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Deterministic and Stochastic Fractional Order Model for Lesser Date Moth

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Abstract: In this paper, a deterministic and stochastic fractional order model for lesser date moth (LDM) using mating disruption and natural enemies is proposed and analysed. The interaction between LDM larvae, fertilized LDM female, unfertilized LDM female, LDM male and the natural enemy is investigated. In order to clarify the characteristics of the proposed deterministic fractional order model, the analysis of existence, uniqueness, non-negativity and boundedness of the solutions of the proposed fractional-order model are examined. In addition, some sufficient conditions are obtained to ensure the local and global stability of equilibrium points. The occurrence of local bifurcation near the equilibrium points is investigated with the help of Sotomayor's theorem. Numerical simulations are conducted to illustrate the properties of the proposed fractional order model with respect to the intrinsic growth rate of the LDM larvae, natural enemy's mortality rate, predation rate, sex pheromone trap parameter, fractional order and environmental noise. The impact of mating disruption on lesser date moth is demonstrated. Also, a numerical approximation method is developed for the proposed stochastic fractional-order model.

Keywords: Lesser date moth; stochastic; stability; natural enemies; Sotomayor's theorem; mating disruption

1 Introduction

The palm tree is considered one of the oldest major and basic crops in Southwest Asia, North Africa and many other places of the world. Palm trees are affected by many agricultural pests that cause significant losses to the palm trees and their fruits, as well as affecting the age and growth of the palm tree if it is left without control [1]. The LDM is one of the most dangerous pests of young palm trees and immature palm fruits. The damage is mainly due to the way in which the LDM larvae feed, as soon as they leave the egg until the date of their entry the virgin dwelling develops feeding and causing tunnels in the affected part of the palm in all directions and depths without any early signs showing the infection [2–5]. One of the most promising strategies for controlling LDM is the use of a mating disruption using the sex pheromone traps [5]. Natural enemies' use to stop pest infestation has long been recommended as a clean

and environmentally friendly way to protect crops. The main natural enemies that are used in agricultural pest control are larval predators. Goniozus swirskiana can be considered as one of the most important natural enemies that can attack the LDM [6,7]. In the real world, plant diseases models are always affected by the environmental noise. Thus, the stochastic models may be a more appropriate way of modelling agricultural pests in many circumstances [8]. Recently, fractional calculus has been applied to describe different mathematical models, and it has been shown to be more accurate in some cases compared to the classical models [9]. The main objective of this paper is to propose and analyse a deterministic and stochastic fractional order LDM model with mating disruption and sex pheromone traps taking into consideration the effect of the natural enemy on LDM.

The paper is arranged as follows: In Section 2, the deterministic mathematical model is described as well as the existence, uniqueness, non-negativity, and boundedness of the solutions of LDM system are verified. The local and global stability of equilibrium points of the LDM system is analyzed in Section 3. Using Sotomayor's theorem, the local bifurcation conditions are derived in Section 4. In Section 5, we extend the deterministic fractional-order LDM model to the stochastic case and a numerical approximation method developed for the proposed stochastic fractional-order case. Some numerical examples are given in Section 6 to illustrate the theoretical findings. Finally, the discussion and conclusion are given in Section 7.

2 Dynamic of the Deterministic Fractional Order Model

Following [10–14], the model of lesser date moth with mating disruption and sex pheromone trap is describing by the following system

$$
\begin{aligned}
{}^{c\alpha}D^{\alpha}x &= rx\left(1 - \frac{x}{k}\right) - \alpha_1 x - \frac{\beta xz}{a + x} - \mu_1 x, \\
{}^{c\alpha}D^{\alpha}y &= \varepsilon\,\alpha_1 x - \alpha_2 y + \delta f - \mu_2 y, \\
{}^{c\alpha}D^{\alpha}f &= \alpha_2 y - \delta f - \mu_3 f, \\
{}^{c\alpha}D^{\alpha}m &= (1 - \varepsilon)\alpha_1 x - \frac{\gamma p m}{y + p} - \mu_4 m, \\
{}^{c\alpha}D^{\alpha}z &= \frac{e\beta xz}{a + x} - \mu_5 z,
\end{aligned}
\tag{1}
$$

where ${}^{C\alpha}D^{\alpha}$ is the Caputo fractional derivative of order α and $0 < \alpha < 1$. The detailed explanation of the variables and parameters for system (1) are listed in Tab. 1.

Table 1: Biological description of parameters used in system (1)

Variables & parameters	Description
x	population density of LDM larvae
y	population density of unfertilized LDM female
f	population density of fertilized LDM female
m	population density of LDM male
z	population density of natural enemy
r	the LDM larvae intrinsic growth rate
k	the carrying capacity
α_1	transfer rate from larvae to unfertilized LDM females
α_2	transfer rate from unfertilized to fertilized LDM females

Table 1 (continued).

Variables & parameters	Description
β	predation intensity between larvae and natural enemy
a	half saturation constant
μ_1	mortality rate of LDM larvae
μ_2	mortality rate of unfertilized LDM female
μ_3	mortality rate of fertilized LDM female
μ_4	mortality rate of LDM male
μ_5	mortality rate of natural enemy
ε	proportion of larvae that emerge to unfertilized females
δ	multiple mating rate of fertilized LDM female
p	the maximal death rate by sex-pheromone trap
γ	the capture rate for male by sex pheromone trap
e	conversion coefficient for predation between LDM larvae and natural enemy.

2.1 Existence and Uniqueness

In this section, we investigate the existence and uniqueness of the solutions of the fractional order system (1) in the region $\Omega \times (0, T]$ where

$$\Omega = \left\{ (x, y, f, m, z) \in \mathbb{R}_+^5 : \max(|x|, |y|, |f|, |m|, |z|) \le \varphi \right\},$$

Theorem 1. *For each $X_0 = (x_0, y_0, f_0, m_0, z_0) \in \Omega$, there exists a unique solution $X(t) \in \Omega$ of the fractional order system (1), which is defined for all $t \ge 0$.*

Proof. Define a mapping $F(X) = (F_1(X), F_2(X), F_3(X), F_4(X), F_5(X))$, in which

$$
\begin{aligned}
F_1(X) &= rx\left(1 - \frac{x}{k}\right) - \alpha_1 x - \frac{\beta xz}{a + x} - \mu_1 x, \\
F_2(X) &= \varepsilon\, \alpha_1 x - \alpha_2 y + \delta f - \mu_2 y, \\
F_3(X) &= \alpha_2 y - \delta f - \mu_3 f, \\
F_4(X) &= (1 - \varepsilon)\alpha_1 x - \frac{\gamma p\, m}{y + p} - \mu_4 m, \\
F_5(X) &= \frac{e\beta xz}{a + x} - \mu_5 z.
\end{aligned}
\tag{2}
$$

