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Radio Labeling Associated with a Class of Commutative Rings Using Zero-Divisor Graph

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Abstract: Graph labeling is useful in networks because each transmitter has a different transmission capacity to send or receive wired or wireless links. An interference of signals can occur when transmitters that are close together receive close frequencies. This problem has been modeled mathematically in the radio labeling problem on graphs, where vertices represent transmitters and edges indicate closeness of the transmitters. For this purpose, each vertex is labeled with a unique positive integer, and to minimize the interference, the difference between maximum and minimum used labels has to be minimized. A radio labeling for a graph $\mathrm{G} = (\mathrm{V}(\mathrm{G}), \mathrm{E}(\mathrm{G}))$ is a function γ from the set of vertices $\mathrm{V}(\mathrm{G})$ to the set of positive integers satisfying the condition $\mathrm{d}(\mathrm{x}, \mathrm{y}) + |\gamma(\mathrm{x}) - \gamma(\mathrm{y})| \geq 1 + \mathrm{diam}(\mathrm{G})$, where $\mathrm{d}(\mathrm{x}, \mathrm{y})$ is the shortest distance between two distinct vertices $\mathrm{x}, \mathrm{y} \in \mathrm{V}(\mathrm{G})$, and $\mathrm{diam}(\mathrm{G})$ is the diameter of the graph G. The minimum span of a radio labeling for G is called the radio number of G. Because the problem of finding radio labeling appears to be difficult in general, many particular cases have been studied. Let R be a commutative ring with nonzero identity, and $\mathrm{Z}(\mathrm{R})$ its set of (nonzero) zero-divisors. The zero-divisor graph of a ring R is the graph $\Gamma(\mathrm{R})$ with vertex set $\mathrm{V}(\Gamma(\mathrm{R})) = \mathrm{Z}(\mathrm{R})$ and edge set $\mathrm{E}(\Gamma(\mathrm{R})) = \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} \cdot \mathrm{y} = 0\}$. In this paper, we investigate the radio number for an associated zero-divisor graph, $\Gamma(Z_{\mathrm{p}^{\mathrm{n}}})$. The study provides some combinatorial properties associated with commutative rings and can be useful for the structures of network communication problems.

Keywords: Radio labeling; radio number; distances in graph; zero divisor graph; commutative ring

1 Introduction

Antennas transmit and receive different frequencies of electromagnetic waves, such as radio waves. By tuning in a radio, we receive signals to access particular frequencies. Each radio station is assigned to a distinct channel. When two radio stations are near each other, the difference between their assigned channels must be greater than a specific number to avoid interference. The task of allocating channels to transmitters is known as channel assignment (CA).

The CA model was introduced in 1980 by William Hale [1]. CA is generally modeled as a graph coloring and labeling representation, where transmitters are represented as nodes (vertices) of the graph. When two nodes are adjacent, their transmitters are close. The labels assigned to nodes are of the channels of the transmitters, where for each pair of labels, there must be an acceptable distance between their nodes. This study aims to find a suitable labeling to minimize the range (span) of the channels.

Let G be a simple and connected graph. The degree of a vertex $x \in V(G)$ is the number of vertices adjacent to x, and is denoted as $d_G(x)$. The shortest distance between two vertices $x, y \in V(G)$ is denoted by $d(x, y)$, and the maximum value of $d(x, y)$ in G is called the diameter of G, denoted as diam(G). Radio labeling of G [2,3], also known as multi-level distance labeling, is a function γ: $V(G) \to \mathbb{N}$ for which the following condition holds for any two distinct vertices x and y:

$$d(x,y) + |\gamma(x) - \gamma(y)| \geq 1 + diam(G), \tag{1}$$

which is referred to as a radio condition.

We denote by $S(G, \gamma)$ the set of consecutive integers $\{m, m+1, \ldots, M\}$, where $m = \min_{x \in V(G)} \gamma(x)$, and $M = \max_{x \in V(G)} \gamma(x)$ is the span of γ, denoted by $span(\gamma) = 1 + M - m$.

