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Inference of Truncated Lomax Inverse Lomax Distribution with Applications

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Abstract: This paper introduces a modified form of the inverse Lomax distribution which offers more flexibility for modeling lifetime data. The new three-parameter model is provided as a member of the truncated Lomax-G procedure. The new modified distribution is called the truncated Lomax inverse Lomax distribution. The density of the new model can be represented as a linear combination of the inverse Lomax distribution. Expansions for quantile function, moment generating function, probability weighted moments, ordinary moments, incomplete moments, inverse moments, conditional moments, and Rényi entropy measure are investigated. The new distribution is capable of monotonically increasing, decreasing, reversed J-shaped and upside-down shaped hazard rates. Maximum likelihood estimators of the population parameters are derived. Also, the approximate confidence interval of parameters is constructed. A simulation study framework is established to assess the accuracy of estimates through some measures. Simulation outcomes show that there is a great agreement between theoretical and empirical studies. The applicability of the truncated Lomax inverse Lomax model is illustrated through two real lifetime data sets and its goodness-of-fit is compared with that of the recent models. In fact, it provides a better fit to these data than the other competitive models.

1 Introduction

The inverse Lomax (IL) distribution is very flexible in analyzing situations of different types of hazard rate function. It is a member of the inverted family of distributions, that is, the IL distribution is the reciprocal of the well-known Lomax distribution. Kleiber et al. [1] reported that the IL distribution can be used in economics and actuarial sciences. Singh et al. [2] the reliability estimators of the IL distribution were

investigated under type-II censoring. Yadav et al. [3] discussed parameter estimators of the IL model in the case of hybrid censored samples. Reyad et al. [4] regarded the Bayesian estimation of a two-component mixture of IL distribution based on a type-I censoring scheme. Bantan et al. [5] provided entropy estimators based on multiple censored scheme. The cumulative distribution function (cdf) of the IL distribution is given by:

$$G(y; b, \delta) = \left(1 + \frac{b}{y}\right)^{-\delta}, \qquad y, b, \delta > 0, \tag{1}$$

where, b and δ are the scale and shape parameters respectively. The probability density function (pdf) of the IL distribution is as follows:

$$g(y; b, \delta) = \frac{b\delta}{y^2}\left(1 + \frac{b}{y}\right)^{-\delta-1}, \qquad y, b, \delta > 0. \tag{2}$$

Recent studies about the generalization and extensions of the IL distribution have been proposed by several authors. Hassan et al. [6] provided the inverse power Lomax distribution. Hassan et al. [7] introduced the Weibull IL distribution. ZeinEldin et al. [8] provided the alpha power transformed IL distribution. Maxwell et al. [9] introduced the Marshall-Olkin IL distribution. ZeinEldin et al. [10] introduced odd Fréchet IL distribution. Hassan et al. [11] proposed Topp-Leone IL distribution.

A classical strategy to generate families of probability distributions consists of adding parameter(s) to baseline distributions. These families have the ability to improve the desirable properties of the probability distributions as well as to extract more information from the several data applied in many areas like engineering, economics, biological studies and environmental sciences. Another useful generator that works with the truncated random variable. The notable studies about the truncated-G families in this regard, are the truncated Fréchet-G [12], truncated inverted Kumaraswamy-G [13], truncated Lomax-G [14], truncated power Lomax-G [15] and truncated Cauchy power-G [16].

Hassan et al. [14] suggested the newly truncated Lomax–G (TL-G) family with the following cdf,

$$F(y; \alpha, \varsigma) = \int_0^{G(y; \varsigma)} \frac{\alpha(1 + t)^{-(\alpha+1)}}{1 - 2^{-\alpha}} dt = A(1 - (1 + G(y; \varsigma))^{-\alpha}), \tag{3}$$

where $\alpha > 0$, $A = (1 - 2^{-\alpha})^{-1}$, ς is the parameter vector and $G(x; \varsigma)$ is the cdf of any distribution and the truncated Lomax pdf is the generator. The pdf associated with Eq. (3) is given by

$$f(y; \alpha, \varsigma) = \alpha A g(y; \varsigma)(1 + G(y; \varsigma))^{-\alpha-1}. \tag{4}$$

The main purpose of this work is to suggest a more flexible and enhance model called the truncated Lomax IL (TLIL) distribution. The key motivations of the TLIL distribution in the practice are (i) to enhance the flexibility of the IL distribution by using TL-G, (ii) to introduce the modified form of the IL distribution whose density can be expressed as a linear combination of the IL distributions, (iii) to develop a new model with increasing, decreasing, reversed J-shaped and upside-down shaped hazard functions and (iv) to provide better fits than the competing models. Further, we study various statistical properties and estimate the model parameters by using the maximum likelihood (ML) method. Finally, simulation studies as well as applications to real data are given.