For any $X, \bar{X} \in \Omega$, it follows from (2) that

$$
\begin{aligned}
\| F(X) - F(\overline{X}) \| ={}& \left| F_1(X) - F_1(\overline{X}) \right| + \left| F_2(X) - F_2(\overline{X}) \right| + \left| F_3(X) - F_3(\overline{X}) \right| + \left| F_4(X) - F_4(\overline{X}) \right| \\
\le{}& \left(r + 2\alpha_1 + \mu_1 + \frac{2r\varphi}{k} + \frac{2\beta\varphi}{a} \right) |x - \bar{x}| + \left(\mu_2 + 2\alpha_2 + \frac{\gamma\varphi}{p} \right) |y - \bar{y}| + (\mu_3 + 2\delta)|f - \bar{f}| \\
&+ (\gamma + \mu_4)|m - \overline{m}| + \left(\varphi + \mu_5 + \frac{\beta\varphi}{a} + e\beta + 1 \right) |z - \bar{z}| \\
\le{}& M_1 \| X - \overline{X} \|,
\end{aligned}
$$

where

$$M_1 = \max\left\{ \left(r + 2\alpha_1 + \mu_1 + \frac{2r\varphi}{k} + \frac{2\beta\varphi}{a}\right),\ \mu_2 + 2\alpha_2 + \frac{\gamma\varphi}{p},\ \mu_3 + 2\delta, \gamma + \mu_4, \varphi + \mu_5 + \frac{\beta\varphi}{a} + e\beta + 1 \right\}.$$

Hence, $F(X)$ satisfies the Lipschitz condition with respect to X. According to Cresson et al. [15], as $F(X)$ locally Lipschitz. Then there exists unique local solution to the fractional order system (1).

2.2 Non-Negativity and Boundedness

The following results show the non-negativity of the solutions of the fractional order system (1). According to Cresson et al. [15], a model of the form $\frac{dX}{dt} = F(X)$ satisfies the positivity property if and only if for all $i = 1, \ldots, 5$, $F_i(X) \geq 0$ for all $X \in R_+^5$ such that $X_i = 0$. Thus, the solution of the integer-order model (1), with nonnegative initial conditions remains nonnegative. Also, the solution satisfies the Lipschitz condition, as stated in Theorem 1. By Theorem 5 and Theorem 6 in Cresson et al. [15], the solution of the fractional-order model (1) also satisfies the non-negativity. The boundedness of the solutions of model (1) are given in the following theorem.

Theorem 2. *All the solutions of the fractional-order LDM Model* (1) *starting in* $\mathbb{R}_+^5$ *are uniformly bounded*.

Proof. The approach of [16,17] is utilized. Let $(x(t), y(t), f(t), m(t), z(t))$ to be any solution of the system (1) with non-negative initial conditions. Let $M(t) = x(t) + y(t) + f(t) + m(t) + z(t)$, then

$$^{c\alpha}D^\alpha M(t) \leq rx\left(1 - \frac{x}{k}\right) - \mu_1 x - \mu_2 y - \mu_3 f - \mu_4 m - \mu_5 z - \frac{\gamma p\, m}{y + p}$$

$$\leq \frac{-r}{k}\left(x - \frac{k}{2}\right)^2 + \frac{rk}{4} - vM,$$

where, $v < \min\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, thus, $^{c\alpha}D^\alpha M(t) + vM \leq \frac{rk}{4}$. In accordance with Lemma 9 in Choi et al. [18], it follows that, $0 \leq M(t) \leq M(0)E_\alpha(-vt^\alpha) + \frac{rk}{4v}t^\alpha E_{\alpha,\alpha+1}(-vt^\alpha)$,

where E_α is the Mittag-Leffler function. According to Lemma 5 and Corollary 6 in Choi et al. [18], it follows

$$0 \leq M(t) \leq \frac{rk}{4v}, \text{ as } t \to \infty.$$

Hence all the solutions of fractional-order LDM model (1) that start in $\mathbb{R}_+^5$ are uniformly bounded in the region, $H = \left\{(x, y, f, m, z) \in \mathbb{R}_+^5 : M(t) \leq \frac{rk}{4v} + \xi, \text{ for any } \xi > 0\right\}$.

3 Equilibria and Stability

The LDM model (1) has the following three equilibrium points:

1) $E_0 = (0, 0, 0, 0, 0)$, which always exists.

2) The free natural enemy equilibrium point $E_1 = (x_1, y_1, f_1, m_1, 0)$, where

$$x_1 = k\left(1 - \frac{1}{R_0}\right),\ y_1 = \frac{\varepsilon(\delta + \mu_3)\left(1 - \frac{1}{R_0}\right)}{\delta\mu_2 + \mu_3(\alpha_2 + \mu_2)},\ f_1 = \frac{\varepsilon\alpha_2\left(1 - \frac{1}{R_0}\right)}{\delta\mu_2 + \mu_3(\alpha_2 + \mu_2)},$$

$$m_1 = \frac{(1-\varepsilon)\alpha_1 k(y_1+p)}{\gamma p + \mu_4(y_1+p)}\left(1 - \frac{1}{R_0}\right).$$

The free natural enemy equilibrium point exists positively if $R_0 > 1$, where, $R_0 = \frac{r}{\alpha_1+\mu_1}$ is the basic offspring number obtained by using the next generation method [19].

3) The coexistence equilibrium point $E_2 = (x_2, y_2, f_2, m_2, z_2)$, where

$$x_2 = \frac{a\mu_5}{\beta - \mu_5}, \quad y_2 = \frac{\varepsilon\alpha_1(\delta+\mu_3)x_2}{\delta\mu_2+\mu_3(\alpha_2+\mu_2)}, \quad f_2 = \frac{\alpha_1\alpha_2\varepsilon x_2}{\delta\mu_2+\mu_3(\alpha_2+\mu_2)},$$

$$m_2 = \frac{(1-\varepsilon)\alpha_1 x_2(y_2+p)}{\gamma p + \mu_4(y_2+p)}, \quad z_2 = \frac{ea(\alpha_1+\mu_1)}{(e\beta-\mu_5)}\left(R_0 - 1 - \frac{ar\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}\right).$$

4) The coexistence equilibrium point E_2 exists if $e\beta > \mu_5$ and $R_0 > 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$.