The minimum span of a radio labeling for G is called the radio number of G, denoted by $rn(G)$. A radio labeling γ of G with $span(\gamma) = rn(G)$ will be called the optimal radio labeling for G. Radio labeling is an interesting graph labeling problem that is the subject of much research. It is complicated to determine the radio number for a general graph. The radio problem is NP-hard, even for a graph with a small diameter, and as a rule, its complication is still unknown [4]. Due to this, researchers have studied this problem, and even for some known families of the graph, the problem is shown to be complex [2,5–12].

2 Applications of Zero-divisor Graphs

The zero-divisor graph over a commutative ring was introduced in 1988 by Beck [13], who discussed the coloring of such graphs.

The interdisciplinary research in algebraic graph theory is excelling, and associated applications are benefiting from such desk research. The study conducted in [14] and [15] serves as an interesting survey to find the relation between the ring-theoretic properties and graph-theoretic properties of $\Gamma(G)$. This study deals with techniques that vary from simple computations to sophisticated ring theory, and in many cases, all the rings or graphs satisfy a certain property. We raise two questions:

i) Could rings with certain theoretic properties have the same physical structures and graphical properties, or vice versa?
ii) Is it possible to determine $\Gamma(R)$ such that $\Gamma(R) \cong G$?

Redmond [16,17] provided all graphs up to 14 vertices that can be realized as the zero-divisor graph of a commutative ring with identity, listed all rings (up to isomorphism) that produce these graphs, and created an algorithm to find all commutative reduced rings with identity (up to isomorphism) that give rise to a zero-divisor graph on n vertices for any $n \geq 1$. A question that naturally arises when studying zero-divisor graphs is whether they are unique. There are some applications and relationships between algebraic theory and chemical graph theory [18,19]. Recent work has been performed on the radio numbers of different algebraic structures [20,21].

Let p be a prime number, and $\Gamma(\mathbb{Z}_{p^n})$ a zero-divisor graph of the commutative rings $\mathbb{Z}_{p^n}$. We investigate the radio number of zero divisor graphs $\Gamma(\mathbb{Z}_{p^n})$ for any positive integer n and prime number p.

3 Main Results

Let R be a commutative ring. A nonzero element $a \in R$ is called a zero-divisor of R if there exists another nonzero element $b \in R$ such that $a.b = 0$. We denote the set of all zero divisors in a ring R by $Z(R)$. Assume that R is a ring with identity 1. Then an element $u \in R$ is called a unit in R if there exists an element $v \in R$ such that $u.v = 1$. We denote the set of all units in R by $U(R)$. A unit element in a ring R cannot be a zero divisor. Similarly, a zero-divisor in R cannot be a unit but $a.u \in Z(R)$ for any $a \in Z(R)$ and $u \in U(R)$.

We consider rings of the form $R = \mathbb{Z}_{\mathbb{k}}$ for a fixed positive integer k, where a nonzero element is either a unit or a zero-divisor. More precisely, an element $a \in \mathbb{Z}_k \backslash \{0\}$ is a zero divisor if and only if $\gcd(a, k) \neq 1$, and an element $u \in \mathbb{Z}_{\mathbb{k}} \backslash \{0\}$ is a unit if and only if $\gcd(u, k) = 1$.

For a fixed prime p and a fixed positive integer n, we consider the ring $R = \mathbb{Z}_{p^n}$. An element $a \in \mathbb{Z}_{p^n} \backslash \{0\}$ is a zero-divisor if and only if p divides a. It is easy to see that the set of zero-divisors $Z(\mathbb{Z}_{p^n}) = \coprod_{i=1}^{n-1} Z_i$, where each $Z_i = \{u.p^i\colon u \text{ is a unit in } \mathbb{Z}_{p^n}\}$ contains those elements of $\mathbb{Z}_{p^n}$ that are multiples of p^i but not of p^{i+1}. Therefore, $|Z_i| = p^{n-i} - p^{n-i-1}$ for each $i = 1, 2, \ldots, n-1$, and hence $|Z(\mathbb{Z}_{p^n})| = \sum_{i=1}^{n-1} |Z_i| = p^{n-1} - 1$.

We associate a zero-divisor graph $\Gamma(\mathbb{Z}_{p^n})$ to a ring $\mathbb{Z}_{p^n}$ with vertex set $V(\Gamma(\mathbb{Z}_{p^n})) = Z(\mathbb{Z}_{p^n})$; note that we consider a zero-divisor to be a nonzero element, so $0 \notin V(\Gamma(\mathbb{Z}_{p^n}))$. The degree of each vertex in Z_i is shown in the following theorem.

