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ARTICLE

# Einstein Weighted Geometric Operator for Pythagorean Fuzzy Hypersoft with Its Application in Material Selection

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## ABSTRACT

Hypersoft set theory is a most advanced form of soft set theory and an innovative mathematical tool for dealing with unclear complications. Pythagorean fuzzy hypersoft set (PFHSS) is the most influential and capable leeway of the hypersoft set (HSS) and Pythagorean fuzzy soft set (PFSS). It is also a general form of the intuitionistic fuzzy hypersoft set (IFHSS), which provides a better and more perfect assessment of the decision-making (DM) process. The fundamental objective of this work is to enrich the precision of decision-making. A novel mixed aggregation operator called Pythagorean fuzzy hypersoft Einstein weighted geometric (PFHSEWG) based on Einstein's operational laws has been developed. Some necessary properties, such as idempotency, boundedness, and homogeneity, have been presented for the anticipated PFHSEWG operator. Multi-criteria decision-making (MCDM) plays an active role in dealing with the complications of manufacturing design for material selection. However, conventional methods of MCDM usually produce inconsistent results. Based on the proposed PFHSEWG operator, a robust MCDM procedure for material selection in manufacturing design is planned to address these inconveniences. The expected MCDM method for material selection (MS) of cryogenic storing vessels has been established in the real world. Significantly, the planned model for handling inaccurate data based on PFHSS is more operative and consistent.

## KEYWORDS

Pythagorean fuzzy soft set; Pythagorean fuzzy hypersoft set; Pythagorean fuzzy hypersoft Einstein weighted geometric operator (PFHSEWG); MCDM

## 1 Introduction

The solution to the problems in our daily lives is based on the classification of information, data, the collection of facts, etc. The critical question in decision analysis is the absence of accurate facts. This statistical difference is bridged by taking a scientific model and applying the appropriate DM. DM's ideas can support the manufacturing enterprise, assemble, and categorize multiple priorities from best to worst viable alternative. As a result, it is a tool to help us select, categorize, and establish our prospects and comprehensively evaluate alternatives. MS is intense in enterprise and product development. The material chosen affects the manufacturer's success and affordability [1]. The manufacturing enterprise suffers from legislation, cost, and penetrating global goals, often inadequate content. The persistence of the manufactured equipment strategy is to select components with state-of-the-art light design standards while providing the best offer at the lowest reasonable price [2]. However, these goals and obstacles are common in conflict situations, so it is essential to address which feature is more important. Suppose the appropriate method is not ready for the design approach. The design method's funding and resource aspects cannot be used in the restructuring or industrial agenda section [3]. Identifying the best materials is essential because design concerns are not correct. Eliminate inappropriate alternatives and manage high quality. Variables that interfere with selecting specific material engineering applications should use logical and straightforward applications [4].

MCDM has deliberated the best applicable procedure for the verdict and the best adequate alternative from all possible choices, ensuing criteria, or attributes. In real-life circumstances, most decisions are taken when the objectives and limitations are usually indefinite or ambiguous. To overcome such ambiguities and anxieties, Zadeh offered the idea of the fuzzy set (FS) [5], a prevailing tool to handle the obscurities and uncertainties in DM. Such a set allocates to all objects a membership value ranging from 0 to 1. Experts mainly consider membership and non-membership value in the DM process that FS cannot handle. Atanassov [6] introduced the generalization of the FS, the idea of the intuitionistic fuzzy set (IFS), to overcome the constraint mentioned above. In 2011, Wang et al. [7] presented numerous operations on IFS, such as Einstein product, Einstein sum, etc., and constructed two aggregation operators (AOs). They also discussed some essential properties of these operators and utilized their proposed operator to resolve multi-attribute decision making (MADM) for the IFS information.