This work can be arranged as follows. In Section 2, we give the model description of the TLIL distribution and provide a linear representation of its density and distribution functions. Section 3 provides some structural properties of the TLIL distribution. Estimation of parameters along with

simulation study is considered in Section 4. Applications are given in Section 5 followed by comments and conclusions.

2 Description of the Model

In this section, we define and provide the density and distribution function of the TLIL distribution. This model is yielded by taking the base-line Eq. (3) to be the cdf of the IL model. A random variable Y is said to have the TLIL, if its cdf is represented as:

$$F(y; \Theta) = A\left[1 - \left(1 + \left(1 + \frac{b}{y}\right)^{-\delta}\right)^{-\alpha}\right], \qquad y, b, \delta, \alpha > 0, \tag{5}$$

where, $\Theta \equiv (b, \alpha, \delta)$ is the set of parameters. The pdf associated with Eq. (5) is given by

$$f(y; \Theta) = \frac{A\alpha\delta b}{y^2}\left(1 + \frac{b}{y}\right)^{-\delta-1}\left(1 + \left(1 + \frac{b}{y}\right)^{-\delta}\right)^{-\alpha-1}, y, b, \delta, \alpha > 0. \tag{6}$$

Expression of hazard rate function (hrf) is given by:

$$h(y; \Theta) = \frac{A\alpha\delta b}{y^2}\left(1 + \frac{b}{y}\right)^{-\delta-1}\left(1 + \left(1 + \frac{b}{y}\right)^{-\delta}\right)^{-\alpha-1}\left\{1 - A\left[1 - \left(1 + \left(1 + \frac{b}{y}\right)^{-\delta}\right)^{-\alpha}\right]\right\}^{-1}. \tag{7}$$

Graphic features of the pdf and hrf plots of $Y \sim$ TLIL (α, δ, b) are represented in Fig. 1
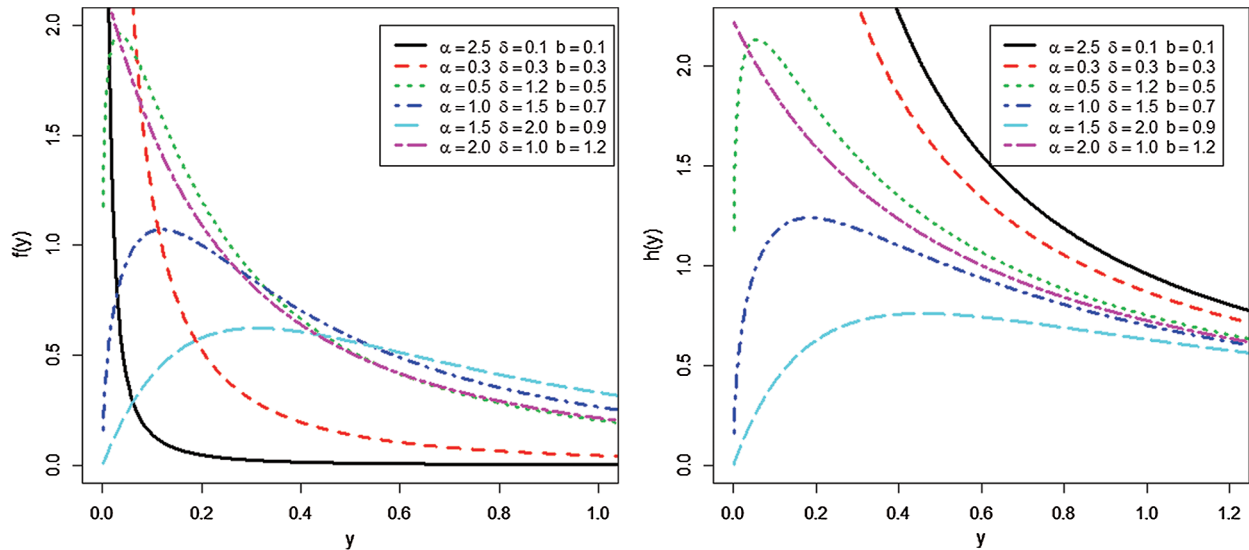

Figure 1: Plots of the pdf and hrf for TLIL model

Fig. 1 gives density and hazard functions plots for specified values of parameters. It is observed that the pdf of the TLIL can be right-skewed, uni-modal and reversed J-shaped. The hazard function can be increasing, decreasing, reversed J-shaped and upside-down bathtub.

2.1 Quantile Function and Median

The quantile function of Y, denoted by $Q(u)$, is defined by inverting Eq. (5) as follows

$$Q(u) = b \left\{ \left[\left(1 - \frac{u}{A} \right)^{\frac{-1}{\alpha}} - 1 \right]^{\frac{-1}{\delta}} - 1 \right\}^{-1}, \tag{8}$$

where $u \sim$ U(0,1). Further, the TLIL distribution can be simply simulated from Eq. (8). The median of Y is obtained by putting $u = 0.5$ in Eq. (8).

