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Parameter Estimation of Alpha Power Inverted Topp-Leone Distribution with Applications

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Abstract: We introduce a new two-parameter lifetime model, referred to alpha power transformed inverted Topp-Leone, derived by combining the alpha power transformation-G family with the inverted Topp-Leone distribution. Structural properties of the proposed distribution are implemented like; quantile function, residual and reversed residual life, Rényi entropy measure, moments and incomplete moments. The maximum likelihood, weighted least squares, maximum product of spacing, and Bayesian methods of estimation are considered. A simulation study is worked out to evaluate the restricted sample properties of the proposed distribution. Numerical results showed that the Bayesian estimates give more accurate results than the corresponding other estimates in the majority of the situations. The flexibility of the suggested model is demonstrated given some applications related to reliability, medicine, and engineering. A real data set is used to illustrate the potentiality of the alpha power transformed inverted Topp-Leone distribution compared to inverted Topp-Leone, inverse Weibull, alpha power inverse Weibull, inverse Lomax, alpha power inverse Lomax, inverse exponential, and alpha power exponential distributions. Criteria measures and their results showed that the suggested distribution is the best candidate for the considered data sets. The alpha power transformed inverted Topp-Leone distribution operates well for lifetime modeling.

Keywords: Inverted Topp-Leone; moments; maximum likelihood; maximum product spacing; weighted least squares; Bayesian estimation; MCMC

1 Introduction

In recent times, probability distributions play a significant role in modeling naturally occurring phenomena. In fact, the statistics literature contains hundreds of continuous univariate distributions and their successful applications. However, there still remain many real-world phenomena involving data, which do not follow any of the traditional probability distributions. So, several attempts are introduced by many researchers to provide more flexibility to a family of distributions. Mahdavi et al. [1] introduced the alpha power transformation (AP) technique by adding an extra shape parameter to well- known baseline distributions. The cumulative distribution function (CDF) of the AP method is defined by:

$$F_{AP}(x) = \begin{cases} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1 \\ G(x), & \alpha > 0,\ \alpha = 1 \end{cases}. \tag{1}$$

Relevant works have been provided based on the AP method, for instance; AP Weibull distribution [2], AP generalized exponential distribution [3], AP extended exponential distribution [4], AP Lindley distribution [5], AP power Lindley distribution [6], AP inverse Lindley distribution [7], AP exponentiated Lomax distribution [8] and AP inverse Lomax distribution [9].

The inverted distributions were suggested in the literature using the inverse transformation of probability distributions. These distributions display different features in the behavior of the density and hazard rate shapes. They allow applicability to the phenomenon in many fields such as; biological sciences, life testing problems, survey sampling, and engineering sciences. Inverted distributions and their applications were discussed by several authors (see [10–18]).

In Hassan et al. [18], the CDF and the probability density function (PDF) of the inverted Topp-Leone (ITL) distribution with shape parameter λ is defined as follows:

$$G(x;\lambda) = 1 - \left\{ \frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}} \right\};\ \ x \geq 0,\ \lambda > 0, \tag{2}$$

and,

$$g(x;\ \lambda) = 2\lambda x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1};\quad x,\ \lambda > 0. \tag{3}$$

In this paper, we propose a new two-parameter related to the ITL distribution depending on the AP family. We call it alpha power inverted Topp Leone (APITL) distribution. The basic motivations to introduce the APITL model are (i) Generalizing a new useful version of the ITL distribution based on the APT method along with deriving its statistical properties, (ii) Providing flexible PDF with right-skewed and uni-modal shapes, (iii) Modeling decreasing, increasing, upside-down hazard rate function (HRF), and (iv) Introducing some real applications in some areas.

This paper is constructed as follows. Section 2 describes the APITL distribution. Section 3 gives some structural properties of the APITL distribution. Section 4 gives the maximum likelihood (ML), the weighted least squares (WLS), the maximum product of spacing (MPS), and Bayesian estimators. Section 5 examines the effectiveness of the proposed estimates through a numerical illustration. Data analyses and some concluding remarks are employed, consequently, in Sections 6 and 7.

2 Description of the Model

In this section, based on the AP family we introduce a new probability distribution related to the ITL distribution. We define the PDF, CDF, HRF and cumulative HRF of the APITL distribution.

Definition 2.1

A random variable X is said to have the APITL distribution when we substitute the CDF (Eq. (2)) and PDF (Eq. (3)) in CDF (Eq. (1)). The CDF of a random variable X has the APITL distribution with parameters α and λ denoted by $X \sim \text{APITL}(\alpha, \lambda)$, is defined by:

$$F(x;\alpha,\lambda) = \begin{bmatrix} \dfrac{\alpha^{1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}}-1}{(\alpha-1)}, & x,\ \alpha,\lambda>0,\alpha\neq 1 \\ 1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}, & x,\alpha,\lambda>0,\ \alpha=1 \end{bmatrix}. \tag{4}$$

The PDF related to Eq. (4) is given by:

$$f(x;\alpha,\lambda) = \begin{bmatrix} \dfrac{2\lambda\log(\alpha)x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1}\alpha^{1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}}}{(\alpha-1)}, & x,\lambda,\alpha>0,\alpha\neq 1 \\ 2\lambda x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1}, & x,\lambda,\alpha>0,\ \alpha=1 \end{bmatrix}. \tag{5}$$

Descriptive PDF plots of the APITL distribution for some choices of parameters are represented in Fig. 1. It can be seen that the PDF of APITL distribution is uni-modal as well as possesses a long tail right-skewed.

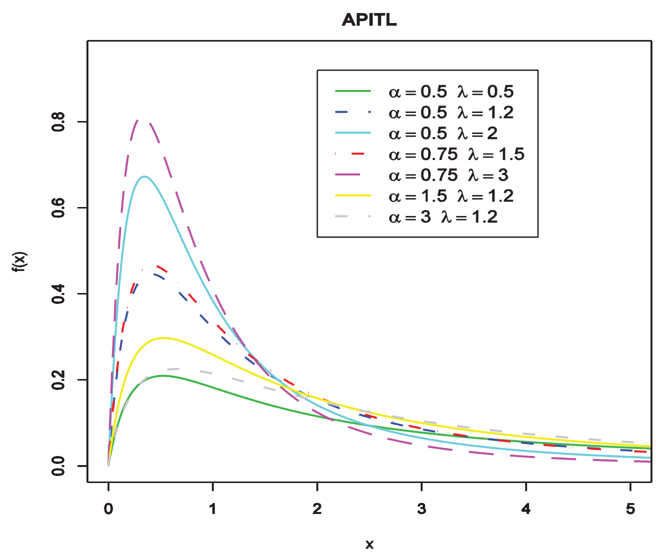


Figure 1: The PDF plots of the APITL distribution

Definition 2.2

The reliability function and the HRF of X are given by:

$$\bar{F}(x;\alpha,\lambda) = \begin{bmatrix} \dfrac{\alpha-\alpha^{1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}}}{\alpha-1}, & x,\lambda,\alpha>0,\alpha\neq 1, \\ (1+2x)^{\lambda}(1+x)^{-2\lambda}, & x,\lambda,\alpha>0,\alpha=1 \end{bmatrix} \tag{6}$$

and

$$h(x;\alpha,\lambda) = \begin{cases} \dfrac{2\lambda\log(\alpha)x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1}\alpha^{1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}}}{\alpha-\alpha^{1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}}}, & x,\lambda,\alpha>0, \alpha\neq 1 \\ 2\lambda x[(1+x)(1+2x)]^{-1}, & x,\lambda,\alpha>0,\ \alpha=1 \end{cases}. \tag{7}$$

An illustration of the HRF plots for the APITL distribution, for some choices of α and λ, is represented in Fig. 2. It describes the HRF plots of the APITL distribution which can be decreasing, increasing, and upside down shaped.