The locally and globally asymptotically stable of equilibrium points of LDM system (1) are now investigated. The stability analysis of the equilibrium point $E_0 = (0,0,0,0,0)$ is not considered because in this case all the population will go to extinction.

The stability of free natural enemy equilibrium point $E_1 = (x_1, y_1, f_1, m_1, 0)$ is investigated as follows.

Theorem 3. If $R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$, then the equilibrium point E_1 is locally asymptotically stable.

Proof. The first three eigenvalues of $J(E_1)$ are $\lambda_1 = r - \frac{2rx_1}{k} - (\alpha_1+\mu_1) = (\alpha_1+\mu_1)(1-R_0)$, $\lambda_2 = -\left(\frac{\gamma p}{y_1+p}+\mu_4\right)$ and $\lambda_3 = \frac{e\beta x_1}{a+x_1} - \mu_5$. The other roots are determined by

$$\lambda^2 + (\alpha_2+\delta+\mu_2+\mu_3)\lambda + \delta\mu_2 + \mu_3(\alpha_2+\mu_2) = 0. \tag{3}$$

The roots of Eq. (3) have negative real parts. It can be observed that $\lambda_3 < 0$, when $\frac{e\beta x_1}{a+x_1} < \mu_5$ which equivalent $R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$. So, E_1 is locally asymptotically stable if $1 < R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$.

Theorem 4. If $R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$, then the natural enemy extinction equilibrium point E_1 is globally asymptotically stable.

Proof. The following positive definite Lyapunov function is considered.

$$L_1 = \frac{ea}{a+x_1}\left(x - x_1 - x_1\ln\frac{x}{x_1}\right) + z.$$

By calculating the time derivative of L_1 along the solution of system (1), one obtains,

$$^{c\alpha}D^\alpha L_1 = \frac{ea(x-x_1)}{a+x_1}\left[r\left(1-\frac{x}{k}\right) - \frac{\beta z}{a+x} - (\alpha_1+\mu_1)\right] + \left[\frac{e\beta x}{a+x} - \mu_5\right]z$$

$$\leq -\frac{ear(x-x_1)^2}{k(a+x_1)} + \left(\frac{e\beta x_1}{a+x_1} - \mu_5\right)z.$$

Then $^{c\alpha}D^\alpha L_1 \leq 0$, when $\frac{e\beta x_1}{a+x_1} < \mu_5$ which equivalent $R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$. According to generalized Lyapunov–Lasalle's invariance principle [20], the natural enemy extinction equilibrium point E_1 is globally asymptotically stable when $R_0 < 1 + \frac{ra\mu_5}{k(\alpha_1+\mu_1)(e\beta-\mu_5)}$.

The stability of the coexistence equilibrium point $E_2 = (x_2, y_2, f_2, m_2, z_2)$ is investigated as follows.

The first eigenvalue of $J(E_2)$ is $\lambda_1 = -\frac{\gamma p}{p+y_2} - \mu_4$. The other eigenvalues are determined by

$$\lambda^4 + \theta_1 \lambda^3 + \theta_2 \lambda^2 + \theta_3 \lambda + \theta_4 = 0, \tag{4}$$

where

$\theta_1 = \alpha_2 + \delta + \mu_2 + \mu_3 + B_{11},$

$\theta_2 = \alpha_2 \mu_3 + \dfrac{ea\beta^2 z_2 x_2}{(a+x_2)^3} + B_{11}(\alpha_2 + \delta + \mu_2 + \mu_3) + \mu_2(\delta + \mu_3),$

$\theta_3 = \dfrac{\alpha_2 \left(B_{11}\mu_3(a+x_2)^3 + ea\beta^2 z_2 x_2\right) + \mu_2 \left(B_{11}(a+x_2)^3(\delta + \mu_3) + ea\beta^2 x_2 z_2\right) + ea\beta^2 x_2 z_2(\delta + \mu_3)}{(a+x_2)^3},$

$\theta_4 = \dfrac{ea\beta^2 x_2 z_2(\alpha_2 \mu_3 + \mu_2(\delta + \mu_3))}{(a+x_2)^3},$

where $B_{11} = \frac{rx_2}{k} - \frac{\beta z_2 x_2}{(a+x_2)^2}$ and $B_{44} = \frac{\gamma p}{p+y_2} + \mu_4$. Then, the proposition proposed in Matouk [21] can be used to determine the stability conditions of the equilibrium point E_2. When $B_{11} > 0$, then $\theta_i > 0, i = 1, 2, 3, 4$. Also,

$\theta_1 \theta_2 \theta_3 - \theta_3^2 - \theta_1^2 \theta_4 = \dfrac{\left(B_{11}(a+x_2)^3\Omega_4 + \Omega_2 \Omega_3\right)\left(B_{11}\delta\mu_2(a+x_2)^3 + \Omega_1(\alpha_2+\mu_2) + ea\beta^2 w_2 x_2(\delta+\mu_3)\right)^2}{(a+x_2)^9}$, where

$\Omega_1 = B_{11}\mu_3(a+x_2)^3 + ea\beta^2 z_2 x_2,$

$\Omega_2 = (a+x_2)^3 - ea\beta^2 z_2 x_2,$

$\Omega_3 = \alpha_2 \mu_3 + \mu_2(\delta + \mu_3) - 1,$

$\Omega_4 = (\alpha_2 + \delta + \mu_2 + \mu_3).$

when $\Omega_2 \Omega_3 > 0$, then $\theta_1 \theta_2 \theta_3 - \theta_3^2 - \theta_1^2 \theta_4 > 0$, therefore all the eigenvalues of the Jacobian matrix $J(E_2)$ near the equilibrium point E_2 have negative real parts. Thus, due to the Routh-Hurwitz criterion the equilibrium point E_2 is locally asymptotically stable. The local stability of the coexistence equilibrium point E_2 is given in the following theorem.

Theorem 5. If $\Omega_2 \Omega_3 > 0$ and $B_{11} > 0$, then the coexistence equilibrium point E_2 is locally asymptotically stable.

The global stability of the coexistence equilibrium point is investigated in the following theorem.

Theorem 6. If $\frac{\beta z_2}{a(a+x_2)} < \frac{r}{k}$, then the coexistence equilibrium point E_2 is globally asymptotically stable.