Theorem 3.1 Let $\Gamma(\mathbb{Z}_{p^n})$ be a zero-divisor graph of $\mathbb{Z}_{p^n}$. Then,

$$d_{Z_i}(x) = \begin{cases} p^i - 1, & for 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ p^i - 2, & for \lceil \frac{n}{2} \rceil \leq i \leq n-1 \end{cases}$$

Proof. For any vertex $x \in Z_i$, we have $x \cdot y = 0$ if and only if $y \in Z_j$ for $j \geq n - i$. For $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, we get $d_{Z_i}(x) = |\coprod_{j=n-i}^{n-1} Z_j| = \sum_{j=n-i}^{n-1} |Z_j| = p^i - 1$. For $\lceil \frac{n}{2} \rceil \leq i \leq n-1$, we get $d_{Z_i}(x) = |\coprod_{j=n-i}^{n-1} Z_j - \{x\}| = \sum_{j=n-i}^{n-1} |Z_j| - 1 = p^i - 1 - 1 = p^i - 2$.

Using the handshaking lemma, after simplification, we obtain the size of $\Gamma(\mathbb{Z}_{p^n})$ in the following theorem.

Theorem 3.2 For $n \geq 2$ and prime number p, the size of $\Gamma(\mathbb{Z}_{p^n})$ is

$\frac{1}{2}\{\sum_{x \in V(\Gamma(\mathbb{Z}_{p^n}))} d(x)\} = \frac{1}{2}\{p^{n-1}(np - n - p) - p^{n - \lceil \frac{n}{2} \rceil} + 2\}$, except $n = p = 2$.

We know that the radio number of complete graph K_n is n. From the definition, it can be seen that the zero-divisor graph of $\Gamma(\mathbb{Z}_{p^n})$ is a complete graph K_{p-1} for n = 2. Therefore, the following theorem holds.

Theorem 3.3 Let $\Gamma(\mathbb{Z}_{p^n})$ be a zero-divisor graph of $\mathbb{Z}_{p^n}$. Then $rn(\Gamma(\mathbb{Z}_{p^n})) = p - 1$ for $n = 2$.

In the next theorem, we determine the radio number of a zero-divisor graph of $\mathbb{Z}_{p^n}$ for $n = 3$.

Theorem 3.4 Let $p \geq 2$ be a prime number, and $\Gamma(\mathbb{Z}_{p^3})$ a zero-divisor graph of $\mathbb{Z}_{p^3}$. Then $rn(\Gamma(\mathbb{Z}_{p^3})) = p^2 + p - 2$.

Proof. For simplicity, we partition the vertex set of $\Gamma(\mathbb{Z}_{p^3})$ into two mutually disjoint sets, $Z_1 = \{u.p\colon u \neq k_1.p, 1 \leq k_1 \leq p-1\} = \{u_i\colon 1 \leq i \leq p^2 - p\}$ and $Z_2 = \{u.p^2 : u = k_2 \, . \, p, 1 \leq k_1 \leq p - 1\} = \{w_j : 1 \leq j \leq p-1\}$. According to the definition of the zero-divisor graph, the vertices of the set Z_1 are not adjacent, and the vertices of the set Z_2 are adjacent. This means that $d(u_1, u_2) \neq 1$ and $d(w_1, w_2) = 1$ for $u_1, u_2 \in Z_1$ and $w_1, w_2 \in Z_2$. Also, all the vertices of set Z_1 are adjacent to each vertex of set Z_2. Hence, $d(u,w) = 1\ \forall u \in Z_1, w \in Z_2$ and $|Z_1| = p^2 - p$, $|Z_2| = p - 1$. From this, we observe that $d(u_1, u_2) = 2$. This indicates that the diameter of $\Gamma(\mathbb{Z}_{p^3})$ is 2. Therefore, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$ must satisfy the radio condition defined in Eq. (1),

$$d(x,y) + |\gamma(x) - \gamma(y)| \geq 3, \text{ for x, y} \in \mathrm{V}(\Gamma(\mathbb{Z}_{\mathrm{p}^3})). \tag{2}$$