2.2 Linear Representation

Here, we provide the representation of cdf Eq. (5) and pdf Eq. (6) of the TLIL distribution. As mentioned in Hassan et al. [14], the pdf of the TL-G family is expressed as follows:

$$f(y; \alpha, \varsigma) = \sum_{j=0}^{\infty} \tau_j g(y; \varsigma) G(y; \varsigma)^j, \tag{9}$$

where $\tau_j = A\alpha(-1)^j \binom{\alpha+1}{j}$. Hence, a useful expression of the TLIL distribution is produced by inserting pdf Eq. (1) and cdf Eq. (2) in Eq. (9) as follows:

$$f(y; \Theta) = \sum_{j=0}^{\infty} \frac{\tau_j b\delta}{y^2} \left(1 + \frac{b}{y} \right)^{-\delta(j+1)-1} = \sum_{j=0}^{\infty} \frac{\tau_j}{(j+1)} g(y; b, \delta(j+1)), \tag{10}$$

where, $g(y; b, \delta(j+1))$ is the pdf of the IL distribution with parameters b and $\delta(j+1)$. That is the pdf Eq. (10) of TLIL is represented as a linear combination of the IL distribution.

Furthermore, Hassan et al. [14] provided a useful expression of cdf $[F(y; \varsigma)]^h$, where h is an integer, as follows:

$$[F(y; \varsigma)]^h = \sum_{l=0}^{\infty} S_l G(y; \varsigma)^l, \tag{11}$$

where $S_l = A^h \sum_{i=0}^{h} (-1)^{i+l} \binom{h}{i} \binom{\alpha i + l - 1}{l}$. Hence, as mentioned in Eq. (11), the cdf $[F(y; \Theta)]^h$, of the TLIL takes the following form:

$$[F(y; \Theta)]^h = \sum_{l=0}^{\infty} S_l \left(1 + \frac{b}{y} \right)^{-\delta l}. \tag{12}$$

3 Statistical Properties

3.1 Probability-Weighted Moments

The probability-weighted moments (PWMs) are less sensitive to outliers. They are utilized to study characteristics of the probability distributions. They are sometimes used when maximum likelihood estimates are unavailable or difficult to compute. The PWMs, denoted by $\Upsilon_{k,h}$, is defined as:

$$\Upsilon_{k,h} = E(Y^k(F(y)^h)) = \int_0^\infty y^k(F(y)^h f(y) dy. \tag{13}$$

where, k and h are positive integers. The class of the PWMs of the TLIL is obtained by substituting Eq. (10) and Eq. (12) in Eq. (13) as follows

$$\Upsilon_{k,h} = \sum_{l,j=0}^\infty S_l \tau_j b\delta \int_0^\infty y^{k-2} \left(1 + \frac{b}{y}\right)^{-\delta(j+l+1)-1} dy = \sum_{l,j=0}^\infty \frac{S_l \tau_j \delta \Gamma(1-k)\Gamma(\delta(j+l+1)+k)}{\Gamma(\delta(j+l+1)+1)}, \tag{14}$$

where; $\Gamma(0) = -\gamma, \quad \Gamma(-k) = \frac{(-1)^k}{k!}\varphi(k) - \frac{(-1)^k}{k!}\gamma \; for \; k = 1, 2, ..., \gamma$ denotes Euler's constant and $\varphi(k) = \sum_{i=1}^k \frac{1}{i}$ (see [17]).

3.2 Moments and Related Measures

Here, we provide an infinite sum representation for the n-th moment about the origin, inverse moments and incomplete moments of the TLIL model, since it has a pivotal role in the study of the distribution and real data applications. The n-th moment for the TLIL is obtained as follows:

$$\mu'_n = \sum_{j=0}^\infty \tau_j b\delta \int_0^\infty y^{n-2} \left(1 + \frac{b}{y}\right)^{-\delta(j+1)-1} dy = \sum_{j=0}^\infty \frac{\tau_j \delta \Gamma(1-n)\Gamma(\delta(j+1)+n)}{\Gamma(\delta(j+1)+1)}, \tag{15}$$

The first four moments are obtained by setting $n =$ 1, 2, 3 and 4 in Eq. (15). The n-th central moment (μ_n) of Y is given by

$$\mu_n = E(Y - \mu'_1)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} (\mu'_1)^i \mu'_{n-i} \tag{16}$$

Based on Eq. (16), we can obtain the skewness and kurtosis measures using the well-known relationships. Further, the moment generating function of the TLIL distribution for $|t| < 1$, is given by

$$M_x(t) = \sum_{j=0}^\infty \frac{t^n \tau_j \delta \Gamma(1-n)\Gamma(\delta(j+1)+n)}{n! \Gamma(\delta(j+1)+1)}, \quad n = 1, 2, ..., \tag{17}$$

The m–th inverse moment, for the TLIL distribution is derived by using pdf Eq. (10) as follows:

$$E(Y^{-m}) = \sum_{j=0}^\infty \frac{\tau_j \delta \Gamma(1+m)\Gamma(\delta(j+1)-m)}{\Gamma(\delta(j+1)+1)}; \; m = 1, 2, ... \tag{18}$$

The harmonic mean of the TLIL distribution can be obtained by using the first inverse moment.

