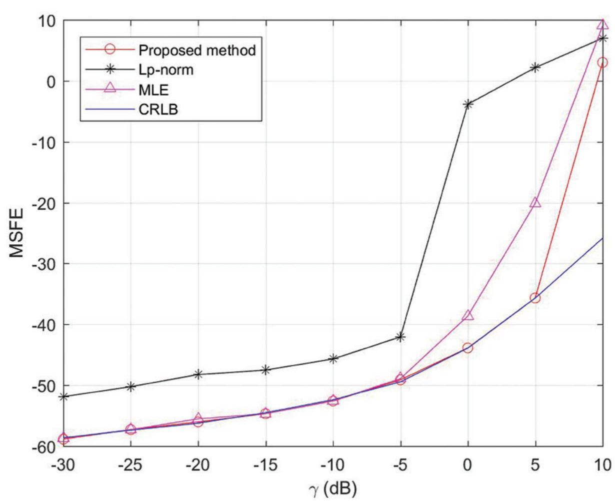

Figure 6: Mean square frequency error of ω *vs.* γ

Finally, the computational complexity of our scheme is studied in different data length. It can be seen in the Tab. 3 that the computational cost of MLE and *lp*-norm is lower than the proposed estimator [38]. However, in the higher data length, our proposed scheme will not increase. This is to say, our method is not sensitive to the data length, indicating the advantage of its application in big data.

Table 3: The computational cost of three methods *vs. N*

N	Proposed	MLE	*lp-norm*
50	5.1275	0.0675	0.0451
150	5.4627	0.0461	0.0190
250	5.9434	0.0605	0.0302
350	4.6475	0.1061	0.0317
450	5.5752	0.1934	0.0433
550	5.3817	0.2515	0.0408

7 Conclusion

In this paper, the improved Bayesian method, namely M-H algorithm, is used to study the accurate frequency estimation method of single sinusoidal signal with ASαSG noise. In order to reduce the computational cost, a new proposal covariance matrix reconstruction criterion and an PDF approximation is designed. Simulation results indicate that the developed method can obtain the unbiased estimates with a stable sampling condition. In addition, MSFE of the proposal estimator can obtain CRLB after discarding burn-in period samples. Our method can be also extended to the other complicated signal models.

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