Since $d(u_1, u_2) = 2$, there are no forbidden values between the vertices of set Z_1. Since $d(w_1, w_2) = 1$, there are $p - 2$ forbidden values between the vertices of set Z_2. Also, $d(u,w) = 1$, which gives only one forbidden value between the vertices of sets Z_1 and Z_2. Therefore, there is a total of $p - 1$ forbidden values in zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$. Hence, the lower bound of the radio number of zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$ is

$$rn(\Gamma(\mathbb{Z}_{p^3})) \leq |V(\Gamma(\mathbb{Z}_{p^3}))| + p - 1 = p^2 + p - 2. \tag{3}$$

To obtain the upper bound of the radio number of zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$, we define the radio labeling γ_1: $V(\Gamma(\mathbb{Z}_{p^3}) \rightarrow \{1, 2, 3, \ldots, p^2 + p - 2\}$ as $\gamma_1(u_i) = i$ for $1 \leq i \leq p^2 - p$, and $\gamma_1(w_j) = p^2 - p + 2j$ for $1 \leq j \leq p - 1$. Without loss of generality, assume that for any two vertices $u_s, u_t \in Z_1$, $w_s, w_t \in Z_2$, $|\gamma_1(u_s) - \gamma_1(u_t)| \geq 1$, $|\gamma_1(w_s) - \gamma_1(w_t)| \geq 2$, and $|\gamma_1(u_j) - \gamma_1(w_i)| = p^2 - p + 2j - i \geq 2$. This demonstrates that the radio labeling γ_1 satisfies the radio condition (2) for zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$. Therefore,

$$rn(\Gamma(\mathbb{Z}_{p^3})) \geq p^2 - p + 2(p - 1) = p^2 + p - 2. \tag{4}$$

Combining Eqs. (3) and (4), we obtain the required result. This completes the proof.

Theorem 3.5 Let p be a prime number, and $\Gamma(\mathbb{Z}_{\mathrm{p}^4})$ a zero-divisor graph of $\mathbb{Z}_{\mathrm{p}^4}$. Then $\mathrm{rn}(\Gamma(\mathbb{Z}_{\mathrm{p}^4})) = \mathrm{p}^3 + \mathrm{p}^2 - 3$.

Proof. For simplicity, we partition the vertex set of $\Gamma(\mathbb{Z}_{p^4})$ into three mutually disjoint sets: $Z_1 = \{u.p\colon u \neq k_1.p^2, 1 \leq k_1 \leq p^2 - 1\} = \{x_i\colon 1 \leq i \leq p^3 - p^2\}$, $Z_2 = \{u.p^2\colon u \neq k_2.p, 1 \leq k_2 \leq p^3 - p^2 + p - 1\} = \{y_j\colon 1 \leq j \leq p^2 - p\}$, and $Z_3 = \{u.p^3\colon u \neq k_3.p, 1 \leq k_3 \leq p^3 - p\} = \{z_t\colon 1 \leq t \leq p - 1\}$. According to the definition of the zero-divisor graph, the vertices of set Z_1 are not adjacent, and the vertices of sets Z_2 and Z_3 are adjacent. This means that $d(x_1, x_2) \neq 1$ and $d(y_1, y_2) = d(z_1, z_2) = 1$ for $x_1,\ x_2 \in Z_1$, $y_1,\ y_2 \in Z_2$, and $z_1,\ z_2 \in Z_3$. Additionally, all the vertices of sets Z_1 and Z_2 are adjacent to each vertex of set Z_3. This means that $d(x,z) = d(y,z) = 1\ \forall x \in Z_1, y \in Z_2, z \in Z_3$, and $|Z_1| = p^3 - p^2$, $|Z_2| = p^2 - p$, $|Z_3| = p - 1$. This implies that $|V(\Gamma(\mathbb{Z}_{p^4}))| = p^3 - 1$. From the above discussion, it is observed that $d(x_1, x_2) = d(x,y) = 2$. This shows that the diameter of $\Gamma(\mathbb{Z}_{p^4})$ is 2. Therefore, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ must satisfy the radio condition defined in Eq. (1), i.e.,

$$d(u,v) + |\gamma(u) - \gamma(v)| \geq 3, \text{ for u, v} \in \mathrm{V}(\Gamma(\mathbb{Z}_{\mathrm{p}^4})). \tag{5}$$