Additionally, the n-th incomplete moment of Y can be obtained from Eq. (10) as follows:

$$\omega_n(t) = \sum_{j=0}^\infty \tau_j b\delta \int_0^t y^{n-2} \left(1 + \frac{b}{y}\right)^{-\delta(j+1)-1} dy = \sum_{j=0}^\infty \tau_j \delta b^n \mathrm{B}\left(1 - n, \delta(j+1) + n, \frac{t}{t+b}\right), \tag{19}$$

where, B(.,.,t) is the incomplete beta function.

3.3 Conditional Moments

The conditional moments and the mean residual lifetime function are of interest for lifetime models to be obtained. The r-th conditional moment is defined as:

$$\Lambda_r(t) = \frac{1}{\bar{F}(t)} \int_t^\infty (y-t)^r f(y)dy = \sum_{i=0}^r \frac{(-1)^{r-i}\binom{n}{i}t^{r-i}}{\bar{F}(t)} \int_t^\infty y^r f(y)dy. \tag{20}$$

The r-th moment of the residual life of the TLIL distribution is obtained by substituting Eq. (10) in Eq. (20) as follows

$$\Lambda_r(t) = \frac{\delta}{\bar{F}(t;\Theta)} \sum_{j=0}^\infty \sum_{i=0}^r (-1)^{r-i}\binom{r}{i}t^{r-i}\tau_j b^r \mathrm{B}\left(1-r, r+\delta(j+1), \frac{b}{t+b}\right) \tag{21}$$

where $\mathrm{B}(.,.,t)$ is the incomplete beta function. In particular, the mean residual life of the TLIL model is obtained by substituting $r=1$ in Eq. (21).

Moreover, the reversed residual life (RRL) is defined as the conditional random variable $ð_i - Y|Y \leq ð_i$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to $ð_i$. The r-th moment of the RRL is defined by:

$$\varpi_r(t) = \frac{1}{F(t)} \int_0^t (x-t)^r f(x)dx = \frac{1}{F(t)} \sum_{i=0}^r (-1)^{r-i}\binom{r}{i}t^{r-i} \int_0^t x^r f(x)dx. \tag{22}$$

The r-th moment of the RRL of TLIL distribution is obtained by substituting Eq. (10) in Eq. (22) as follows

$$\varpi_r(t;\Theta) = \frac{\delta}{F(t;\Theta)} \sum_{i=0}^r (-1)^{r-i}\binom{r}{i}t^{r-i}b^r \mathrm{B}\left(1-r, r+\delta(j+1), \frac{t}{t+b}\right). \tag{23}$$

The mean of RRL (mean waiting time) represents the waiting time elapsed since the failure of an item on condition that this failure had occurred. For, $r = 1$ in Eq. (23) we obtain the mean of RRL of the TLIL distribution.

3.4 Rényi Entropy

The entropy affords a great tool to evaluate the amount of information (or uncertainty) exists in a random observation relating to its parent distribution. A small value of entropy provides the smaller uncertainty in the data. Hassan et al. [14] mentioned that the Rényi entropy of the TL-G distributions is defined as:

$$\Xi_v(Y) = (1-v)^{-1} \log \sum_{i=0}^\infty \eta_i^* \int_{-\infty}^\infty g(y;\xi)^v G(y;\xi)^i dy, \tag{24}$$

where $\eta_i^* = (-1)^i(A\alpha)^v \binom{v(\alpha+1)+i-1}{i}$. The Rényi entropy of the TLIL distribution is obtained by substituting pdf Eq. (1) and cdf Eq. (2) in Eq. (24) as follows:

$$\Xi_v(Y) = (1-v)^{-1} \log \int\limits_0^\infty \left(\frac{\delta b}{y^2}\right)^v \left(1+\frac{b}{y}\right)^{-v(\delta+1)-\delta i} dy. \tag{25}$$

So, after some manipulation, the Rényi entropy of the TLIL distribution is given by

$$\Xi_v(Y) = (1-v)^{-1} \log \left[\sum_{i=0}^\infty \eta_i^* b^{1-v} \delta^v \mathrm{B}(\delta i + v\delta - v + 1, 2v - 1)\right], \tag{26}$$

where, B(.,.) is the beta function.

4 Estimation and Numerical Study

In this section, the ML estimators of the model parameters are derived in the case of complete samples. Also, approximate confidence intervals (CIs) are obtained. Furthermore, a numerical study is conducted.