The models mentioned above have been well-recognized by specialists. Still, the existing IFS cannot handle the inappropriate and vague data because it is deliberate to envision the linear inequality concerning the membership and non-membership grades. For example, if decision-makers choose membership and non-membership values 0.9 and 0.6, respectively, then $0.9 + 0.6 \geq 1$. The IFS mentioned above theory cannot be applied to this data. To resolve the limitation described above, Yager [8] presented the idea of the Pythagorean fuzzy set (PFS) by improving the basic circumstance $a + b \leq 1$ to $a^2 + b^2 \leq 1$ and developed some results associated with score function and accuracy function. Ejegwa [9] extended the notion of PFS and presented a decision-making technique. Rahman et al. [10] formed the Pythagorean fuzzy Einstein weighted geometric operator and presented a multi-attribute group decision making (MAGDM) methodology utilizing the proposed operator. Zhang et al. [11] developed some basic operational laws and prolonged the TOPSIS method to resolve MCDM complications for PFS information. Pythagorean fuzzy power AOs along with essential characteristics were introduced by Wei et al. [12]. They also recommended a DM technique to resolve MADM difficulties based on presented operators. Wang et al. [13] offered the interaction operational laws for PFNs, and developed power Bonferroni mean operators under the PFS environment. They also discussed some definite cases of developed operators and their basic characteristics. IIbahar et al. [14] offered the Pythagorean fuzzy proportional risk assessment technique to assess professional health risk. Zhang [15] proposed a novel decision-making (DM) approach based on similarity measures to resolve multi-criteria group decision making (MCGDM) difficulties for the PFS information.

Peng et al. [16] introduced the division and subtraction operations for Pythagorean fuzzy numbers (PFNs), proved their basic properties, and presented a superiority and inferiority ranking approach under the considered environment. Garg [17] introduced operational laws based on Einstein norms for PFNs, proposed weighted average and ordered weighted average operators, and then utilized these operators for DM. Garg [18] presented the series of generalized geometric AOs for PFS. Garg [19] introduced logarithmic operational laws for the PFS and constructed various weighted operators based on the proposed logarithm operational laws. Gao et al. [20] developed numerous interaction AOs under the PFS environment. Wang et al. [21] offered the interactive Hamacher operations for the PFS and settled on a DM method to solve MCDM difficulties. Wang et al. [22] utilized the interval-valued PFS, presented some novel PFS operators, and offered a DM approach to resolve the MCGDM complications. Moreover, to deal with the MCDM complexities. Peng et al. [23] explored some new inequalities for AOs under PFS. They introduced some point operators under the PFS environment. They combined the Pythagorean fuzzy point operators with the generalized AOs and offered a MADM approach based on settled operators. Moreover, Arora et al. [24] presented basic operational laws and suggested several selected AOs for linguistic IFSs. Ma et al. [25] modified the existing score function and accuracy function for PFNs and defined novel AOs for PFS.

All the methods mentioned above have too many applications in many fields. But due to their inefficiency, these methods have many limitations in terms of parameterization tools. In presenting the solution to obscurity and ambiguity, Molodtsov [26] introduced the basic notions of soft sets (SS) and debated some elementary operations with their possessions. Maji et al. [27] protracted the idea of SS and defined several basic operations for SS. Maji et al. [28] combined two prevailing notions, such as FS and SS. They developed the idea of FSS, which is a more robust and reliable tool. They also presented basic operations and established and applied this concept in DM. Maji et al. [29] demonstrated the intuitionistic fuzzy soft set (IFSS) theory and offered some basic operations with their essential properties. Arora et al. [30] developed the AOs for IFSS and discussed their basic properties. Nowadays, the conception and application consequences of soft sets and the earlier-mentioned several research developments are evolving speedily. Peng et al. [31] established the concept of PFSS by merging two prevailing models, PFS and SS. Athira et al. [32] established entropy measures for the PFSS. They also offered Euclidean distance and hamming distance for the PFSS and utilized their methods for DM [33]. Naeem et al. [34] developed the TOPSIS and VIKOR methods for PFSNs and presented an approach to the stock exchange investment problem. Zulqarnain et al. [35] introduced the AOs under the PFSS environment and presented an application for green supplier chain management. Zulqarnain et al. [36,37] formed the Einstein-ordered weighted average and geometric AOs for PFSS. They also proposed the MAGDM techniques using their developed operators for sustainable supplier selection and a business to finance money.