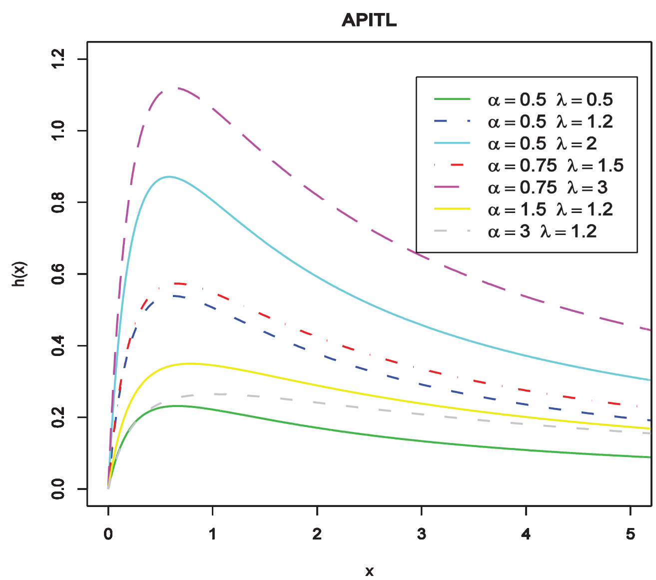


Figure 2: The HRF of the APITL distribution

3 Distributional Properties

Here, we give some statistical properties.

3.1 Quantile Function

The APITL distribution is simulated by inverting CDF Eq. (4) as follows:

$$x = Q(x) = F^{-1}(u), \qquad 0 < u < 1\,. \tag{8}$$

The u^{th} quantile for the APITL random variable is obtained by solving $F(x) = u$ for x as follows:

$$x_u = \frac{-2[(1-\mathrm{K})^{\frac{1}{\lambda}}-1]+\sqrt{4[(1-\mathrm{K})^{\frac{1}{\lambda}}-1]^2-4((1-\mathrm{K})^{\frac{1}{\lambda}})(((1-\mathrm{K})^{\frac{1}{\lambda}}-1)}}{2(1-\mathrm{K})^{\frac{1}{\lambda}}}, \tag{9}$$

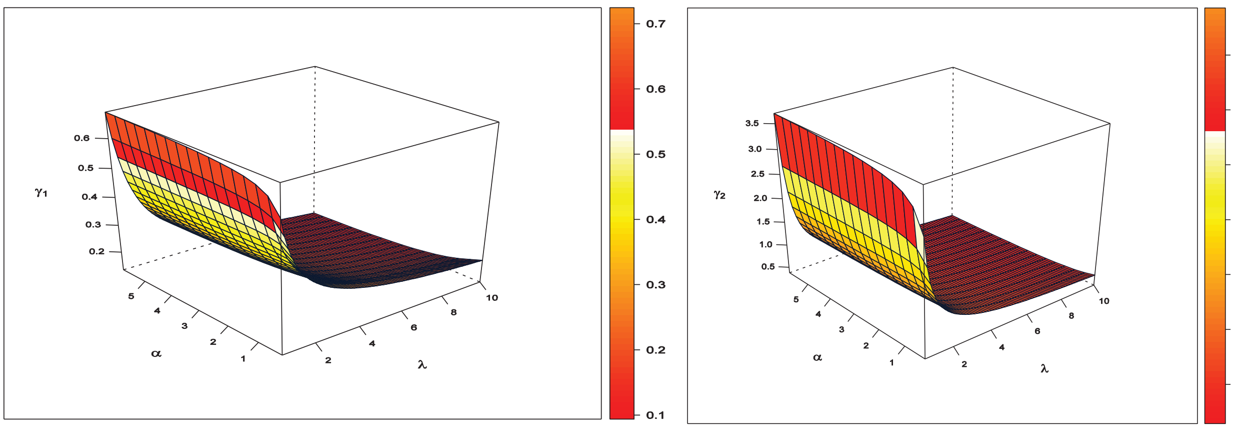
where, $\mathrm{K} = (\log u(\alpha-1)+1)(\log\alpha)^{-1}$. The 25^{th}, 50^{th}, and 75^{th} percentiles for the random variable X is obtained by setting u = 0.25, 0.5 and 0.75 in Eq. (9). The Bowley's skewness, depends on quartiles, is defined as follows:

$$\mathrm{SK}_B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/2)}, \tag{10}$$

where $Q(.)$ is the APITL quantile function. The Moor's kurtosis is given as:

$$\mathrm{KU}_m = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{11}$$

Skewness and kurtosis plots of the APITL model, based on the quantiles, are exhibited in Fig. 3.



(a) (b)

Figure 3: 3D Plots of (a) Skewness and (b) Kurtosis of the APITL distribution

3.2 Moments

Moments of a probability distribution are crucial to deduce its properties such as measures of central tendency, dispersion, skewness, and kurtosis. The ordinary rth moment of the APITL distribution is derived. The rth moment of the APITL distribution is obtained from Eq. (5) as follows:

$$\mu_r' = \frac{2\lambda \log(\alpha)}{(\alpha - 1)} \int_0^\infty x^{r+1}(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1} \alpha^{1-\left\{\frac{(1+2x)^\lambda}{(1+x)^{2\lambda}}\right\}} dx. \tag{12}$$

Since the power series representation is written as:

$$\alpha^x = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} x^i. \tag{13}$$

Using the power series expansion Eq. (13) in Eq. (12), then we get

$$\mu_r' = \frac{2\lambda}{(\alpha - 1)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty x^{r+1}(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1} \left[1 - \left\{\frac{(1+2x)^\lambda}{(1+x)^{2\lambda}}\right\}\right]^i dx. \tag{14}$$

Using the binomial expansion in Eq. (14) then μ_r' takes the form

$$\mu_r' = \frac{2\lambda}{(\alpha-1)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{(\log\alpha)^{i+1}}{i!} \int_0^{\infty} x^{r+1} (1+x)^{-2\lambda(j+1)-1} (1+2x)^{\lambda(j+1)-1} dx. \tag{15}$$

After using binomial expansion, then the r^th moment is given by:

$$\begin{aligned} \mu'_r &= \frac{2\lambda}{(\alpha-1)} \sum_{i,m=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\lambda(j+1)-1}{m} \frac{(\log\alpha)^{i+1}}{i!} \int_0^{\infty} x^{r+m+1} (1+x)^{-\lambda(j+1)-m-2} dx. \\ &= \sum_{i,m=0}^{\infty} \xi_{i,j,m} \mathrm{B}(r+m+2, \lambda(j+1)-r), \qquad \lambda(j+1) > r,\ r = 1, 2, \ldots. \end{aligned} \tag{16}$$

where, $\xi_{i,j,m} = \frac{2\lambda}{(\alpha-1)} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\lambda(j+1)-1}{m} \frac{(\log\alpha)^{i+1}}{i!}$.

The first four moments, for $r = 1, 2, 3$ and 4, of the APITL distribution are obtained from Eq. (16). The r^th central moment (μ_r) of X is given by:

$$\mu_r = E(X - \mu_1')^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (\mu_1')^i \mu'_{r-i}. \tag{17}$$

Values of mean (μ_1'), variance (σ^2), skewness (γ_1), and kurtosis (γ_2) of the APITL distribution for certain values of parameters are given in Tab. 1.