4.1 ML Estimators

Suppose that $x_1, x_2,\ldots,x_n$ be the observed values from TLIL distribution whose lifetimes have the pdf Eq. (6). The log-likelihood based on complete sample, is given by

$$\log \mathrm{L} = \mathrm{n}\log(A\alpha\delta b) - 2\sum_{i=1}^n \log y_i - (\delta+1)\sum_{i=1}^n \log W_i - (\alpha+1)\sum_{i=1}^n \log(1 + W_i^{-\delta}), \tag{27}$$

where, $W_i = \left(1 + \dfrac{b}{y_i}\right)$. Then, the first partial derivatives of the log-likelihood are given by

$$\frac{\partial \log \mathrm{L}}{\partial \alpha} = \frac{n}{\alpha} - nA\, 2^{-\alpha}\log 2 - \sum_{i=1}^n \log(1 + W_i^{-\delta}), \tag{28}$$

$$\frac{\partial \log \mathrm{L}}{\partial b} = \frac{n}{b} - (\delta+1)\sum_{i=1}^n (y_i\, W_i)^{-1} + \delta(\alpha+1)\sum_{i=1}^n y_i^{-1}\, W_i^{-(\delta+1)}\left[1 + W_i^{-\delta}\right]^{-1} \tag{29}$$

$$\frac{\partial \log \mathrm{L}}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^n \log W_i + (\alpha+1)\sum_{i=1}^n W_i^{-\delta}\log W_i\left[1 + W_i^{-\delta}\right]^{-1}. \tag{30}$$

For interval estimation of the parameters, it is known that under regularity conditions, the asymptotic distribution of ML estimators of elements of unknown parameters for α, b and δ is given

$$(\hat{\alpha} - \alpha), \left(b - \hat{b}\right)\left(\hat{\delta} - \delta\right) \rightarrow \mathrm{N}\left(0, \mathrm{I}^{-1}(\alpha, b, \delta)\right), \tag{31}$$

where, $\mathrm{I}^{-1}(\alpha, b, \delta)$ is the variance-covariance matrix of the unknown parameters α, b and δ. The elements of the Fisher information matrix are obtained. Therefore, the two-sided approximate $\in 100$ percent limits for the ML estimators of the population parameters for α, b and δ can be obtained, respectively, as follows:

$$\hat{\alpha} \pm z_{\in/2}\sqrt{\mathrm{var}(\hat{\alpha})}, \ \hat{b} \pm z_{\in/2}\sqrt{\mathrm{var}\left(\hat{b}\right)}, \ \hat{\delta} \pm z_{\in/2}\sqrt{\mathrm{var}\left(\hat{\delta}\right)}, \tag{32}$$

where $z_{\in/2}$ is the $100\left(1 - z_{\in/2}\right)\%$ standard normal percentile and var(.)'s denotes the diagonal elements of the variance covariance matrix corresponding to the model parameters.

4.2 Simulation Study

In this sub-section, we perform a simulation study to evaluate the behavior of the ML estimates (MLEs). The attitude of the different estimates is checked in terms of their mean square errors (MSEs), standard errors (SEs) and average lengths (ALs) of the CIs. 10000 random samples of sizes 30, 50, 75 and 100 are generated from TLIL distribution. Four sets of parameter values are chosen as; $(i) \equiv (\alpha = 2.5, b = 0.5, \delta = 1.25)$, $(ii) \equiv (\alpha = 2, b = 0.25, \delta = 0.75)$, $(iii) \equiv (\alpha = 1.5, b = 0.25, \delta = 0.5)$ and $(iv) \equiv (\alpha = 1.25, b = 0.25, \delta = 0.25)$. We generate the random number from TLIL distribution using Eq. (8). Then, we obtain the ML estimators by solving numerical Eqs. (28)–(30). The AL of CIs for all estimates is obtained with confidence level of 95 % using Eq. (32). We compute MSEs, SEs and ALs of the CIs of all estimates and their results are listed in Tabs. 1 and 2.

Table 1: MSEs, SEs and ALs of the MLEs for TLIL distribution

n	Measures	(i)			(ii)		
		$\alpha = 2.5$	$b = 0.5$	$\delta = 1.25$	$\alpha = 2$	$b = 0.25$	$\delta = 0.75$
30	MSE	2.0307	1.8101	0.7903	1.3591	0.4968	0.3187
	SE	0.2181	0.1476	0.0633	0.1649	0.0603	0.0346
	AL	0.8551	0.5785	0.2483	0.6463	0.2363	0.1357
50	MSE	1.8644	1.3712	0.7645	1.3516	0.3468	0.3089
	SE	0.1635	0.0866	0.0380	0.1309	0.0267	0.0206
	AL	0.6411	0.3396	0.1491	0.5132	0.1047	0.0808
75	MSE	1.8632	0.9086	0.7416	1.3083	0.3037	0.3012
	SE	0.1318	0.0411	0.0273	0.1040	0.0166	0.0137
	AL	0.5165	0.1612	0.1070	0.4079	0.0652	0.0537
100	MSE	1.6464	0.8515	0.7291	1.3006	0.2941	0.2981
	SE	0.1166	0.0320	0.0251	0.0958	0.0124	0.0117
	AL	0.4570	0.1255	0.0984	0.3755	0.0486	0.0459
150	MSE	1.5751	0.7546	0.6981	1.2597	0.2764	0.2949
	SE	0.0923	0.0195	0.0160	0.0753	7.3890*	7.9447*
	AL	0.3619	0.0765	0.0626	0.2950	0.0290	0.0311

Note: * Indicate that the value multiply 10^{-3}.