Proof. The following positive definite Lyapunov function is considered.

$$L_2 = H\left(x - x_2 - x_2 \ln\frac{x}{x_2}\right) + z - z_2 - z_2 \ln\frac{z}{z_2}.$$

By calculating the time derivative of L_2 along the solution of system (1), one obtains,

$$^{c\alpha}D^{\alpha}L_2 = H(x - x_2)\left[r\left(1 - \frac{x}{k}\right) - \frac{\beta z}{a+x} - (\alpha_1 + \mu_1)\right] + (z - z_2)\left[\frac{e\beta x}{a+x} - \mu_5\right]$$

$$\leq H\left[\frac{\beta z_2}{a(a+x_2)} - \frac{r}{k}\right] + \frac{\beta(x - x_2)(z - z_2)}{(a+x)(a+x_2)}(ea - H(a+x_2)).$$

Choosing $H = \frac{ea}{a+x_2}$, then $^{c\alpha}D^{\alpha}L_2 < 0$. According to generalized Lyapunov–Lasalle's invariance principle [20], the coexistence equilibrium point E_2 is globally asymptotically stable when $\frac{\beta z_2}{a(a+x_2)} < \frac{r}{k}$.

4 Bifurcation Analysis

In this section the local bifurcations near the equilibrium points of LDM model (1) are inves-tigated with the help of Sotomayor's theorem [22]. The Hopf bifurcation theorem given in Liu [23] is also presented to discuss the bifurcation analysis of the underlying system. One can compute

$$D^2 F(x, \varphi)(U, U) = \begin{pmatrix} \left(\dfrac{2a\beta z}{(a+x)^3} - \dfrac{2r}{k} \right) \zeta_1^2 - \dfrac{2a\beta \zeta_1 \zeta_5}{(a+x)^2} \\ 0 \\ 0 \\ -\dfrac{2\gamma p m \zeta_2^2}{(p+y)^3} + \dfrac{2\gamma p \zeta_2 \zeta_4}{(p+y)^2} \\ \dfrac{ea\beta \zeta_1 \zeta_5}{(a+x)^2} - \dfrac{2ea\beta z \zeta_1^2}{(a+x)^3} \end{pmatrix}, \tag{5}$$

where φ is any bifurcation parameter and $U = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^T$ is any eigenvector.

Theorem 7. *The LDM system (1) undergoes a transcritical bifurcation with respect to the bifurcation parameter r around E_0 if $r = r^* = \alpha_1 + \mu_1$.*

Proof. Let $V_1 = (v_1, v_2, v_3, v_4, v_5)^T$ be the eigenvector corresponding to the zero eigenvalue of the matrix $J(E_0)$ $r^* = \alpha_1 + \mu_1$, hence $J(E_0)V_1 = 0$, gives

$$V_1 = \begin{pmatrix} v_1 \\ \dfrac{\varepsilon \alpha_1 (\delta + \mu_3) v_1}{\alpha_2 \mu_3 + \mu_2 (\delta + \mu_3)} \\ \dfrac{\varepsilon \alpha_1 \alpha_2 v_1}{\alpha_2 \mu_3 + \mu_2 (\delta + \mu_3)} \\ \dfrac{(1 - \varepsilon)\alpha_1 v_1}{\gamma + \mu_4} \\ 0 \end{pmatrix},$$

where v_1 is any non zero real number. Similarly, suppose $V_2 = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^T$ be the eigenvector corresponding to the zero eigenvalue of the matrices $J(E_0)$, thus $J(E_0)^T V_2 = 0$ gives $V_2 = (1, 0, 0, 0, 0)^T$. Consider, $\frac{\partial F}{\partial r} = F_r(X, r) = (x(1 - \frac{x}{k}), 0, 0, 0, 0)^T$, thus, $V_2^T F_r(E_0, r^*) = 0$. Therefore, according to Sotomayor's theorem for local bifurcation, the LDM model (1) has no saddle-node bifurcation near E_0 at $r^* = \alpha_1 + \mu_1$. One can note that $r = \alpha_1 + \mu_1$ is equivalent to $R_0 = 1$. Now,

$$DF_r(E_0, r^*) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $V_2^T DF_r(E_0, r^*)V_1 = v_1 \neq 0$. Using (5), one obtains

$$V_2^T D^2 F(X, r)(V_1, V_1) = -\frac{2r}{k} v_1^2 - \frac{2\beta}{a} v_1 v_5 \neq 0.$$

Thus, according to Sotomayor's theorem, the LDM system (1) has a transcritical bifurcation at $r^* = \alpha_1 + \mu_1$ as the parameter r passes through the value r^*, thus the proof is complete.

Theorem 8. *The LDM system (1) undergoes a transcritical bifurcation with respect to the bifurcation parameter μ_5 around $E_1 = (x_1, y_1, f_1, m_1, 0)$ if $\mu_5 = \mu_5^* = \frac{e\beta x_1}{a+x_1}$.*

Proof. The Jacobian matrix of the LDM system (1) at the free enemy equilibrium point E_1 with $\mu_5 = \mu_5^*$ has zero eigenvalue takes the form

$$J(E_1, \mu_5^*) = \begin{pmatrix} A_{11} & 0 & 0 & 0 & \frac{-\beta x_1}{a+x_1} \\ \alpha_1 \varepsilon & -A_{22} & \delta & 0 & 0 \\ 0 & \alpha_2 & -A_{33} & 0 & 0 \\ (1-\varepsilon)\alpha_1 & \frac{\gamma p m_1}{(p+y_1)^2} & 0 & -A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A_{11} = r - \frac{2rx_1}{k} - (\alpha_1 + \mu_1)$, $A_{22} = (\alpha_2 + \mu_2)$, $A_{33} = (\delta + \mu_3)$ and $A_{44} = \frac{\gamma p}{y_1 + p} + \mu_4$. $J(E_1)V_3 = 0$, gives the eigenvector corresponding to the zero eigenvalue of the matrix $J(E_1)$, hence

$$V_3 = \begin{pmatrix} \frac{\mu_5 v_5}{eA_{11}} \\ \frac{\varepsilon A_{33}\alpha_1 \mu_5 v_5}{eA_{11}(A_{22}A_{33} - \delta\alpha_2)} \\ \frac{\varepsilon \alpha_1 \alpha_2 \mu_5 v_5}{eA_{11}(A_{22}A_{33} - \delta\alpha_2)} \\ \frac{\alpha_1\left(\left((1-\varepsilon)A_{22}(y_1 + p)^2 + \varepsilon\gamma p m_1\right)A_{33} - (1-\varepsilon)\delta(y_1 + p)^2 \alpha_2\right)\mu_5 v_5}{e(y_1 + p)^2 A_{11}A_{44}(A_{22}A_{33} - \delta\alpha_2)} \\ v_5 \end{pmatrix},$$