Since $d(x_1, x_2) = 2$, there is no forbidden value between the vertices of set Z_1 and $d(y_1, y_2) = 1$. Therefore, there are $p^2 - p - 1$ forbidden values between the vertices of set Z_2 and $d(z_1, z_2) = 1$. Similarly, there are $p - 2$ forbidden values between the vertices of set Z_3. Since $d(x,y) = 2$, this shows that there is no forbidden value between the vertices of sets Z_1 and Z_2. Also $d(x,z) = d(y,z) = 1$, which gives only one forbidden value between the vertices of set Z_3 and sets Z_1, Z_2. Thus there are $p^2 - p - 1 + p - 2 + 1 = p^2 - 2$ forbidden values in the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$. Hence, the lower bound of the radio number of zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ is

$$rn(\Gamma(\mathbb{Z}_{p^4})) \leq |V(\Gamma(\mathbb{Z}_{p^4}))| + p^2 - 2 = p^3 + p^2 - 3. \tag{6}$$

To obtain the upper bound of the radio number of zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$, we define the radio labeling γ_2: $V(\Gamma(\mathbb{Z}_{p^4})) \rightarrow \{1, 2, 3, \ldots, p^3 + p^2 - 3\}$ as follows.

$\gamma_2(x_i) = i$ for $1 \leq i \leq p^3 - p^2$, $\gamma_2(y_j) = p^3 - p^2 + 2j - 1$ for $1 \leq j \leq p^2 - p$, and $\gamma_2(z_t) = p^3 + p^2 - 2p + 2t - 1$ for $1 \leq t \leq p - 1$. For any two vertices $x_s, x_t \in Z_1$, $y_s, y_t \in Z_2$, $z_s, z_t \in Z_3$, $|\gamma_2(x_s) - \gamma_2(x_t)| \geq 1$, $|\gamma_2(y_s) - \gamma_2(y_t)| \geq 2$, $|\gamma_2(z_s) - \gamma_2(z_t)| \geq 2$, $|\gamma_2(x_i) - \gamma_2(y_j)| \geq 1$, $|\gamma_2(x_i) - \gamma_2(z_t)| \geq 2$, and $|\gamma_2(y_j) - \gamma_2(z_t)| \geq 2$. In addition, $d(x_s, x_t) = d(x, y) = 2$ and $d(x_i, z_t) = d(y_j, z_t) = d(y_s, y_t) = d(z_s, z_t) = 1$. This shows that the radio labeling γ_2 satisfies the radio condition (2) for zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$. Therefore, we arrive at

$$rn(\Gamma(\mathbb{Z}_{p^4})) \geq p^3 + p^2 - 3. \tag{7}$$

Combining Eqs. (6) and (7), we obtain the required result. This completes the proof.

The above theorems lead us to establish general results related to the radio number of zero-divisor graphs associated to rings $\mathbb{Z}_{p^n}$ for $n \geq 5$.

In the following proposition, we determine the lower bound of the radio number for a zero-divisor graph of $\mathbb{Z}_{p^n}$ for $n \geq 5$.

Proposition 3.6 Let p be a prime number and $n \geq 5$. Then $\mathrm{rn}(\Gamma(\mathbb{Z}_{p^n})) \geq p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$.

Proof. From the above discussion, we have $V(\Gamma(\mathbb{Z}_{p^n})) = Z_i = \{u.p^i\colon u \text{ is a unit in } \mathbb{Z}_{p^n}\}$. This means that it contains those elements of $\mathbb{Z}_{p^n}$ which are multiples of p^i but not of p^{i+1}, which implies that $|Z_i| = p^{n-i} - p^{n-i-1}$ for $1 \leq i \leq n-1$. Therefore, $V(\Gamma(\mathbb{Z}_{p^n})) = p^{n-1} - 1$. Let $d_{Z_i}(x)$ denote the degree of a vertex x in set Z_i, and $d(Z_i, Z_j)$ the distance between the vertices of sets Z_i and Z_j. For any vertex $x_1^i \in Z_i$, we have $d_{Z_i}(x_1^i) = p^i - 1$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $d_{Z_i}(x_1^i) = p^i - 2$ for $\lceil \frac{n}{2} \rceil \leq i \leq n-1$. Additionally,