Table 2: MSEs, SEs and ALs of the MLEs for TLIL distribution

n	Measures	(iii)			(iv)		
		$\alpha = 1.5$	$b = 0.25$	$\delta = 0.5$	$\alpha = 1.25$	$b = 0.25$	$\delta = 0.25$
30	MSE	0.8450	0.1031	0.1109	0.4564	3.1732*	0.3244
	SE	0.1057	0.0273	0.0316	0.1189	0.0101	0.1032
	AL	0.4144	0.1071	0.1238	0.4660	0.0397	0.4045

(Continued)

Table 2 (continued).

n	Measures	(iii)			(iv)		
		$\alpha = 1.5$	$b = 0.25$	$\delta = 0.5$	$\alpha = 1.25$	$b = 0.25$	$\delta = 0.25$
50	**MSE**	0.8080	0.0776	0.1006	0.4435	1.6097*	0.0669
	SE	0.0814	0.0136	0.0180	0.0894	5.6538*	0.0364
	AL	0.3190	0.0534	0.0705	0.3506	0.0222	0.1426
75	**MSE**	0.7385	0.0684	0.0952	0.3987	1.1765*	0.0273
	SE	0.0653	8.6751*	0.0122	0.0702	3.9515*	0.0191
	AL	0.2558	0.0340	0.0479	0.2752	0.0155	0.0747
100	**MSE**	0.7243	0.0649	0.0938	0.3672	0.7097*	0.0221
	SE	0.0562	6.4092*	9.8561*	0.0591	2.6644*	0.0148
	AL	0.2204	0.0251	0.0386	0.2315	0.0104	0.0579
150	**MSE**	0.6873	0.0633	0.0932	0.3567	0.5559*	0.0123
	SE	0.0454	4.1558*	6.4674*	0.0477	1.9260*	9.0701*
	AL	0.1778	0.0163	0.0254	0.1870	7.5499*	0.0356

Note: * Indicate that the value multiply 10^{-3}.

Based on the above tables, we conclude the following

- It is clear that MSEs, SEs and ALs decrease as sample size increases for all estimates.
- The MSEs for α and b estimates at set (iv) take the smallest values corresponding to the MSEs of estimates of the other sets for all sample sizes.
- The MSEs for δ estimates at set (iii) get the smallest values corresponding to the MSEs for the estimates of the other sets for different sample sizes.
- As the value of α, b and δ decreases, the MSEs, SEs of MLEs and their ALs are decreasing.
- As the value of α, b and δ increases, the MSEs, SEs of MLEs and their ALs are increasing.
- The SEs and ALs for α, δ estimates based on set (iii) are smaller than the corresponding SEs of estimates of the other sets in majority of situations. While, the SEs and ALs of $b$estimates at set (iv) are the smallest values corresponding to SEs and ALs for other estimates from other sets for different sample sizes.

5 Applications to Real Data

In this section, we provide the applicability of TLIL distribution by using two real data sets. These data have been used by several authors to show the superiority of other competing models. We also provide a formative evaluation of the goodness of-fit of the models and make comparisons with other distributions. The measures of goodness of fit including; the Akaike information criterion (AIC), Consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling(A^*) and Cramér- von Mises (W^*) are calculated to compare the fitted models. Generally, the best fit to the data that is correspond to the lowest values of these statistics.

5.1 Aircraft Windshield Data

The first data set was proposed by Murthy et al. [18], which represents the failure times for a particular windshield device. For these data, we shall compare the TLIL model with the following models: the Zubair Lomax (ZL) [19], gamma-Lomax (GL) [20], exponentiated transmuted generalized Rayleigh (ETGR) [21], transmuted Weibull Lomax (TWL) [22], Kumaraswamy Lomax (KwL) [23], Kumaraswamy Weibull (KwW) [24], McDonald Weibull (McW) [25], beta Weibull (BW) [26] and transmuted Marshall-Olkin Fréchet (TMOFr) [27]. The estimated parameters of these models and the corresponding SEs for windshield data are provided in Tab. 3. The statistics AIC, CAIC, HQIC, A^* and W^* are mentioned in Tab. 4. The estimated pdf, cdf, sf and PP plots for Aircraft Windshield data of the fitted models are displayed in Fig. 2.