where v_5 is any non zero real number. Similarly, $J(E_1)^T V_4 = 0$ gives the eigenvector corresponding to the zero eigenvalue of the matrix $J(E_1)^T V_4 = 0$, hence $V_4 = (0, 0, 0, 0, 1)^T$. Consider, $\frac{\partial F}{\partial \mu_5} = F_{\mu_5}(X, \mu_5) = (0, 0, 0, 0, -z)^T$, thus, $V_4^T F_{\mu_5}(E_1, \mu_5^*) = 0$. Therefore, according to Sotomayor's theorem for local bifurcation, the LDM model (1) has no saddle-node bifurcation near E_1 at $\mu_5 = \mu_5^*$. Now,

$$DF_{\mu_5}(E_1, \mu_5^*) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

then, $V_4^T DF_{\mu_5}(E_1, \mu_5^*)V_3 = -v_5 \neq 0$. Using (5), one obtains

$$V_4^T D^2 F(X, \mu_5)(V_3, V_3) = \frac{ea\beta v_1 v_5}{(a + x_1)^2} \neq 0.$$

Thus, according to Sotomayor's theorem, the LDM system (1) has a transcritical bifurcation at $\mu_5^* = \frac{e\beta x_1}{a+x_1}$ as the parameter μ_5 passes through the value μ_5^*, thus the proof is complete.

In this part, we shall show that as the coexistence equilibrium loses stability, periodic solutions can bifurcate from the positive equilibrium. We first give the following lemma.

Lemma 9 . *The characteristic Eq.(4) has a pair of purely imaginary roots and the remaining roots have negative real parts if and only if $z_2 = \frac{r(a+x_2)^2}{k\beta}$ and $\theta_1\theta_2\theta_3 - \theta_3^2 - \theta_1^2\theta_4 = 0$.*

Suppose (4) has two eigenvalues which have negative real parts and two complex conjugates eigenvalues (call them $\lambda = m(\varphi) \pm in(\varphi)$) such that $m(\varphi^*) = 0$, $n(\varphi^*) > 0$, $\frac{dm}{d\varphi}|_{\varphi=\varphi^*} \neq 0$. Substituting

$\lambda = m(\varphi) \pm in(\varphi)$ into (4), and separating the real and imaginary, we get

$$m^4 + \theta_1 m^3 + \theta_2 m^3 + \theta_3 m + \theta_4 - (6m^2 + 3\theta_1 m + \theta_2)n^2 + n^4 = 0, \tag{6}$$

$$4m^3 + 3\theta_1 m^2 + 2\theta_2 m + \theta_3 - (4m + \theta_1)n^2 = 0, \tag{7}$$

Substituting (6) into (7), differentiating with respect to φ and utilizing $m(\varphi^*) = 0$ and $n(\varphi^*) \neq 0$, we have

$$\frac{dm}{d\varphi} = \left[\frac{\frac{d}{d\varphi}(\theta_1\theta_2\theta_3 - \theta_3^2 - \theta_1^2\theta_4)}{2\theta_1(4\theta_4 - \theta_1\theta_3 - \theta_2^2)} \right]_{\varphi=\varphi^*} \neq 0.$$

Theorem 10. *For the coexistence equilibrium point E_2 of the integer order LDM system (1), the system around E_2 enters into the Hopf bifurcation when φ passes φ^* if the coefficients $\theta_j(\varphi)(j = 1, 2, 3, 4)$ at $\varphi = \varphi^*$ satisfying the following condition:*

1. $\Phi(\varphi^*) = \left[\theta_1(\varphi)\theta_2(\varphi)\theta_3(\varphi) - \theta_3^2(\varphi) - \theta_1^2(\varphi)\theta_4(\varphi)\right]|_{\varphi=\varphi^*} = 0,$
2. $(4\theta_4 - \theta_1\theta_3 - \theta_2^2)|_{\varphi=\varphi^*} \neq 0,$
3. $\frac{d\Phi(\varphi)}{d\varphi}|_{\varphi=\varphi^*} \neq 0.$

According to Theorem 10, there exists a Hopf bifurcation in the LDM model (1) with $\alpha = 1$, where the Hopf bifurcation is controlled by φ.

5 Stochastic Fractional Order Model

This section extends the deterministic fractional-order LDM model (1) to the following stochastic fractional-order model.

$$^{C\alpha}D^\alpha x = rx\left(1 - \frac{x}{k}\right) - \alpha_1 x - \frac{\beta xz}{a+x} - \mu_1 x + \sigma_1 x\frac{dW_1}{dt},$$

$$^{C\alpha}D^\alpha y = \varepsilon\,\alpha_1 x - \alpha_2 y + \delta f - \mu_2 y + \sigma_2 y\frac{dW_2}{dt},$$

$$^{C\alpha}D^\alpha f = \alpha_2 y - \delta f - \mu_3 f + \sigma_3 f\frac{dW_3}{dt}, \tag{8}$$

$$^{C\alpha}D^\alpha m = (1 - \varepsilon)\alpha_1 x - \frac{\gamma p\,m}{y+p} - \mu_4 m + \sigma_4\,m\frac{dW_4}{dt},$$

$$^{C\alpha}D^\alpha z = \frac{e\beta xz}{a+x} - \mu_5 z + \sigma_5 z\frac{dW_5}{dt},$$

where $W_i(i = 1, 2, 3, 4, 5)$ are independent standard Brownian motions with $W_i(0) = 0$ and $\sigma_i > 0$ denote the intensities of the white noise. The stochastic fractional-order LDM model (8) can be written in the general form:

$$^{C\alpha}D^\alpha X(t) = F(X) + g(X)\frac{dW}{dt}, \tag{9}$$

where $F(x)$ is given in (2), $g(x) = (\sigma_1 x, \sigma_2 y, \sigma_3 f, \sigma_4 m, \sigma_5 z)$ and $\frac{dW}{dt} = (\frac{dW_1}{dt}, \frac{dW_2}{dt}, \frac{dW_3}{dt}, \frac{dW_4}{dt}, \frac{dW_5}{dt})^T$. Applying Riemann–Liouville integral to both sides of (9), one can obtain the following stochastic Volterra integral equation.

$$X(t) = X_0 + \int_0^t \frac{F(X)(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^t \frac{g(X)(t-s)^{\alpha-1} dW(s)}{\Gamma(\alpha)} ds. \tag{10}$$