Table 1: Some moments values of the APITL distribution for specific values of parameters

(α, λ)	μ_1'	σ^2	γ_1	γ_2
(3,4.5)	0.458	0.684	4.339	89.518
(3,6)	0.347	0.335	2.936	21.996
(3,10)	0.226	0.121	2.053	9.306
(10,4.5)	0.564	0.932	3.866	72.824
(10,6)	0.421	0.446	2.588	17.809
(10,10)	0.269	0.157	1.785	7.526

We concluded from Tab. 1 that, as the value of λ increases and for fixed value of α, then values of μ_1', σ^2, γ_1, and γ_2 are decreasing. As the value of α increases and for fixed value of λ, then values of μ_1', and σ^2 are increasing, while the values of γ_1 and γ_2 decrease. The distribution is positively skewed and leptokurtic.

3.3 The Probability Weighted Moments

The class of probability-weighted moments (PWMs), denoted by $\Upsilon_{s,q}$, s and q are positive integers, of a random variable X is defined as follows:

$$\Upsilon_{s,q} = E[X^s F(x)^q] = \int_{-\infty}^{\infty} x^s f(x) (F(x))^q dx. \tag{18}$$

Substituting Eq. (4), and Eq. (5) in Eq. (18), further using expansion Eq. (13) and binomial expansion in Eq. (18), then the PWM of the APITL distribution is obtained as follows:

$$\Upsilon_{s,q} = \sum_{i,d=0}^{\infty} v_{m,k,i,d} \mathrm{B}(s+d+2, \lambda(m+1)-s). \tag{19}$$

where, $v_{m,k,i,d} = \sum_{m=0}^{i}\sum_{k=0}^{q} \binom{\lambda(m+1)-1}{d}\binom{i}{m}\binom{q}{k} \frac{(-1)^{q-k+m} 2\lambda(\log(\alpha))^{i+1}(q+1)^{i}}{(\alpha-1)^{q+1} i!}$.

3.4 Residual and Reversed Residual Life

The m^{th} moment of the residual life (RLe), say $\Pi_m(t) = E[(X-t)^m | X > t], m = 1, 2, ..$ uniquely determines $F(x)$. The m^{th} moment of the residual life of X is defined by:

$$\Pi_m(t) = (\bar{F}(t))^{-1} \int_t^{\infty} (x-t)^m dF(x). \tag{20}$$

Therefore, the m^{th} moment of RLe for APITL distribution is obtained by substituting PDF Eq. (5) in Eq. (20), then employing binomial expansions and Eq. (13) as follows:

$$\Pi_m(t) = (\bar{F}(t))^{-1} \sum_{k,\ell=0}^{\infty} \Psi_{k,i,\ell,d}(t)^{m-i} \mathrm{B}(i+\ell+2, \lambda(d+1)-i, (1+t)^{-1}), \tag{21}$$

where $\Psi_{i,k,\ell,d} = \sum_{i=0}^{m}\sum_{d=0}^{k} \frac{(-1)^{m-i+d} 2\lambda(\log\alpha)^{k+1}}{k!(\alpha-1)} \binom{m}{i}\binom{k}{d}\binom{\lambda(d+1)-1}{\ell}$, and $\mathrm{B}(.,.,(1+t)^{-1})$ is the incomplete beta function. For $m = 1$, in Eq. (21) we obtain the mean RLe of the APITL distribution. Also, m^{th} moment of the reversed RLe (RRLe) for the APITL distribution is given by:

$$\varepsilon_m(t) = (F(t))^{-1} \sum_{k,\ell=0}^{\infty} \Psi_{i,k,\ell,d}(t)^{m-i} \mathrm{B}(i+\ell+2, \lambda(d+1)-i, t/(1+t)), \tag{22}$$

where, $\Psi_{i,k,\ell,d} = \sum_{i=0}^{m}\sum_{d=0}^{k} \frac{(-1)^{m-i+d} 2\lambda(\log\alpha)^{k+1}}{k!(\alpha-1)} \binom{m}{i}\binom{k}{d}\binom{\lambda(d+1)-1}{\ell}$.

The mean of RRLe serves as the waiting time elapsed since the failure of an item on the condition that this failure had occurred.

3.5 Rényi Entropy

The entropy of a random variable measures the amounts of information (or uncertainty) contained in a random observation; i.e., large value of entropy indicates higher uncertainty in the data. Rényi entropy of X, for $\upsilon > 0$ and $\upsilon \neq 1$, is defined by:

$$R_e(x) = (1-\upsilon)^{-1} \log\left(\int_0^{\infty} (f(x))^{\upsilon} dx\right). \tag{23}$$

Substituting PDF Eq. (5) in Eq. (23), we obtain Rényi entropy of the APTIL model as follows:

$$R_e(x) = (1-\upsilon)^{-1} \log\left[\left(\frac{2\lambda \log(\alpha)}{(\alpha-1)^{\upsilon}}\right)^{\upsilon} \int_0^{\infty} x^{\upsilon}(1+x)^{-\upsilon(2\lambda+1)}(1+2x)^{\upsilon(\lambda-1)}\alpha^{\upsilon\left[1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}\right]} dx\right]. \tag{24}$$

From Eq. (24), and after simplification, the Rényi entropy of the APITL distribution will be:

$$R_e(x) = (1-\upsilon)^{-1} \log\left\{\sum_{j,k=0}^{\infty} \Phi_{j,m,k} \mathrm{B}(\upsilon+k+1, \lambda\upsilon+\lambda m+\upsilon-1)\right\}, \tag{25}$$

where, $\Phi_{j,m,k} = \sum_{m=0}^{j} \frac{(\upsilon log\alpha)^j(-1)^m}{j!}\left(\frac{2\lambda\log(\alpha)}{(\alpha-1)^{\upsilon}}\right)^{\upsilon}\binom{j}{m}\binom{\upsilon(\lambda-1)+\lambda m}{k}$.

4 Parameter Estimation of the APITL Model

In this section, we deal with parameter estimators of the APITL distribution based on ML, WLS, MPS, and Bayesian estimation methods.

4.1 ML Estimators

Let $X_1,\ldots, X_n$ e the observed values from the APITL distribution with parameters α and λ. The likelihood function, say $L(\underline{x}|\alpha, \lambda)$ of the APITL distribution is expressed as:

$$L(\underline{x}|\alpha,\lambda) = 2^n\lambda^n(\log(\alpha))^n(\alpha-1)^{-n}\alpha^{n-\sum_{i=1}^{n}\frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}}}\prod_{i=1}^{n}\frac{x_i(1+2x_i)^{\lambda-1}}{(1+x_i)^{2\lambda+1}}. \tag{26}$$

Then the log-likelihood function, say ℓ, of the APITL distribution is given as:

$$\begin{aligned}\ell = {} & n\log(2\lambda)+n\log(\log(\alpha))-n\log(\alpha-1)+n\log(\alpha)-\log(\alpha)\sum_{i=1}^{n}\frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}} \\ & +\sum_{i=1}^{n}\log(x_i)+(\lambda-1)\sum_{i=1}^{n}\log(1+2x_i)-(2\lambda+1)\sum_{i=1}^{n}\log(1+x_i).\end{aligned} \tag{27}$$

Therefore, the ML equations are given by:

$$\frac{\partial\ell}{\partial\alpha} = \frac{1}{\alpha}\left[n+\frac{n}{\log(\alpha)}-\sum_{i=1}^{n}\frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}}\right]-\frac{n}{\alpha-1}, \tag{28}$$

and,

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda}-\log(\alpha)\sum_{i=1}^{n}\frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}}\log\left(\frac{(1+2x_i)}{(1+x_i)^2}\right)+\sum_{i=1}^{n}\log(1+2x_i)-2\sum_{i=1}^{n}\log(1+x_i). \tag{29}$$

Solving the non-linear equations $(\partial\ell/\partial\alpha)=0$ and $(\partial\ell/\partial\lambda)=0$ numerically using optimization algorism as conjugate-gradient optimization, we get the ML estimators of α and λ.