Smarandache [38] proposed the idea of the hypersoft set (HSS), which penetrates multiple sub-attributes in the parameter function $f$, which is a characteristic of the cartesian product with the $n$ attribute. Compared with SS and other existing concepts, Samarandche HSS is the most suitable theory which handles the multiple sub-attributes of the considered parameters. Rahman et al. [39] settled the DM techniques based on similarity measures for IFHSS. Zulqarnain et al. [40] prolonged the notion of IFHSS to PFHSS with fundamental operations and their properties. Zulqarnain et al. [41] expanded the AOs under the IFHSS environment and developed a DM approach based on their presented AOs. Zulqarnain et al. [42] extended the PFSS to interval-valued PFSS and developed the

AOs for interval-valued PFSS. They developed the MAGDM approach to resolve DM complications. The method designated in [43] is inadequate to examine the data with a reflective intellect for higher commencement and perfect decisions. For example, $O = \{O^1, O^2\}$ be a set of two professionals and $d_1, d_2$ are two parameters with their corresponding sub-attributes $d_1 = \{d_{11}, d_{12}\}$ and $d_2 = \{d_{21}\}$. Then $d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}\} = \{(d_{11}, d_{21}), (d_{12}, d_{21})\} = \{\check{d}_1, \check{d}_2\}$, where $\mathfrak{H}$ an alternative, then preferences of experts be can be summarized as $\mathfrak{H} = \begin{bmatrix} (0, 0.7) & (0.6, 0.7) \\ (0.8, 0.7) & (0.7, 0.2) \end{bmatrix}$. Let $\theta_i = (0.7, 0.3)^T$ and $\lambda_j = (0.4, 0.6)^T$ indicate the weights of experts and sub-parameters, respectively. Then, we attained the aggregated assessment expending the PFHSWG [43] operator is $\langle 0, 0.6638 \rangle$. This clearly shows that there is no influence on the collective result $\mu_e$. Because $a_{\mathcal{F}(\check{d}_k)} = a_{\mathcal{F}(\check{d}_{11})} = 0, a_{\mathcal{F}(\check{d}_{12})} = 0.8, a_{\mathcal{F}(\check{d}_{21})} = 0.6$, and $a_{\mathcal{F}(\check{d}_{22})} = 0.7$, which is unreasoning. PFHSS is a hybrid intellectual structure of PFSS. An enhanced sorting process fascinates investigators to crack baffling and inadequate information. Rendering to the investigation outcomes, PFHSS plays a vital role in decision-making by collecting numerous sources into a single value. According to the most generally known knowledge, the emergence of PFSS and hypersoft set (HSS) hybridization has not been combined with the PFSS background. Therefore, to inspire the current research of PFHSS, we will state AO based on rough data, the fundamental objectives of the following study are given as follows:

- The PFHSS competently deals the complex issues considering the multi sub-attributes of the considered parameters in the DM process. To keep this advantage in mind, we establish the AO for PFHSS.
- The Einstein operator is a well-known charming guesstimate AO. It is noticed that the prevailing Einstein AOs look unenthusiastic in marking the exact judgment through the DM procedure in some circumstances. To overwhelm these particular difficulties, these AOs need to be modified. We demonstrate advanced operational laws based on Einstein norms for Pythagorean fuzzy hypersoft numbers (PFHSNs).
- Establish the PFHSEWG operator using the above-mentioned Einstein operational laws with fundamental properties.
- A novel MCDM technique was established based on the proposed PFHSEWG to cope with DM issues under the PFHSS environment.
- MS is a significant aspect of engineering as it sees the practical standards of all constituents. MS is a time-consuming but significant step in the enterprise procedure. The industrialist's productivity, effectiveness, and character will suffer as an outcome deprived of material selection.
- Comparative analysis of the developed MCDM technique is proposed with current approaches to deliberate the practicality and supremacy of the planned model.