According to Wang et al. [24,25], under some conditions on the coefficient functions, the global existence and uniqueness of solutions for the stochastic fractional-order system (8) can be investigated. Because Grunwald-Letnikov's definition is the most straightforward from the point of view of numerical implementation, so we will use it to solve the LDM system of fractional order stochastic differential equations. Grunwald-Letnikov ($^{GL}D^\alpha$) fractional derivative of order α defined by Aminikhah et al. [26,27]

$$^{GL\alpha}D^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{11}$$

where $\left[\frac{t-a}{h}\right]$ means the integer part of $\frac{t-a}{h}$. This formula (11) can be reduced to

$$^{GL\alpha}D^\alpha f(t_n) \approx h^{-\alpha} \sum_{j=0}^{n} w_j^\alpha f(t_{n-j}), \tag{12}$$

where h is the time step, $t_n = nh$ and w_j^α are the Grunwald-Letnikov coefficients satisfy the following recurrence relationship

$$w_0^\alpha = 1, \ w_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right) w_{j-1}^\alpha, \ j = 1, 2, 3, \ldots$$

If $f(t)$ is continuous function and $f'(t)$ is integrable function in the interval $[0, T]$, then the relation between Caputo and Grunwald-Letnikov fractional derivative takes the form [28,29]

$$\begin{aligned}
^{C\alpha}D^\alpha f(t) &= {}^{GL\alpha} D^\alpha f(t) - \frac{f(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} \\
&= {}^{GL\alpha} D^\alpha f(t) - \lim_{n\to\infty} \frac{(-1)^n}{h^\alpha} \binom{\alpha-1}{n} f(0) \\
&\approx \frac{1}{h^\alpha} \sum_{j=0}^{n} w_j^\alpha \big(f(t_{n-j}) - f(0)\big).
\end{aligned} \tag{13}$$

Now, the fractional order stochastic LDM model (8) can be written as

$$\begin{aligned}
x_n &= x_0 + h^\alpha \left(r x_{n-1}\left(1 - \frac{x_{n-1}}{k}\right) - (\alpha_1 + \mu_1)x_{n-1} - \frac{\beta x_{n-1}z_{n-1}}{a+x_{n-1}} + \sigma_1 x_{n-1}\sqrt{h}\zeta_{1n} \right) - \sum_{j=1}^{n} w_j^\alpha (x_{n-j} - x_0) \\
y_n &= y_0 + h^\alpha \left(\varepsilon\alpha_1 x_n - (\alpha_2 + \mu_2)y_{n-1} + \delta f_{n-1} + \sigma_2 y_{n-1}\sqrt{h}\zeta_{2n} \right) - \sum_{j=1}^{n} w_j^\alpha (y_{n-j} - y_0) \\
f_n &= f_0 + h^\alpha \left(\alpha_2 y_n - (\delta + \mu_3)f_{n-1} + \sigma_3 f_{n-1}\sqrt{h}\zeta_{3n} \right) - \sum_{j=1}^{n} w_j^\alpha (f_{n-j} - f_0) \\
m_n &= m_0 + h^\alpha \left((1-\varepsilon)\alpha_1 x_n - \frac{\gamma p\, m_{n-1}}{y_n + p} - \mu_4 m_{n-1} + \sigma_4 m_{n-1}\sqrt{h}\zeta_{4n} \right) - \sum_{j=1}^{n} w_j^\alpha (m_{n-j} - m_0) \\
z_n &= z_0 + h^\alpha \left(\frac{e\beta x_n z_{n-1}}{a+x_n} - \mu_5 z_{n-1} + \sigma_5 z_{n-1}\sqrt{h}\zeta_{5n} \right) - \sum_{j=1}^{n} w_j^\alpha (z_{n-j} - z_0),
\end{aligned} \tag{14}$$

where, σ_i and ζ_{in} represent real constants and a 5D Gaussian white noise processes, respectively, $i = 1, 2, 3, 4, 5$. ζ_i satisfy the follows:

$$\langle \zeta_j(t) \rangle = 0, \, (j = 0, 1, 2, 3, 4, 5) \, \text{and} \, \langle \zeta_i(t_1) \zeta_j(t_2) \rangle = \delta_{ij} \delta(t_1 - t_j),$$

δ_{ij} is Kronecker delta and $\delta(t_1 - t_j)$ is the Dirac delta function.

6 Numerical Simulations

In this section, we simulate the fractional-order LDM system (1) and stochastic fractional-order LDM (8) by the following parameters:

$r = 1.8, \, \varepsilon = 0.6, \, k = 30, \, \mu_1 = 0.01, \, \mu_2 = 0.01, \, \mu_3 = 0.01, \, \mu_4 = 0.01, \, \mu_5 = 0.01,$

$\alpha_1 = 0.5, \, \alpha_2 = 0.5, \, \delta = 0.5, \, a = 15, \, \gamma = 0.4, \, p = 0.2, \, e = 0.9, \, \beta = 0.1.$

To show the effect of the intrinsic growth rate of the LDM larvae, we draw the bifurcation diagram concerning r as a bifurcation parameter. It can be seen that a transcritical bifurcation occurs at $r = 0.51$ as shown in Fig. 1 and stated in Theorem 7. It can also be observed that when $r > 0.51$, the coexistence equilibrium point $E_2 = (1.6667, 25.2475, 24.7525, 25.36, 157.407r - 85)$ is locally asymptotically stable. According to Theorem 10, it can be seen that the supercritical Hopf bifurcation value localized at $r = 1.31143$. For $r > 1.31143$ the LDM system (1) undergoes limit cycle behaviour.
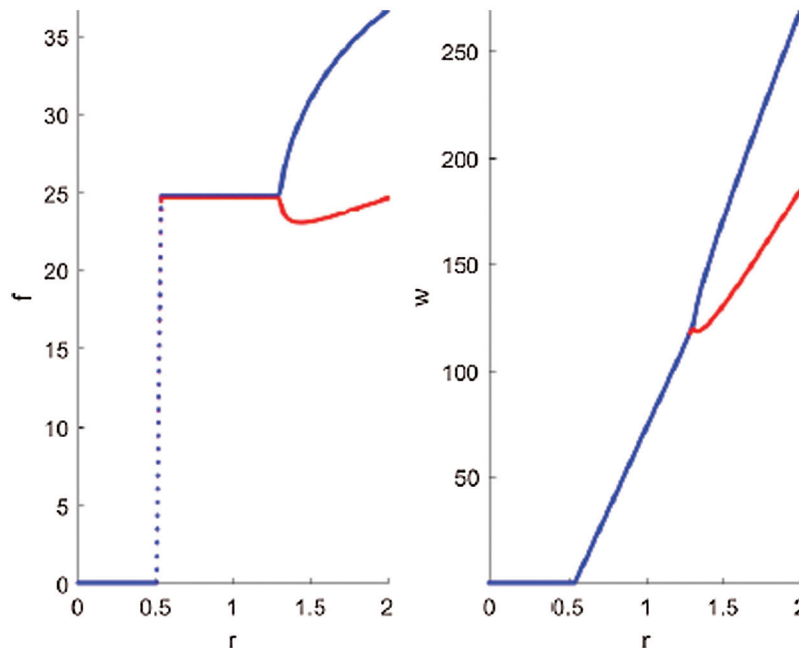