4.2 WLS Estimators

Let $X_1,\ldots, X_n$ be a simple random sample from the APITL distribution and let $X_{(1)}< X_{(2)}<\ldots<X_{(n)}$ be the associated order statistics. The WLS estimators of α and λ are attained by minimizing the following:

$$W(\underline{x}|\alpha,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\Xi\left(x_{(i)}\right)-\frac{i}{n+1}\right)^2, \tag{30}$$

$\Xi\left(x_{(i)}\right)=(\alpha-1)^{-1}\left(\alpha^{1-\left\{\frac{(1+2x_{(i)})^{\lambda}}{(1+x_{(i)})^{2\lambda}}\right\}}-1\right)$. Furthermore, the WLS estimators followed by solving the following nonlinear equations:

$$2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\Xi\left(x_{(i)}\right)-\frac{i}{n+1}\right)\Lambda_s\left(x_{(i)}\right);\quad s=\alpha,\lambda. \tag{31}$$

where $\Lambda_\alpha\left(x_{(i)}\right)=\dfrac{\alpha^{-\left\{\frac{(1+2x_{(i)})^{\lambda}}{(1+x_{(i)})^{2\lambda}}\right\}}\left[\frac{(1+2x_{(i)})^{\lambda}}{(1+x_{(i)})^{2\lambda}}(1-\alpha)-1\right]+1}{(\alpha-1)^2}$ and

$$\Lambda_\lambda\left(x_{(i)}\right)=\frac{\alpha^{1-\left\{\frac{(1+2x_{(i)})^{\lambda}}{(1+x_{(i)})^{2\lambda}}\right\}}}{(\alpha-1)}\left(\frac{(1+2x_{(i)})}{(1+x_{(i)})^2}\right)^{\lambda}\ln(\alpha)\ln\left(\frac{(1+2x_{(i)})}{(1+x_{(i)})^2}\right).$$

4.3 MPS Estimators

The MPS method is an alternative procedure of the ML method which provides a parameter estimate of continuous distribution. The MPS estimators of α and λ are attained by maximizing the following:

$$\ell(g)=\frac{1}{n+1}\left\{\ln\left(\Xi\left(x_{(1)}\right)\right)+\sum_{i=2}^{n}\ln\left(\Xi\left(x_{(i)}\right)-\Xi\left(x_{(i-1)}\right)\right)+\ln\left(1-\Xi\left(x_{(n)}\right)\right)\right\}. \tag{32}$$

Solving the non-linear equations $(\partial\ell(g)/\partial\alpha)=0$ and$(\partial\ell(g)/\partial\alpha)=0$ via numerical technique, we obtain the MPS estimators of α and λ. To obtain the MPS estimators, we differentiate natural logarithm of the product spacing function of APITL distribution partially for α and λ.

$$\frac{\partial\ell(g)}{\partial\alpha}=\frac{1}{n+1}\left[\frac{\Lambda_\alpha\left(x_{(1)}\right)}{\Xi\left(x_{(1)}\right)}+\sum_{i=2}^{n}\frac{\left(\Lambda_\alpha\left(x_{(i)}\right)-\Lambda_\alpha\left(x_{(i-1)}\right)\right)}{\Xi\left(x_{(i)}\right)-\Xi\left(x_{(i-1)}\right)}-\frac{\Lambda_\alpha\left(x_{(n)}\right)}{1-\Xi\left(x_{(n)}\right)}\right], \tag{33}$$

and,

$$\frac{\partial\ell(g)}{\partial\lambda}=\frac{1}{n+1}\left[\frac{\Lambda_\lambda\left(x_{(1)}\right)}{\Xi\left(x_{(1)}\right)}+\sum_{i=2}^{n}\frac{\left(\Lambda_\lambda\left(x_{(i)}\right)-\Lambda_\lambda\left(x_{(i-1)}\right)\right)}{\Xi\left(x_{(i)}\right)-\Xi\left(x_{(i-1)}\right)}-\frac{\Lambda_\lambda\left(x_{(n)}\right)}{1-\Xi\left(x_{(n)}\right)}\right]. \tag{34}$$

Equating Eqs. (33) and (34) with zero and using optimization algorism (conjugate-gradient or Newton-Raphson optimization) we get the solution.

4.4 Bayesian Estimators

Here, we get the Bayesian estimator of the APITL parameters. The Bayesian estimator is regarded under symmetric (squared error loss function (SELF)) which is defined as $L(\tilde{\alpha},\alpha)=E(\tilde{\alpha}-\alpha)^2$, $L(\tilde{\lambda},\lambda)=E(\tilde{\lambda}-\lambda)^2$. Also, the Bayesian estimator is considered under asymmetric (Linear exponential (LINEX) loss function) which is expressed as $L(\breve{\alpha},\alpha)=e^{h(\breve{\alpha}-\alpha)}-h(\breve{\alpha}-\alpha)-1$,

$L(\breve{\lambda}, \lambda) = e^{h(\breve{\lambda}, \lambda)} - h(\breve{\lambda}, \lambda) - 1, h \neq 0, \breve{\alpha} = \frac{-1}{h} \ln E(e^{-h\alpha}),\ \breve{\lambda} = \frac{-1}{h} \ln E(e^{-h\lambda})$, where h reflects the direction and degree of asymmetry. Assuming that the prior distribution of α and λ, denoted by $\pi(\alpha), \pi(\lambda)$, has an independent gamma distribution. The joint gamma prior density of α and λ can be written as:

$$\pi(\alpha, \lambda) \propto \alpha^{a_1-1} e^{-b_1\alpha} \lambda^{a_2-1} e^{-b_2\lambda}; a_1, b_1, a_2, b_2 > 0. \tag{35}$$

To elicit the hyper-parameters of the informative priors, the ML estimator for α and λ is obtained by equating the estimates and their variances by the inverse of Fisher information matrix of $\hat{\alpha}$ and $\hat{\lambda}$.

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{n}{(\alpha-1)^2} - \frac{n}{\alpha^2 (\log(\alpha))^2} - \frac{1}{\alpha^2}\left[1 + \frac{n}{\log(\alpha)} - \sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}}\right], \tag{36}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{-n}{\lambda^2} - \log(\alpha) \sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}} \left[\log\left(\frac{(1+2x_i)}{(1+x_i)^2}\right)\right]^2, \tag{37}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = -\frac{1}{\alpha} \sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}} \log\left(\frac{(1+2x_i)}{(1+x_i)^2}\right), \tag{38}$$

where, $V(\alpha, \lambda) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}^{-1}_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}}$.

From Eqs. (26) and (35), the joint posterior of the APITL distribution with parameters α and λ is

$$\pi(\alpha, \lambda | \underline{x}) \propto e^{-b_1\alpha} \lambda^{n+a_2-1} e^{-b_2\lambda} (\log(\alpha))^n (\alpha-1)^{-n} \alpha^{n - \sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}} + a_1 - 1} \prod_{i=1}^{n} \frac{(1+2x_i)^{\lambda-1}}{(1+x_i)^{2\lambda+1}}. \tag{39}$$

To obtain the Bayesian estimators, we can use the Markov Chain Monte Carlo (MCMC) approach. A useful sub-class of the MCMC techniques is the Gibbs sampling and more general Metropolis within Gibbs samplers. The Metropolis-Hastings (MH) algorithm along with the Gibbs sampling are the two most popular examples of the MCMC method. We use the MH within Gibbs sampling steps to generate random samples from conditional posterior densities of α and λ as follows:

$$\pi(\alpha | \lambda, \underline{x}) \propto e^{-b_1\alpha} (\log(\alpha))^n (\alpha-1)^{-n} \alpha^{n - \sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}} + a_1 - 1}, \tag{40}$$

and

$$\pi(\lambda | \alpha, \underline{x}) = \lambda^{n+a_2-1} e^{-b_2\lambda} \alpha^{-\sum_{i=1}^{n} \frac{(1+2x_i)^{\lambda}}{(1+x_i)^{2\lambda}}} \prod_{i=1}^{n} \frac{(1+2x_i)^{\lambda-1}}{(1+x_i)^{2\lambda+1}}. \tag{41}$$

The Bayesian estimators are obtained via SELF and LINEX loss function (for more information see [19–23]).