Table 3: MLEs and SEs (in parentheses) for Aircraft Windshield data

Distribution	MLEs				
TLIL(δ, b, α)	2.459 (0.251)	47.134 (14.289)	1151 (0.044)		
ZL(α, β, λ)	826.196 (3505)	785.944 (3343)	4.471 (0.886)		
KwL(a, b, α, β)	2.615 (1.343)	100.276 (404.095)	5.277 (37.988)	78.677 (799.338)	
ETGR $(\alpha, \beta, \lambda, \delta)$	0.034 (0.048)	0.379 (0.025)	−0.354 (0.815)	26.430 (40.252)	
KwW(a, b, α, β)	34.660 (17.527)	81.846 (52.014)	14.433 (27.095)	0.204 (0.042)	
McW (a, b, α, β, c)	17.686 (6.222)	33.639 (19.994)	1.940 (1.011)	0.306 (0.045)	16.721 (9.622)
BW(a, b, α, β)	34.180 (14.838)	11.496 (6.730)	1.360 (1.002)	0.298 (0.060)	
TMOFr$(\alpha, \beta, \sigma, \delta)$	200.747 (87.275)	1.952 (0.125)	0.102 (0.017)	−0.869 (0.101)	

Table 4: The measures of goodness of fit for Aircraft Windshield data

Distribution	AIC	CAIC	HQIC	A*	W*
TLIL	267.323	267.623	270.255	0.6702	0.0639
ZL	268.515	269.022	271.447	0.708	0.072
KwL	270.296	270.802	280.019	0.868	0.097
ETGR	269.975	270.481	279.700	0.786	0.085
KwW	281.434	281.941	291.158	1.506	0.185
McW	283.899	284.669	296.053	1.591	0.199
BW	305.028	305.534	314.751	3.220	0.465
TMOFr	309.472	309.978	319.195	2.404	0.320

It is clear from Tab. 3, Tab. 4 and Fig. 2 that the TLIL provides a better fit to the data and therefore could be chosen as the best model.

5.2 Cancer Patient Data

This data set describes there mission times (in months) of a random sample of 128 bladder cancer patients studied by Lee et al. [28]. For these data, we compare the fit of the TLIL with some other distributions like; the ZL, transmuted modified Weibull (TMW) distribution [29], transmuted additive Weibull (TAW) distribution [30], generalized inverse gamma distribution distribion [31], beta exponentiated Burr XII (BEBXII) distribution [32], beta Fréchet (BFr) distribution [33] and ETGR

distribution The estimated parameters along with their SEs are provided in Tab. 5. The statistics of the fitted models are presented in Tab. 6. We note from Tab. 6 that the TLIL gives the lowest values of AIC, CAIC, HQIC, A* and W* as compared to the other competitive models. Therefore, TLIL distribution provides the best fit for the cancer patient data. More information can be found in Fig. 3.
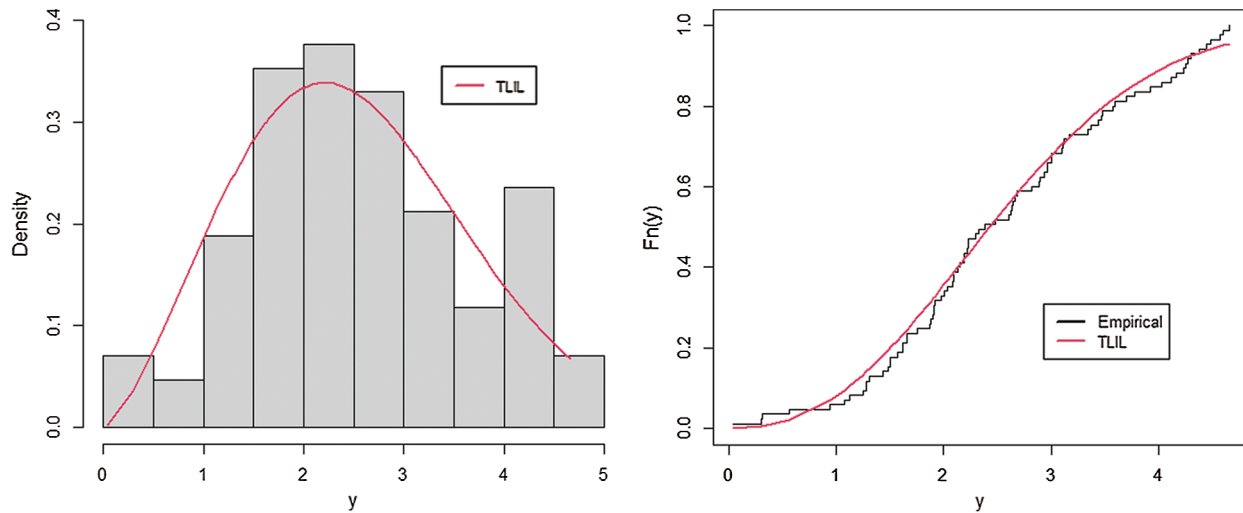