$$d(x_1^i, x_2^j) = \begin{cases} 1, & \text{if } i = j \text{ and } \lceil \frac{n}{2} \rceil \leq i \leq n-1 \\ 2, & \text{if } i = j \text{ and } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \end{cases}$$

$$d(x_1^i, x_2^j) = \begin{cases} 2, & \text{if } i \neq j \text{ and } 1 \leq i, j \leq \lceil \frac{n}{2} \rceil - 1 \\ 1, & \text{if } i \neq j \text{ and } \lceil \frac{n}{2} \rceil \leq i, j \leq n-1 \end{cases}$$

$$d(x_1^1, x_2^j) = \begin{cases} 2, & \text{if } 2 \leq j \leq n-2 \\ 1, & \text{if } j = n-1 \end{cases},$$

and $d(x_1^i, x_2^j) = 1$ for $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $\lceil \frac{n}{2} \rceil \leq j \leq n-1$. This implies that $diam(\Gamma(\mathbb{Z}_{p^n})) = 2$. Any radio labeling γ of $\Gamma(\mathbb{Z}_{p^n})$ must satisfy the following radio condition:

$$d(x_1^i, x_2^j) + |\gamma(x_1^i) - \gamma(x_2^j)| \geq diam(\Gamma(\mathbb{Z}_{p^n})) + 1 = 3,$$

for any distinct vertices $x_1^i, x_2^j \in V(\Gamma(\mathbb{Z}_{p^n}))$. We now count the forbidden values for $\Gamma(\mathbb{Z}_{p^n})$. If $d(x_1^i, x_2^j) = 2$, then it is possible to assign consecutive labels between those vertices. This means there is no forbidden value between them. Therefore, for n-even, there are no forbidden values between the vertices of $Z_i(1 \leq i \leq \frac{n}{2} - 1)$ and $Z_j(\frac{n}{2} \leq j \leq n-1)$, and for n-odd, there are no forbidden values between the vertices of $Z_i(1 \leq i \leq \frac{n-1}{2})$ and $Z_j(\frac{n+1}{2} \leq j \leq n-1)$. Now, if $d(x_1^i, x_2^j) = 1$, then $|\gamma(x_1^i) - \gamma(x_2^j)|$ must be greater than 2. This means there must be a forbidden value between those vertices. Therefore, for n-even, there are $\sum_{i=\frac{n}{2}}^{n-1} |Z_i| - 1$ forbidden values between the vertices of $Z_i(\frac{n}{2} \leq i \leq n-1)$, and for n-odd, there are $\sum_{i=\frac{n+1}{2}}^{n-1} |Z_i| - 1$ forbidden values between the vertices of $Z_i(\frac{n+1}{2} \leq i \leq n-1)$. By adding the forbidden values to the order of the graph, we obtain the total number of labels.

Hence, for n-even,

$$rn(\Gamma(\mathbb{Z}_{p^n})) \geq |V(\Gamma(\mathbb{Z}_{p^n}))| + \sum_{i=\frac{n}{2}}^{n-1} |Z_i| - 1 = \sum_{i=1}^{\frac{n}{2}-1} |Z_i| + 2\sum_{i=\frac{n}{2}}^{n-1} |Z_i| - 1 = p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3, \quad (8)$$

and for n-odd,

$$rn(\Gamma(\mathbb{Z}_{p^n})) \geq |V(\Gamma(\mathbb{Z}_{p^n}))| + \sum_{i=\frac{n+1}{2}}^{n-1} |Z_i| - 1 = \sum_{i=1}^{\frac{n-1}{2}-1} |Z_i| + 2\sum_{i=\frac{n+1}{2}}^{n-1} |Z_i| - 1 = p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3. \quad (9)$$

Combining Eqs. (8) and (9), we arrive at

$$rn(\Gamma(\mathbb{Z}_{p^n})) \geq p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3. \quad (10)$$

This completes the proof.

The next proposition determines the upper bound of the radio number for a zero-divisor graph of $\mathbb{Z}_{p^n}$ for $n \geq 5$.