Figure 1: Bifurcation diagram of LDM model (1) with respect to r

To show the effect of the natural enemy's mortality rate around the coexistence equilibrium points, we draw the bifurcation diagram for μ_5 as a bifurcation parameter. It can be seen that the Hopf bifurcation value localized at $\mu_5 = 0.0284692$ as shown in Fig. 2. It can also be observed that when $\mu_5 > 0.0284692$, the coexistence equilibrium point E_2 is locally asymptotically stable. For $\mu_5 < 0.3235$, the system undergoes limit cycle behaviour. According to Theorem 7, for $\mu_5 > 0.0642346$ the natural enemy goes extinct from

the system and a transcritical bifurcation occurs at $\mu_5 = 0.0642346$ and the equilibrium point $E_1 = (26.94, 408.101, 400.009, 528.446, 0)$ is locally asymptotically stable.
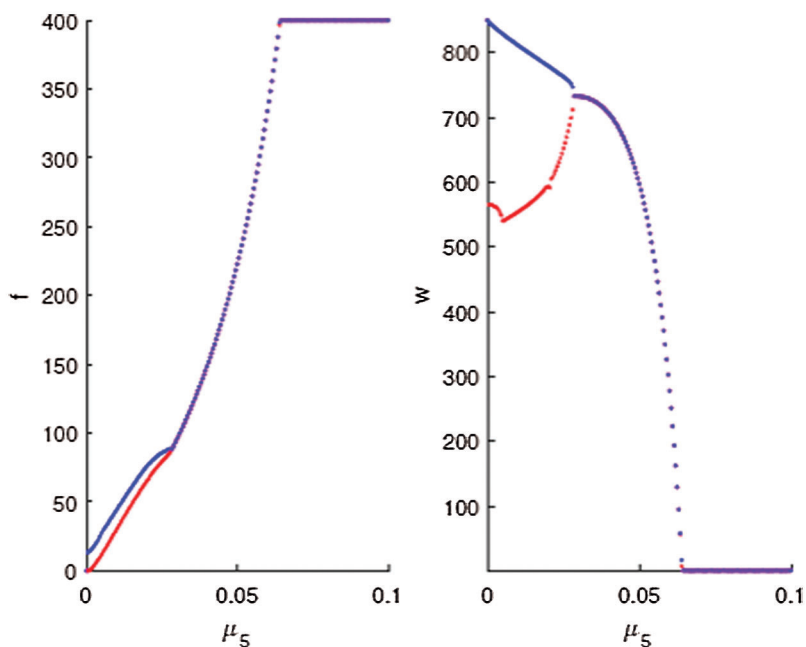

Figure 2: Bifurcation diagram of LDM model (1) with respect to μ_5

In order to show the effect of the predation rate, we draw the bifurcation diagram with respect to β as a bifurcation parameter. It can be seen that a transcritical bifurcation occurs at $\beta = 0.0501342$ as shown in Fig. 3. It can also be observed that when $\beta < 0.0501342$, the free enemy equilibrium point $E_1 = (22.35, 338.569, 331.93, 436.688, 0)$ is locally asymptotically stable as indicated in Theorem 8. For $0.0501342 < \beta < 0.152449$, the coexistence equilibrium point E_2 is locally asymptotically stable. It can be seen that the supercritical Hopf bifurcation value localized at $\beta = 0.152449$. For $\beta > 0.152449$ the LDM system (1) undergoes limit cycle behaviour.

From Fig. 4a, it can be seen that the sex pheromone trap parameter p is important in that it affects the population density of the LDM male. One can observe from Fig. 4b. that the population density of LDM male decrease with increasing p. We conclude that the dynamics of LDM can be controlled by sex pheromone trap parameters p.

Fig. 5 shows that the deterministic fractional-order remains stable for different values of fractional-order α though solutions reach to equilibrium point $E_2(7.5, 113.614, 111.386, 140.149, 189)$ more slowly for a smaller value of fractional-order α. It is important to notice that when $\alpha = 1$ the fractional order model for lesser date moth (1) reduces to the classical integer-order model [10–14].

When the strength of environmental noise is very close to zero, the fractional-order stochastic LDM system (8) behaves like the deterministic model. Fig. 6a indicates the dynamical behavior of stochastic LDM (8) without noise (i.e., $\sigma_i = 0$), which gives the deterministic model (1). In Fig. 6a, the dynamical behavior of the LDM system was unstable, which corresponds with the results of Theorem 5. Fig. 6b shows that the LDM smooth oscillations' dynamical behavior when the strength of the noise was low ($\sigma_i = 0.02$). However, with an increase in the strength of noise, such as the medium-noise situation shown in Fig. 6c, the dynamical behavior of the LDM became more complex, and they tended to extinction.