5 Simulation Study

A Monte-Carlo simulation study was conducted to evaluate and compare the behavior of the different estimates based on mean square errors (MSEs) and biases. Generate 10000 random samples of sizes $n = 50$, 100, 150 and 200 from APITL distribution. Different actual parameter values were considered.

We calculated the ML estimate (MLE), WLS estimate (WLSE), MPS estimate (MPSE), and Bayesian estimate (BE) of α and λ. Then, the biases and MSEs of the different estimates were determined. Simulated results were scheduled in Tabs. 2 and 3 and we noticed the following:

Table 2: Biases and MSEs of the APITL distribution under different methods

				ML		WLS		MPS		SELF		LINEX (0.5)		LINEX (1.5)	
n	α	λ		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
50	0.75	0.5	α	−0.0548	0.9675	−0.0215	0.1834	−0.0550	0.3838	0.12695	0.08138	0.09019	0.06391	0.02555	0.04295
			λ	−0.0716	0.0227	−0.0282	0.0148	−0.0718	0.0295	0.00794	0.00117	0.00648	0.00113	0.00358	0.00108
		1.2	α	0.9620	1.0917	1.1126	0.7401	0.9194	1.0035	0.13249	0.08978	0.09817	0.07064	0.03771	0.04739
			λ	0.0192	0.0634	−0.0667	0.0547	−0.1264	0.0797	0.00529	0.00043	0.00317	0.00041	−0.00103	0.00039
		3	α	−0.0465	0.8156	0.2324	0.7940	−0.0467	0.3477	0.12012	0.09971	0.09288	0.08479	0.04281	0.06343
			λ	−0.3475	0.7940	−0.0114	0.6331	−0.3481	0.7195	0.00191	0.00012	−0.00039	0.00011	−0.00498	0.00014
	2	0.5	α	−0.2319	0.6502	−0.1322	0.2170	−0.2317	0.4993	0.06612	0.02823	0.00341	0.02121	−0.10808	0.02961
			λ	−0.0332	0.0072	−0.0134	0.0053	−0.0333	0.0074	0.00422	0.00034	0.00360	0.00034	0.00236	0.00033
		1.2	α	0.4612	1.0682	−0.1374	0.7401	−0.3306	1.0032	0.24996	0.29895	0.06710	0.17261	−0.20445	0.15091
			λ	0.0192	0.0634	−0.0667	0.0547	−0.1279	0.0799	0.01179	0.00326	0.00667	0.00310	−0.00341	0.00296
		3	α	−0.0302	0.9561	0.0334	0.9424	−0.3011	0.9032	0.36851	0.63164	0.12254	0.35078	−0.21574	0.24823
			λ	−0.2719	0.4773	−0.1700	0.4296	−0.2723	0.3900	0.03683	0.02339	0.00285	0.02093	−0.06248	0.02296
100	0.75	0.5	α	−0.0825	0.3758	−0.0312	0.1116	−0.0826	0.2327	0.12948	0.06681	0.09599	0.05252	0.03725	0.03545
			λ	−0.0550	0.0124	−0.0214	0.0081	−0.0551	0.0179	0.00833	0.00116	0.00712	0.00113	0.00471	0.00108
		1.2	α	1.1799	0.6729	1.1556	0.6264	0.9860	0.6087	0.11617	0.06892	0.09008	0.05684	0.04303	0.04026
			λ	0.0055	0.0371	−0.0456	0.0322	−0.0798	0.0367	0.00533	0.00058	0.00334	0.00056	−0.00064	0.00054
		3	α	−0.0781	0.6382	0.1700	0.5510	−0.0782	0.2098	0.08784	0.06171	0.06821	0.05404	0.03166	0.04277
			λ	−0.2766	0.4818	−0.0039	0.4971	−0.2771	0.4474	0.00182	0.00017	−0.00044	0.00017	−0.00496	0.00019
	2	0.5	α	−0.1630	0.6384	−0.0451	0.0974	−0.1629	0.2653	0.07283	0.03301	0.01439	0.02512	−0.08987	0.02942
			λ	−0.0201	0.0044	−0.0023	0.0027	−0.0202	0.0036	0.00229	0.00037	0.00179	0.00036	0.00080	0.00036
		1.2	α	−0.0701	0.7286	−0.0944	0.6264	−0.2640	0.6087	0.22919	0.26288	0.07037	0.15881	−0.16927	0.13156
			λ	0.0055	0.0371	−0.0456	0.0322	−0.0798	0.0371	0.00925	0.00343	0.00511	0.00333	−0.00309	0.00322
		3	α	−0.2413	0.7090	0.0892	0.6865	−0.2411	0.5432	0.35177	0.45415	0.14393	0.31028	−0.15169	0.20198
			λ	−0.1764	0.2568	−0.0968	0.2831	−0.1767	0.1923	0.02796	0.02341	0.00021	0.02178	−0.05398	0.02321

1. The bias and MSE for all estimates decrease as n increases (see Tabs. 2 and 3).
2. As values of λ near to one and for fixed α values, the biases and MSEs of α and λ estimates increase.
3. For a fixed value of α as well as the value of λ increases, the biases and MSEs of λ estimates are increasing, in approximately most of the cases.
4. For a fixed value of λ as well as the value of α increases, the biases, and MSEs for all estimates increase, in approximately most of the situations.
5. The measures of WLSEs are better than MLEs and MPSEs with decreasing sample size.
6. The measures of MPSEs are preferable to MLEs and WLSEs with sample sizes.
7. The BEs under the LINEX loss function is preferable to the other estimates.