Proposition 3.7 Let p be a prime number and n $\geq$ 5. Then $\mathrm{rn}(\Gamma(\mathbb{Z}_{p^n})) \leq p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3$.

Proof. We provide a radio labeling of $\Gamma(\mathbb{Z}_{p^n})$ with span $p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3$, which implies that $rn(\Gamma(\mathbb{Z}_{p^n})) \leq p^{n-1} + p^{\lfloor\frac{n}{2}\rfloor} - 3$.

The radio labeling γ: $V(\Gamma(\mathbb{Z}_{p^n})) \to \mathbb{Z}^+$ is defined as follows.

Case 1: n-even

For $x_j^i \in Z_i$, where $|Z_i| = p^{n-i} - p^{n-i-1}$ and $|Z_0| = 0$,

$$\gamma\left(x_j^i\right) = \begin{cases} j + \sum\limits_{s=1}^{i} |Z_{s-1}|, & \text{if } 1 \leq j \leq |Z_i| \text{ and } 1 \leq i \leq \frac{n}{2} - 1 \\ 2j - 1 + \sum\limits_{s=1}^{\frac{n}{2}-1} |Z_s| + 2\sum\limits_{s=\frac{n}{2}}^{i} |Z_{s-1}| - 2|Z_{\frac{n}{2}-1}|, & \text{if } 1 \leq j \leq |Z_i| \text{ and } \frac{n}{2} \leq i \leq n-1 \end{cases}$$

Case 2: n-odd

For $x_j^i \in Z_i$, where $|Z_i| = p^{n-i} - p^{n-i-1}$ and $|Z_0| = 0$,

$$\gamma(x_j^i) = \begin{cases} j - 1 + \sum_{s=1}^{i} |Z_{s-1}| + 2\sum_{s=\frac{n+1}{2}}^{n-1} |Z_s|, & \text{if } 1 \leq j \leq |Z_i| \text{ and } 1 \leq i \leq \frac{n-1}{2} \\ 2j - 1 + 2\sum_{s=\frac{n+1}{2}}^{n-1} |Z_s| - 2\sum_{s=\frac{n+1}{2}}^{i} |Z_s|, & \text{if } 1 \leq j \leq |Z_i| \text{ and } \frac{n+1}{2} \leq i \leq n-1 \end{cases}$$

From case 1, it can be seen that $\Gamma(\mathbb{Z}_{p^n})$ attains an upper bound if and only if $i = n-1$ and $j = |Z_{n-1}|$, i.e.,

$$2|Z_{n-1}| - 1 + \sum_{s=1}^{\frac{n}{2}-1} |Z_s| + 2\sum_{s=\frac{n}{2}}^{n-1} |Z_{s-1}| - 2|Z_{\frac{n}{2}-1}|$$

$$= 2(p-1) - 1 + \sum_{s=1}^{\frac{n}{2}-1} |Z_s| + 2\sum_{s=\frac{n}{2}+1}^{n-1} |Z_{s-1}| + 2|Z_{\frac{n}{2}-1}| - 2|Z_{\frac{n}{2}-1}|$$

$$= 2p - 3 + \sum_{s=1}^{\frac{n}{2}-1} |Z_s| + 2\sum_{s=\frac{n}{2}}^{n-2} |Z_s|$$

$$= 2p - 3 + \sum_{s=1}^{n-2} |Z_s| + \sum_{s=\frac{n}{2}}^{n-2} |Z_s|$$

$= 2p - 3 + p^{n-1} - p^{n-n+2-1} + p^{\frac{n}{2}} - p$

$= p^{n-1} + p^{\frac{n}{2}} - 3 = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3.$

Similarly, from case 2, it can be seen that $\Gamma(\mathbb{Z}_{p^n})$ attains an upper bound if and only if $i = \frac{n-1}{2}$ and $j = |Z_{\frac{n-1}{2}}|$, i.e.,

$|Z_{\frac{n-1}{2}}| - 1 + \sum_{s=1}^{\frac{n-1}{2}} |Z_{s-1}| + 2\sum_{s=\frac{n+1}{2}}^{n-1} |Z_s|$

$= |Z_{\frac{n-1}{2}}| - 1 + \sum_{s=1}^{\frac{n-3}{2}} |Z_s| + 2\sum_{s=\frac{n+1}{2}}^{n-1} |Z_s|$

$= -1 + \sum_{s=1}^{\frac{n-1}{2}} |Z_s| + 2\sum_{s=\frac{n+1}{2}}^{n-1} |Z_s|$

$= -1 + \sum_{s=1}^{n-1} |Z_s| + \sum_{s=\frac{n+1}{2}}^{n-1} |Z_s|$

$= -1 + p^{n-1} - 1 + p^{\frac{n-1}{2}} - 1 = p^{n-1} + p^{\frac{n-1}{2}} - 3 = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3.$

One can see that for both cases, the span of γ is equal to $p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$.