This study is systematized as follows: Basic knowledge of some important notions like SS, HSS, IFHSS, PFHSS, and Einstein norms have been deliberate in Section 2. Section 3 demarcated some basic operational laws for PFHSNs based on Einstein norms and established the PFHSEWG operator. Also, the planned operator's dynamic properties will be present in the same section. An MCDM approach is introduced using the PFHSEWG operator in Section 4. In the same section, a case study has been presented for material selection in the manufacturing industry. In Section 5, a comparison with some standing approaches is provided.

## 2 Preliminaries

This section remembers some essential concepts such as SS, HSS, IFHSS, and PFHSS.

***Definition 2.1*** [26] Let $X$ and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively. Let $\mathcal{P}(X)$ be the power set of $X$ and $\mathcal{A} \subseteq \mathbb{N}$. A pair $(\Omega, \mathcal{A})$ is called a SS over $X$, and its mapping is expressed as follows:

$$\Omega : \mathcal{A} \rightarrow \mathcal{P}(X)$$

Also, it can be defined as follows:

$$(\Omega, \mathcal{A}) = \{\Omega(e) \in \mathcal{P}(X) : e \in \mathbb{N},\ \Omega(e) = \emptyset \mathit{if}\ e \notin \mathcal{A}\}$$

***Definition 2.2*** [38] Let $X$ be a universe of discourse and $\mathcal{P}(X)$ be a power set of $X$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (\mathrm{n} \geq 1)$ and $K_i$ represented the set of attributes and their corresponding sub-attributes, such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and, $i, j \in \{1, 2, 3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$ , and $\alpha, \beta, \gamma \in$ N. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \cdots \times K_n = (\Omega, \ddot{\mathcal{A}})$ is known as HSS and defined as follows:

$$\Omega : K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} \rightarrow \mathcal{P}(X).$$

It is also defined as

$$(\Omega, \ddot{\mathcal{A}}) = \left\{\check{d},\ \Omega_{\ddot{\mathcal{A}}}\left(\check{d}\right) :\ \check{d} \in \ddot{\mathcal{A}},\ \Omega_{\ddot{\mathcal{A}}}\left(\check{d}\right) \in \mathcal{P}(X)\right\}.$$

***Definition 2.3*** [38] Let $X$ be a universe of discourse and $\mathcal{P}(X)$ be a power set of $X$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (\mathrm{n} \geq 1)$ and $K_i$ represented the set of attributes and their corresponding sub-attributes, such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1, 2, 3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \cdots \times K_n = \ddot{\mathcal{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in$ N , and $IFS^X$ be a collection of all fuzzy subsets over $X$. Then the pair $(\Omega, K_1 \times K_2 \times K_3 \times \ldots \times K_n = (\Omega, \ddot{\mathcal{A}})$ is known as IFHSS and defined as follows:

$$\Omega : K_1 \times K_2 \times K_3 \times \cdots \times K_n = \ddot{\mathcal{A}} \rightarrow IFS^X.$$

It is also defined as

$(\Omega, \ddot{\mathcal{A}}) = \left\{\left(\check{d},\ \Omega_{\ddot{\mathcal{A}}}\left(\check{d}\right)\right) :\ \check{d} \in \ddot{\mathcal{A}},\ \Omega_{\ddot{\mathcal{A}}}\left(\check{d}\right) \in IFS^X \in [0,\ 1]\right\}$, where $\Omega_{\ddot{\mathcal{A}}}\left(\check{d}\right) = \left\{\left\langle\delta,\ a_{\Omega(\check{d})}(\delta),\ b_{\Omega(\check{d})}(\delta)\right\rangle :\ \delta \in X\right\}$, where $a_{\Omega(\check{d})}(\delta)$ and $b_{\Omega(\check{d})}(\delta)$ signifies the Mem and NMem values of the attributes:

$$a_{\Omega(\check{d})}(\delta),\ b_{\Omega(\check{d})}(\delta) \in [0,\ 1], \text{ and } 0 \leq a_{\Omega(\check{d})}(\delta) + b_{\Omega(\check{d})}(\delta) \leq 1.$$