Table 3: Biases and MSEs of the APITL distribution under different methods

				ML		WLS		MPS		SELF		LINEX (0.5)		LINEX(1.5)	
n	α	λ		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
150	0.75	0.5	α	−0.0499	0.24593	0.0021	0.10933	−0.0495	0.15739	0.11137	0.06125	0.08280	0.04947	0.03186	0.03468
			λ	−0.0337	0.00768	−0.0123	0.00590	−0.0337	0.00955	0.00761	0.00109	0.00658	0.00106	0.00451	0.00103
		1.2	α	0.1631	0.32274	0.0962	0.27620	−0.0499	0.17235	0.07791	0.04444	0.05725	0.03753	0.01968	0.02845
			λ	0.0257	0.02466	−0.0102	0.06043	−0.0844	0.05781	0.00754	0.00074	0.00562	0.00071	0.00179	0.00067
		3	α	−0.0498	0.32287	0.1404	0.38843	−0.0494	0.13954	0.07904	0.04635	0.06217	0.04073	0.03098	0.03273
			λ	−0.1777	0.28429	0.0135	0.38771	−0.1773	0.25787	0.00289	0.00021	0.00065	0.00020	−0.00382	0.00022
	2	0.5	α	−0.0958	0.38265	0.0116	0.27172	−0.0953	0.15978	0.06533	0.03428	0.01100	0.02764	−0.08656	0.03170
			λ	−0.0118	0.00240	−0.0008	0.00189	−0.0117	0.00183	0.00173	0.00037	0.00130	0.00037	0.00045	0.00037
		1.2	α	0.2403	0.62056	0.0029	0.61890	−0.1635	0.37899	0.23018	0.24647	0.08626	0.15308	−0.13520	0.11692
			λ	0.0124	0.02279	−0.0250	0.02251	−0.0472	0.01905	0.00947	0.00338	0.00591	0.00328	−0.00116	0.00317
		3	α	−0.1492	0.59608	0.1421	0.55189	−0.1485	0.33969	0.29826	0.24158	0.11988	0.24502	−0.14413	0.16560
			λ	−0.1057	0.15444	−0.0431	0.14067	−0.1055	0.10120	0.03659	0.02650	0.01246	0.02303	−0.03511	0.02274
200	0.75	0.5	α	−0.0541	0.15230	0.0255	0.10660	−0.0544	0.12684	0.10701	0.06040	0.08064	0.04936	0.03347	0.03510
			λ	−0.0324	0.00575	−0.0070	0.00484	−0.0325	0.00758	0.00738	0.00103	0.00646	0.00102	0.00462	0.00101
		1.2	α	0.1065	0.22339	0.0826	0.23848	−0.0547	0.13868	0.07764	0.03694	0.05978	0.03163	0.02700	0.02435
			λ	0.0044	0.01361	−0.0151	0.05126	−0.0808	0.04575	0.00711	0.00081	0.00528	0.00078	0.00164	0.00074
		3	α	−0.0525	0.22652	0.0986	0.27801	−0.0528	0.11287	0.07426	0.03685	0.06035	0.03287	0.03426	0.02684
			λ	−0.1730	0.21711	−0.0140	0.28686	−0.1737	0.20680	0.00385	0.00024	0.00163	0.00023	−0.00279	0.00023
	2	0.5	α	−0.0900	0.16462	−0.0461	0.14601	−0.0902	0.13328	0.07360	0.03368	0.02149	0.02634	−0.07237	0.02793
			λ	−0.0124	0.00144	−0.0061	0.00165	−0.0125	0.00136	0.00231	0.00035	0.00194	0.00035	0.00118	0.00034
		1.2	α	−0.0190	0.46434	−0.0108	0.41257	−0.1521	0.31712	0.20131	0.23086	0.07490	0.15318	−0.12522	0.11925
			λ	−0.0050	0.01542	−0.0215	0.01560	−0.0463	0.01419	0.00809	0.00329	0.00496	0.00321	−0.00128	0.00312
		3	α	−0.1362	0.40894	0.1118	0.39202	−0.1365	0.28282	0.29218	0.39220	0.13049	0.24178	−0.11010	0.16256
			λ	−0.1035	0.11330	−0.0447	0.10584	−0.1039	0.07466	0.03851	0.02562	0.01718	0.02244	−0.02495	0.02136

6 Real Data Illustration

Here, we fit the APITL distribution under three real data taken from fields of survival times of medicine, engineering, and reliability. The APITL model is compared to other competitive models as, ITL, inverse Weibull (IW), alpha power IW (APIW), inverse Lomax (ILo), alpha power ILo (APILo), inverse exponential (IEx), and alpha power inverse exponential (APIEx) distributions.

Tabs. 4–6 provide values of Akaike information criterion (AIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov- Smirnov (KS) statistic along with its *P*-value for all fitted models for three real data. In addition, these tables contain the MLEs and standard errors (SEs) (appear in parentheses) of the parameters for the considered models. We compared the fits of the APITL model with the ITL, IW, APIW, ILo, APILo, IEx, and APIEx models (see Tabs. 4–6). The fitted APITL PDF, CDF, PP-plot and QQ-plot of the three real data were displayed in Figs. 4–6, respectively. These figures indicated that the APITL distribution has the smallest values of AIC, CAIC, HQIC, KS and the largest P-value among all fitted models.

The first (I) set of data was studied in Bjerkedal [24]. It represents the rth survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. These data were analyzed in Refs. [25,26]. Tab. 4 listed values of MLEs and statistic measures for data I. Fig. 4 provided estimated PDF, CDF, PP-plot, and QQ-plot of APITL distribution for data **I**.

Table 4: MLEs, SEs, and statistic measures for data **I**

Model	α	λ	β	KS	P-Value	AIC	CAIC	HQIC
APITL	204.3711 (186.05)	4.5486 (0.4218)	–	0.082	0.7179	191.1228	191.2967	192.9355
APIW	416.1724 (377.35)	1.5890 (0.1117)	0.2231 (0.0431)	0.1449	0.09728	217.6459	217.9988	220.3649
APILo	0.0026 (0.0018)	12.3172 (3.5600)	0.2714 (0.0906)	0.1141	0.3056	209.0081	209.361	211.7271
APIEx	0.0199 (0.0195)	2.3775 (0.3039)	–	0.1582	0.05434	229.4396	229.6135	231.2523

Table 5: MLEs, SEs, and statistic measures for data **II**

Model	α	λ	β	KS	P-Value	AIC	CAIC	HQIC
ITL		1.2533 (0.1982)		0.14799	0.3450	188.9506	189.0559	189.5613
APITL	6.9168 (6.6745)	1.8963 (0.3783)		**0.10909**	**0.7278**	**187.4425**	**187.7668**	**188.6638**
ILo		3.7381 (2.1288)	0.4900 (0.3351)	0.11064	0.7092	189.0165	189.3408	190.2378
APILo	10.9198 (9.0519)	23.9997 (5.6358)	0.0293 (0.0444)	0.1123	0.6945	191.5962	192.2629	193.4282

Table 6: MLEs, SEs, and statistic measures for data **III**

Model	α	λ	β	KS	P-Value	AIC	CAIC	HQIC
APITL	301.696 (339.011)	0.8271 (0.1006)		0.150	0.5082	311.0474	311.4918	311.9439
IW		0.7238 (0.0927)	6.9663 (1.8023)	0.159	0.4304	314.2288	314.6733	315.1253
APIW	42.7187 (70.859)	0.8576 (0.1391)	3.1712 (1.8568)	0.158	0.4906	312.3953	313.3184	313.7401
IE		11.1799 (2.0412)		0.233	0.07706	320.1239	320.2668	320.5722
APIE	23.767 (23.4501)	5.1597 (1.6219)		0.1591	0.4598	311.4365	311.8809	312.333

The second (II) set of data represents the active repair times (hr) for an airborne communication transceiver [27]. The data are as follows: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00 and 24.50. The MLEs and statistic measures for data **II** listed in Tab. 5. Fig. 5 provided estimated PDF, CDF, PP-plot, and QQ-plot of the APITL model for data **II**.

The third (III) set of data was studied in Aarset [28]. It refers to 30 failure times of air-conditioning system of an airplane. Tab. 6 listed values of MLEs and statistic measures for data **III**. Fig. 6 provided estimated PDF, CDF, PP-plot and QQ-plot of the APITL for data **III**.

According to tables and Figs. 4–6, we observed that the APITL distribution provides well overall the fitted model and consequently could be selected as the more suitable model than other models.