Fig. 6c represents the dynamical behavior of the model (1) when the noise strength was high ($\sigma_i = 0.05$). The natural enemy can die out due to the white noise stochastic disturbance. By comparing Figs. 6a and 6b, one can realize that if the noise is not strong, the stochastic perturbation does not cause sharp changes in the LDM model (8). However, when the environmental noise σ_i is sufficiently large (see Fig. 6c), the noise can force the natural enemy to become extinct.
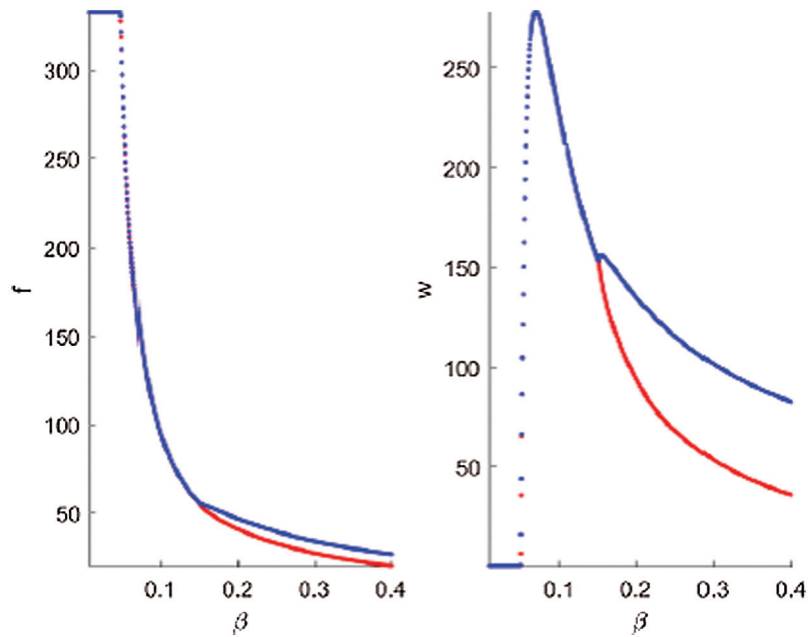

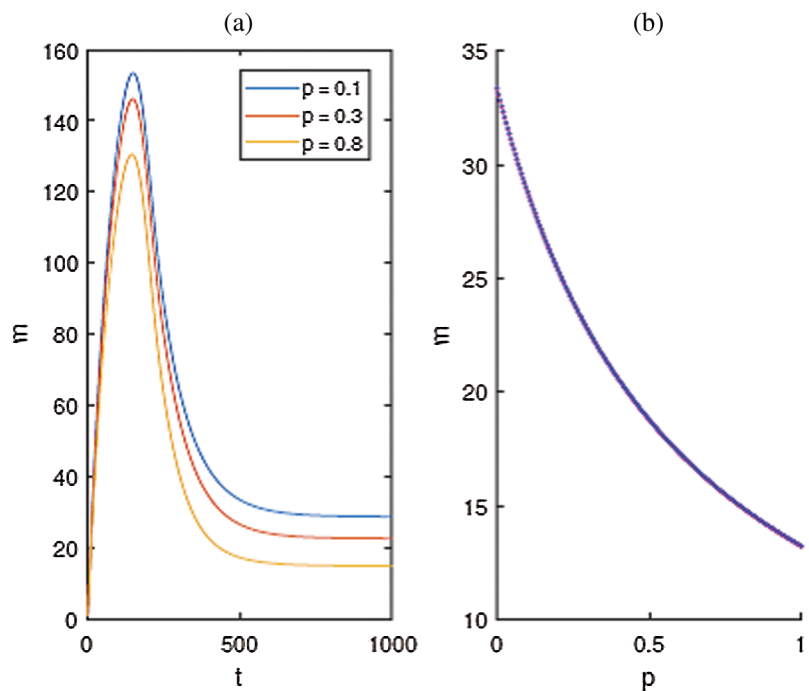**Figure 3:** Bifurcation diagram of LDM model (1) with respect to β

Figure 4: (a) Plot of the LDM male versus time with $p = 0.1, 0.3, 0.8$, (b) Plot of the LDM male versus time with $0 \leq p \leq 1$
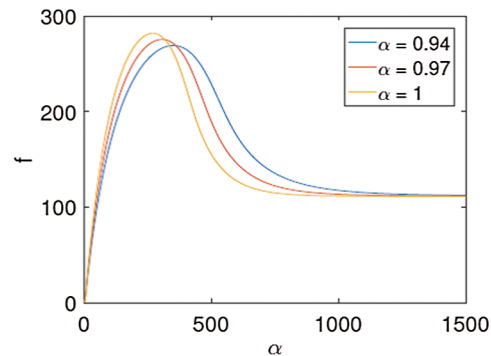
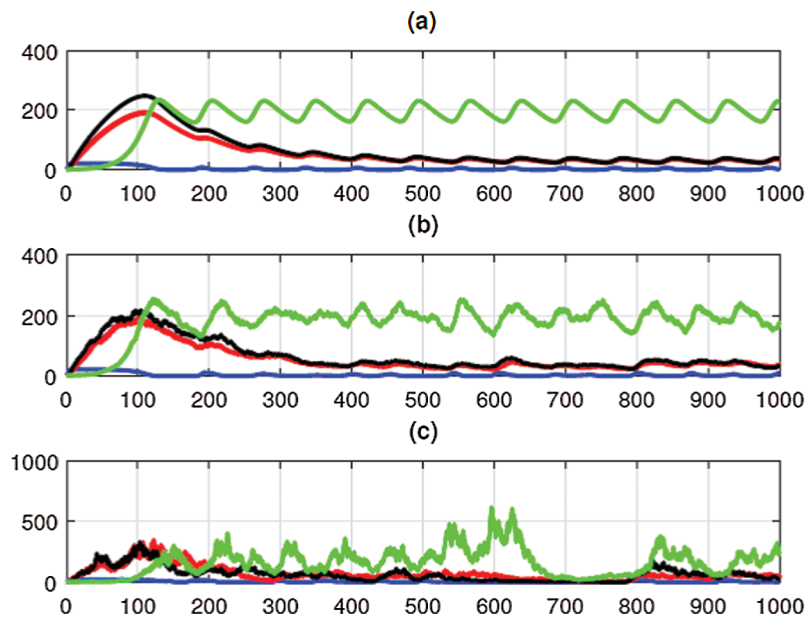**Figure 5:** Time series of the LDM female with α = 0.94, 0.97, 1

Figure 6: Time series of the fractional order stochastic model (10) with σ = 0, 0.02, 0.05 (a) σ_i = 0 (b) σ_i = 0.02 (c) σ_i = 0.05

7 Discussion and Conclusion

In this paper, we consider a deterministic and stochastic fractional order model for LDM using mating disruption and natural enemies. We obtain some sufficient conditions that ensure the local and global stability of equilibrium points. We conclude that sex pheromone trap parameters can control dynamics of LDM. The occurrence of local bifurcation near the equilibrium point is performed using Sotomayor's theorem. Numerical simulations are performed to support and illustrate the theoretical findings. From the numerical results, one can realize that if the noise is not strong, the stochastic perturbation does not cause sharp changes in the dynamics of the LDM model. However, when the environmental noise is sufficiently large, the noise can force the population to become extinct.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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