***Definition 2.4*** [40] Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}, (\mathrm{n} \geq 1)$ and $K_i$ represented the set of attributes and their corresponding sub-attributes, such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1, 2, 3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathbb{N}$. and $PFS^{\mathcal{U}}$ be a collection of all fuzzy subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \cdots \times K_n = (\mathcal{F}, \ddot{\mathcal{A}}))$ is known as PFHSS and defined as follows:

$$\mathcal{F} : K_1 \times K_2 \times K_3 \times \cdots \times K_n = \ddot{\mathcal{A}} \rightarrow PFS^{\mathcal{U}}.$$

It is also defined as $(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\check{d}, \mathcal{F}_{\ddot{\mathcal{A}}}\left(\check{d}\right)\right) : \check{d} \in \ddot{\mathcal{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}\left(\check{d}\right) \in PFS^{\mathcal{U}} \in [0, 1]\right\}$, where $\mathcal{F}_{\ddot{\mathcal{A}}}\left(\check{d}\right) = \left\{\left\langle \delta, a_{\mathcal{F}(\check{d})}(\delta), b_{\mathcal{F}(\check{d})}(\delta)\right\rangle : \delta \in \mathcal{U}\right\}$, where $a_{\mathcal{F}(\check{d})}(\delta)$ and $b_{\mathcal{F}(\check{d})}(\delta)$ signifies the Mem and NMem values of the attributes:

$a_{\mathcal{F}(\check{d})}(\delta)$, $b_{\mathcal{F}(\check{d})}(\delta) \in [0, 1]$, and $0 \le \left(a_{\mathcal{F}(\check{d})}(\delta)\right)^2 + \left(b_{\mathcal{F}(\check{d})}(\delta)\right)^2 \le 1$.

A Pythagorean fuzzy hypersoft number (PFHSN) can be stated as $\mathcal{F} = \left\{\left(a_{\mathcal{F}(\check{d})}(\delta), b_{\mathcal{F}(\check{d})}(\delta)\right)\right\}$, where $0 \le \left(a_{\mathcal{F}(\check{d})}(\delta)\right)^2 + \left(b_{\mathcal{F}(\check{d})}(\delta)\right)^2 \le 1$.

***Remark 2.1*** If $\left(a_{\mathcal{F}(\check{d})}(\delta)\right)^2 + \left(b_{\mathcal{F}(\check{d})}(\delta)\right)^2$ and $a_{\mathcal{F}(\check{d})}(\delta) + b_{\mathcal{F}(\check{d})}(\delta) \le 1$ both are holds. Then, PFHSS is condensed to IFHSS [41].

For readers' aptness, the PFHSN $\mathcal{F}_{\delta_i}\left(\check{d}_j\right) = \left\{\left(a_{\mathcal{F}(\check{d}_j)}(\delta_i), b_{\mathcal{F}(\check{d}_j)}(\delta_i)\right) | \delta_i \in \mathcal{U}\right\}$ can be written as $\mathfrak{J}_{\check{d}_{ij}} = \left\langle a_{\mathcal{F}(\check{d}_{ij})}, b_{\mathcal{F}(\check{d}_{ij})}\right\rangle$. The score function [43] for $\mathfrak{J}_{\check{d}_{ij}}$ is expressed as follows:

$$\mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) = a_{\mathcal{F}(\check{d}_{ij})}{}^2 - b_{\mathcal{F}(\check{d}_{ij})}{}^2, \mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [-1, 1] \tag{1}$$