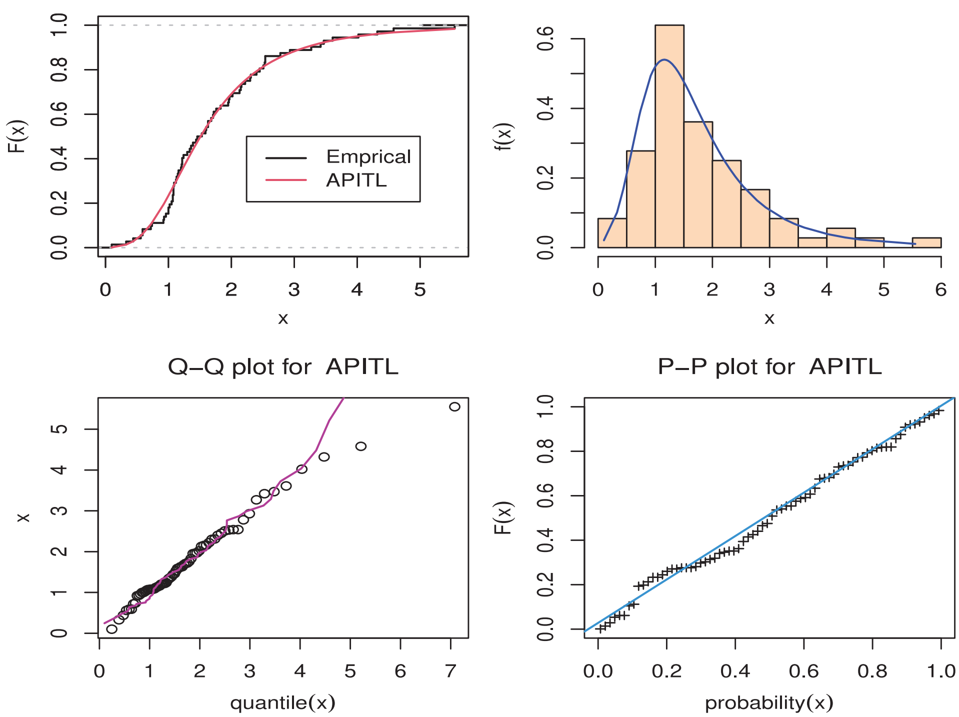


Figure 4: Estimated PDF, CDF, PP-plot and QQ-plot of the APITL model for data **I**

Furthermore, the suggested methods of estimation (see Section 4) for the APITL parameters were considered based on the three data. Tab. 7 displayed different estimates of the APITL parameters for the three data sets. In these data, we cannot use the MPS method because there are equal observations in the data and consequently the spacing followed by the product will be zeros. For more information about this method see [29,30].

The convergence of the MCMC estimation of α and λ are shown in Figs. 7–9 for all datasets, respectively.

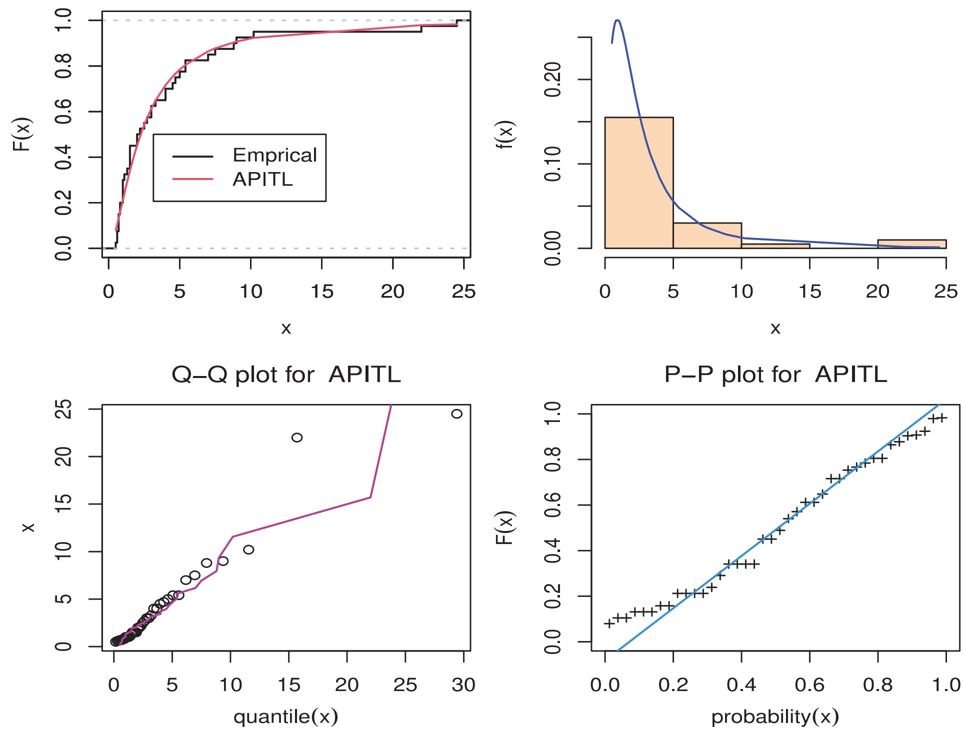


Figure 5: Estimated PDF, CDF, PP-plot, and QQ-plot of the APITL model for data **II**

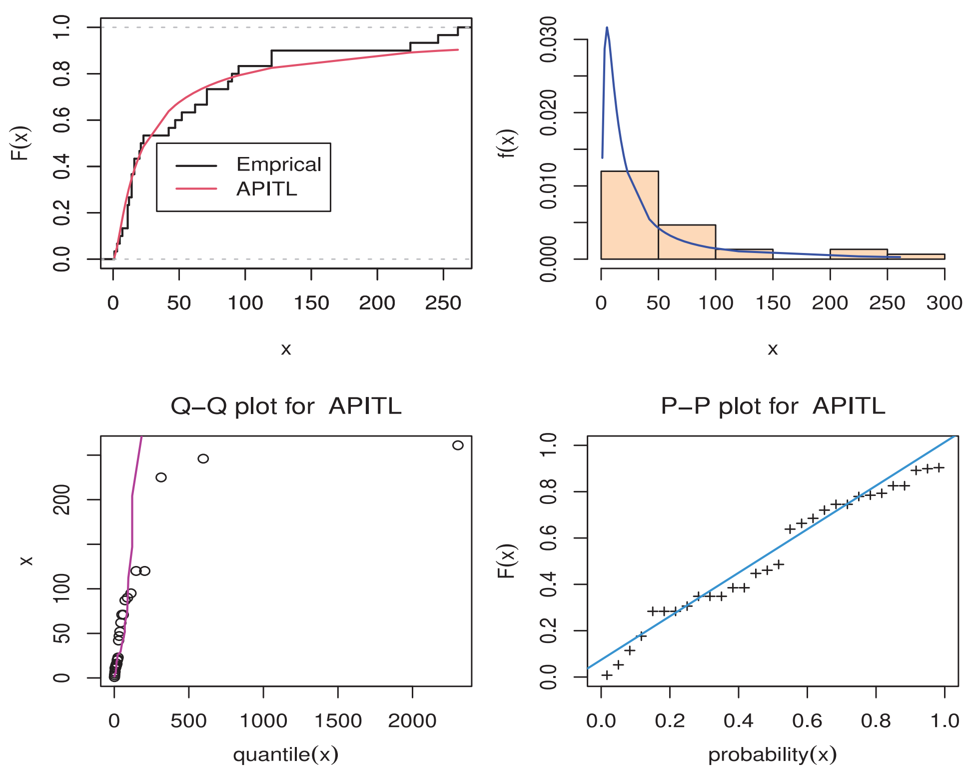


Figure 6: Estimated PDF, CDF, PP-plot, and QQ-plot of the APITL model for data **III**

Table 7: Different estimates of the APITL parameters for real datasets

Data			ML	WLS	SELF	LINEX (1.5)	LINEX(-1.5)
I	α	**Estimate**	204.3711	230.2754	319.536	321.5280	303.5203
		SE	86.0516	49.2177	83.123	82.9465	82.9622
	λ	**Estimate**	4.5486	4.5407	4.6113	4.5071	4.7201
		SE	0.4218	0.0895	0.3089	0.3092	0.2081
II	α	**Estimate**	6.9168	5.7518	7.1372	2.6879	16.9063
		SE	6.6745	1.2317	4.2794	2.8905	2.2560
	λ	**Estimate**	1.8963	1.7875	1.8793	1.8011	1.9589
		SE	0.3783	0.0953	0.3248	0.2595	0.1925
III	α	**Estimate**	301.6959	364.1921	555.4224	517.0317	519.0317
		SE	33. 9011	34.264	33. 7555	24. 9852	26. 3894
	λ	**Estimate**	0.8271	0.8195	0.8457	0.8386	0.8504
		SE	0.1006	0.1248	0.0973	0.0935	0.0897