Claim: The labeling γ is a valid radio labeling. We must show that the following radio condition holds for all pair of vertices $x_1^i, x_2^j \in V(\Gamma(\mathbb{Z}_{p^n}))$, where $x_1^i \neq x_2^j$:

$$d(x_1^i, x_2^j) + |\gamma(x_1^i) - \gamma(x_2^j)| \geq diam(\Gamma(\mathbb{Z}_{p^n})) + 1 = 3. \quad (11)$$

Case 1: n-even

1: Consider the pair (x_1^i, x_2^j) with $1 \leq i,j \leq \frac{n}{2} - 1$. Note that $d(x_1^i, x_2^j) = 2$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 1$. Hence, radio Eq. (11) is satisfied.

2: Consider the pair (x_1^i, x_2^j) with $\frac{n}{2} \leq i,j \leq n-1$. We have $d(x_1^i, x_2^j) = 1$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 2$. Hence, radio Eq. (11) is satisfied.

3: Consider the pair (x_1^i, x_2^j) with $i = 1, j = n-1$. We have $d(x_1^i, x_2^j) = 1$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 2$. Hence, radio Eq. (11) is satisfied.

4: Consider the pair (x_1^i, x_2^j) with $i = 1, \frac{n}{2} \leq j \leq n-2$. We have $d(x_1^i, x_2^j) = 2$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 2$. Hence, radio Eq. (11) is satisfied.

5: Consider the pair (x_1^i, x_2^j) with $2 \leq i \leq \frac{n}{2} - 2, \frac{n}{2} \leq j \leq n-1$. We have $d(x_1^i, x_2^j) = 1$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 2$. Hence, radio Eq. (11) is satisfied.

6: Consider the pair (x_1^i, x_2^j) with $i = \frac{n}{2} - 1, j = \frac{n}{2}$. We have $d(x_1^i, x_2^j) = 2$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 1$. Hence, radio Eq. (11) is satisfied.

7: Consider the pair (x_1^i, x_2^j) with $i = \frac{n}{2} - 1, \frac{n}{2} + 1 \leq j \leq n-1$. We have $d(x_1^i, x_2^j) = 1$ and $|\gamma(x_1^i) - \gamma(x_2^j)| \geq 2$. Hence, radio Eq. (11) is satisfied.

We have shown that condition (11) is satisfied for all pairs. This means that $rn(\Gamma(\mathbb{Z}_{p^n})) \leq \text{span}(\gamma) = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$ for n-even. Similarly, it is easy to show that $rn(\Gamma(\mathbb{Z}_{p^n})) \leq \text{span}(\gamma) = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$ for n-odd. This implies that $rn(\Gamma(\mathbb{Z}_{p^n})) \leq \text{span}(\gamma) = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$ for $n \geq 5$. This completes the proof.

Theorem 3.8 Let p be a prime number, and n $\geq$ 5 a positive integer. The radio number for a zero-divisor graph of $\mathbb{Z}_{p^n}$ is $p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$, i.e., $rn(\Gamma(\mathbb{Z}_{p^n})) = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$.

Proof. Combining Propositions 3.6 and 3.7, we obtain $rn(\Gamma(\mathbb{Z}_{p^n})) = p^{n-1} + p^{\lfloor \frac{n}{2} \rfloor} - 3$ for any $n \geq 5$.

4 Conclusion

We determined the radio number for the zero-divisor graph $\Gamma(\mathbb{Z}_{p^n})$ of the commutative ring $\mathbb{Z}_{p^n}$. In addition to the importance of the study on combinatorial properties associated with algebraic structure, these results can also be useful for circuit design and communication problems such as channel assessment.

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