Figure 2: Plots of estimated pdf and cdf for Aircraft Windshield data

Table 5: Estimates and SE (in parentheses) for cancer data

Distribution	MLEs				
TLIL (δ, b, α)	1.429 (0.151)	41.006 (10.109)	12.589 (0.053)		
ZL (α, β, λ)	2.677 (1.005)	7.235 (7.009)	1.32 (1.855)		
BEBXII (a, b, c, β, k)	22.186 (21.956)	20.277 (17.296)	0.224 (0.144)	1.780 (1.076)	1.306 (1.079)
GIG (a, b, c, β, k)	2.327 (0.369)	0.0002 (0.0002)	17.931 (7.385)	0.543 (0.042)	0.001 (0.0003)
BFr (a, b, α, β)	12.526 (24.469)	33.342 (36.348)	27.753 (71.507)	0.169 (0.104)	
ETGR $(\alpha, \beta, \lambda, \delta)$	7.376 (5.389)	0.047 (0.004)	0.118 (0.260)	0.049 (0.036)	
TMW $(a, \alpha, \beta, \lambda)$	0.0002 (0.011)	0.1208 (0.024)	0.8955 (0.626)	0.407 (0.407)	
TAW $(a, b, \alpha, \beta, \lambda)$	0.00003 (0.0061)	1.0065 (0.035)	0.1139 (0.032)	0.9722 (0.125)	−0.1630 (0.280)

Table 6: The measures of goodness of fit for cancer data

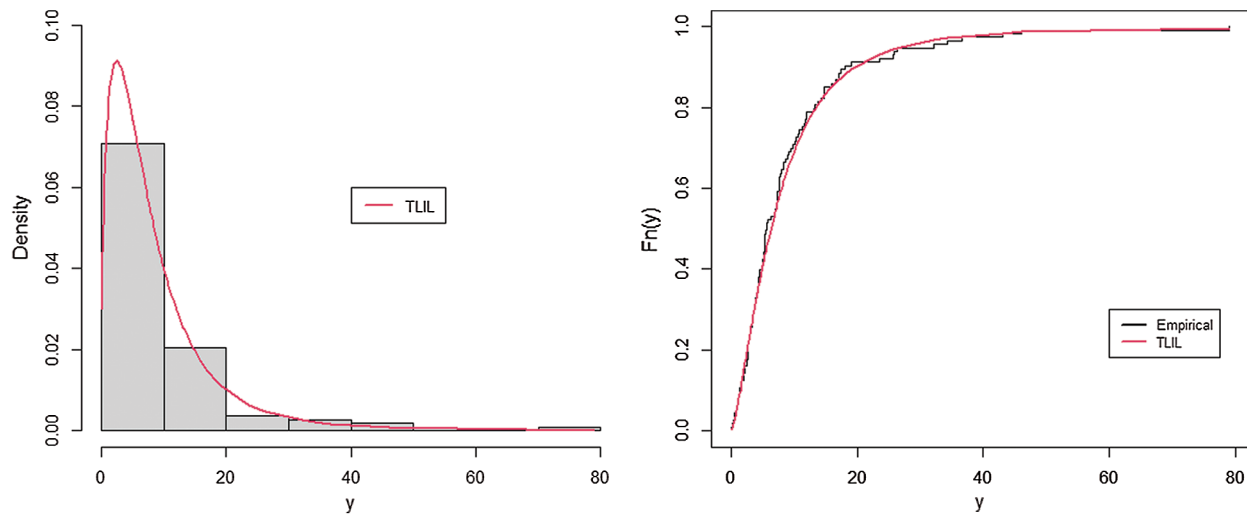| Distribution	AIC	CAIC	HQIC	A*	W*
TLIL	825.827	826.02	828.99	0.1514	0.023
ZL	827.465	827.659	830.942	0.340	0.048
BEBXII	841.268	841.760	855.528	0.900	0.134
GIG	839.824	840.316	854.085	2.618	0.410
BFr	842.965	843.290	854.373	1.121	0.168
ETGR	866.350	866.675	877.758	2.361	0.398
TMW	836.450	836.775	847.858	3.125	0.760
TAW	838.478	838.970	852.739	3.113	0.703

Figure 3: Plots of estimated pdf and cdf plots for cancer data

As seen from Fig. 3 that the TLIL provides a better fit to cancer data and so could be preferred than the other competitive model.

6 Concluding Remarks

We provide a new three-parameter lifetime distribution depends on the recent truncated Lomax-G family. The new truncated Lomax inverse Lomax model offers flexibility for modeling lifetime data. Expressions of density and distribution functions are obtained as a linear combination of the inverse Lomax distribution. Several mathematical properties of the new model are derived like; probability weighted moments, quantile function, moment generating function, ordinary and incomplete moments, inverse moments, conditional moments, and Rényi entropy. The maximum likelihood method of estimation is employed to obtain the point and approximate confidence interval of population parameters. We assess the accuracy of estimates viz simulation study. Applications to two real data are utilized to illustrate the importance and usefulness of the new model compared to some models.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present review.

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