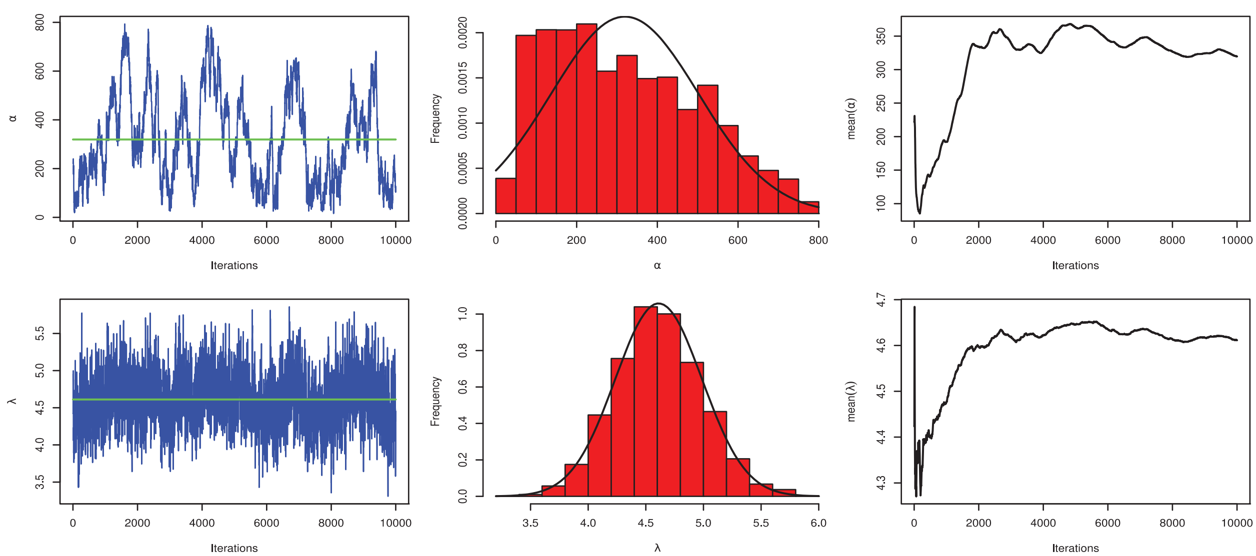


Figure 7: Convergence of the MCMC estimation of α and λ for data **I**

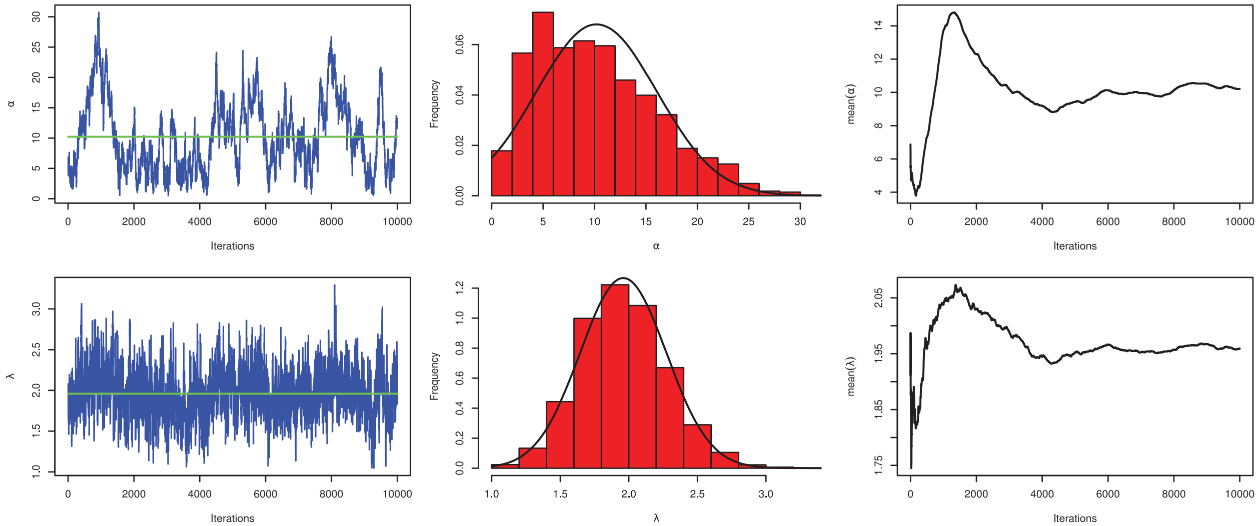


Figure 8: Convergence of the MCMC estimation of α and λ for data **II**

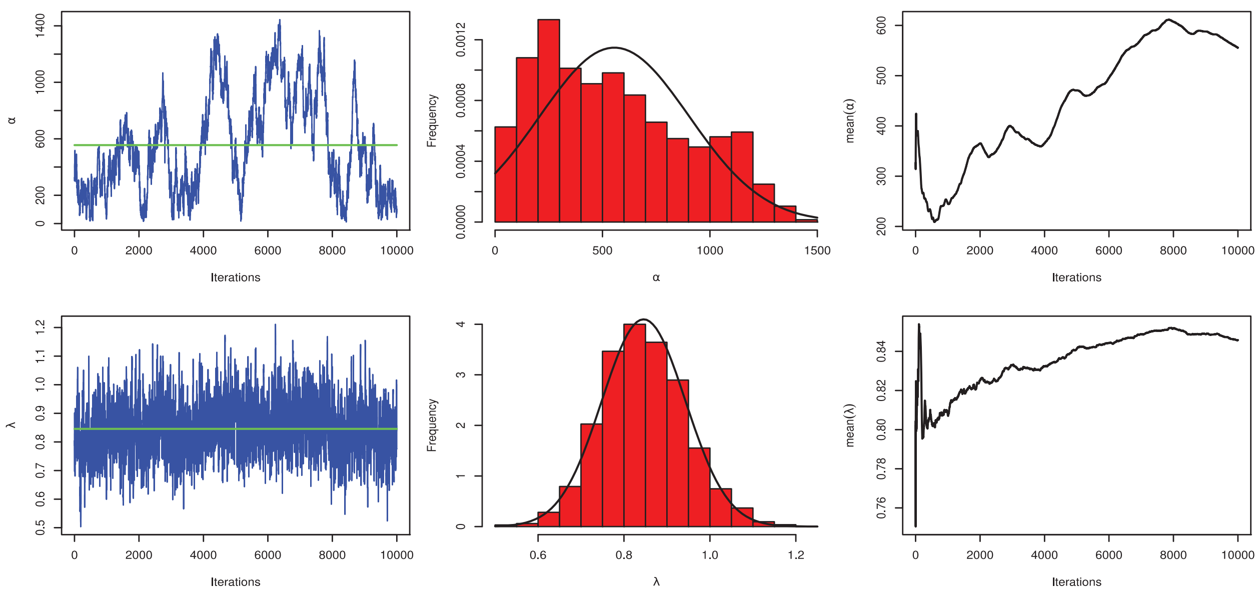


Figure 9: Convergence of the MCMC estimation of α and λ for data **III**

7 Concluding Remarks

We proposed and studied the alpha power transformed inverted Topp-Leone distribution. Some structural properties of the APITL distribution were provided. Bayesian and non-Bayesian methods of estimation were considered. We obtained the ML, WLS, and MPS estimators of the population parameters. The Bayesian estimator was deduced under LINEX and SELF. The Monte Carlo simulation study was worked out to assess the behavior of estimates. Generally, we concluded that the Bayes estimates are preferable to the corresponding other estimates in approximately most of the situations. We proved empirically that the APITL model reveals its superiority over other competitive models for different real data.

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