But, in some cases, the above-defined score function cannot handle the scenario. For example, if we consider the two PFHSNs, such as $\mathfrak{J}_{\check{d}_{11}} = \langle .4, .7\rangle$ and $\mathfrak{J}_{\check{d}_{12}} = \langle .5, .8\rangle$. The score function cannot deliver relevant results to subtract the PFHSNs. So, in such situations, it is tough to achieve the most suitable alternative $\mathbb{S}\left(\mathfrak{J}_{\check{d}_{11}}\right) = .3 = \mathbb{S}\left(\mathfrak{J}_{\check{d}_{12}}\right)$. The accuracy function [43] had been developed.

$$H\left(\mathfrak{J}_{\check{d}_{ij}}\right) = a_{\mathcal{F}(\check{d}_{ij})}{}^2 + b_{\mathcal{F}(\check{d}_{ij})}{}^2, H\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [0, 1] \tag{2}$$

The consequent comparative laws will be used $\mathfrak{J}_{\check{d}_{ij}}$ and $\mathfrak{T}_{\check{d}_{ij}}$.

1. If $\mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) > \mathbb{S}\left(\mathfrak{T}_{\check{d}_{ij}}\right)$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$.
2. If $\mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) = \mathbb{S}\left(\mathfrak{T}_{\check{d}_{ij}}\right)$, then

- If $H\left(\mathfrak{J}_{\check{d}_{ij}}\right) > H\left(\mathfrak{T}_{\check{d}_{ij}}\right)$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$
- If $H\left(\mathfrak{J}_{\check{d}_{ij}}\right) = H\left(\mathfrak{T}_{\check{d}_{ij}}\right)$, then $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{T}_{\check{d}_{ij}}$.

***Definition 2.5*** Einstein's sum $\oplus_\varepsilon$ and Einstein product $\otimes_\varepsilon$ are good alternatives of algebraic t-norm and t-conorm, respectively, given as follows:

$$a \oplus_\varepsilon b = \frac{a + b}{1 + (a \cdot b)} \text{ and } a \otimes_\varepsilon b = \frac{a \cdot b}{1 + (1 - a) \cdot (1 - b)}, \forall (a, b) \in [0, 1]^2$$

Under the Pythagorean fuzzy environment, Einstein sum $\oplus_\varepsilon$ and Einstein product $\otimes_\varepsilon$ are defined as:

$$a \oplus_\varepsilon b = \sqrt{\frac{a^2 + b^2}{1 + \left(a^2 \cdot b^2\right)}}, \ a \otimes_\varepsilon b = \frac{a \cdot b}{\sqrt{1 + (1 - a^2) \cdot \left(1 - b^2\right)}}, \forall (a, b) \in [0, 1]^2$$

where $a \oplus_\varepsilon b$ and $a \otimes_\varepsilon b$ is known as t-norm and t-conorm, respectively, satisfying the bounded, monotonicity, commutativity, and associativity properties.

## 3 Einstein Weighted Geometric Aggregation Operator for Pythagorean Fuzzy Hypersoft Set

This section will introduce a novel Einstein-weighted AO such as the PFHSEWG operator for PFHSNs with essential properties.

### 3.1 Operational Laws for PFHSNs

***Definition 3.1*** [44] Let $\mathfrak{J}_{\check{d}_k} = \left(a_{\check{d}_k},\ b_{\check{d}_k}\right)$, $\mathfrak{J}_{\check{d}_{11}} = \left(a_{\check{d}_{11}},\ b_{\check{d}_{11}}\right)$, and $\mathfrak{J}_{\check{d}_{12}} = \left(a_{\check{d}_{12}},\ b_{\check{d}_{12}}\right)$ represents the PFHSNs and $\partial$ is a positive real number. Then, operational laws for PFHSNs based on Einstein norms can be expressed as follows:

1. $\mathfrak{J}_{\check{d}_{11}} \otimes_\varepsilon \mathfrak{J}_{\check{d}_{12}} = \left\langle \frac{\sqrt{\left(1+a^2_{\check{d}_{1j}}\right)-\left(1-a^2_{\check{d}_{1j}}\right)}}{\sqrt{\left(1+a^2_{\check{d}_{1j}}\right)+\left(1-a^2_{\check{d}_{1j}}\right)}},\ \frac{\sqrt{2b^2_{\check{d}_{1j}}}}{\sqrt{\left(2-b^2_{\check{d}_{1j}}\right)+b^2_{\check{d}_{1j}}}} \right\rangle$

2. $\mathfrak{J}_{\check{d}_{11}} \otimes_\varepsilon \mathfrak{J}_{\check{d}_{12}} = \left\langle \frac{\sqrt{2\alpha^2_{\check{d}_{1j}}}}{\sqrt{\left(2-\alpha^2_{\check{d}_{1j}}\right)+\alpha^2_{\check{d}_{1j}}}},\ \frac{\sqrt{\left(1+b^2_{\check{d}_{1j}}\right)-\left(1-b^2_{\check{d}_{1j}}\right)}}{\sqrt{\left(1+b^2_{\check{d}_{1j}}\right)+\left(1-b^2_{\check{d}_{1j}}\right)}} \right\rangle$

3. $\partial\mathfrak{J}_{\check{d}_k} = \left\langle \frac{\sqrt{\left(1+a_{\check{d}_k}{}^2\right)^\partial-\left(1-a_{\check{d}_k}{}^2\right)^\partial}}{\sqrt{\left(1+a_{\check{d}_k}{}^2\right)^\partial+\left(1-a_{\check{d}_k}{}^2\right)^\partial}},\ \frac{\sqrt{2\left(b_{\check{d}_k}{}^2\right)^\partial}}{\sqrt{\left(2-b_{\check{d}_k}{}^2\right)^\partial+\left(b_{\check{d}_k}{}^2\right)^\partial}} \right\rangle$

4. $\mathfrak{J}_{\check{d}_k}{}^\partial = \left\langle \frac{\sqrt{2\left(a_{\check{d}_k}{}^2\right)^\partial}}{\sqrt{\left(2-a_{\check{d}_k}{}^2\right)^\partial+\left(a_{\check{d}_k}{}^2\right)^\partial}},\ \frac{\sqrt{\left(1+b_{\check{d}_k}{}^2\right)^\partial-\left(1-b_{\check{d}_k}{}^2\right)^\partial}}{\sqrt{\left(1+b_{\check{d}_k}{}^2\right)^\partial+\left(1-b_{\check{d}_k}{}^2\right)^\partial}} \right\rangle$

***Definition 3.2*** Let $\mathfrak{J}_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}},\ b_{\check{d}_{ij}}\right)$ be a collection of PFHSNs. Then the PFHSEWG operator is defined as

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\lambda_j\left(\otimes_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{ij}}\right) \tag{3}$$

where $(i = 1,2,\ldots,n)$, $(j = 1,2,\ldots,m)$ and $\theta_i,\ \lambda_j$ signify the weighted vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$.

***Theorem 3.1*** Let $\mathfrak{J}_{\check{d}_{ij}} = \langle\left(a_{\check{d}_{ij}},\ b_{\check{d}_{ij}}\right)\rangle$ be a collection of PFHSNs, then the aggregated value attained by Eq. (3) is given as

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\lambda_j\left(\otimes_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{ij}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle \tag{4}$$

where ($i = 1, 2, \ldots, \text{n}$), ($j = 1, 2, \ldots, \text{m}$) and $\theta_i$, $\lambda_j$ signify the weight vectors such that $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$.

Proof: We will use mathematical induction to demonstrate the above result.

For n = 1, we get $\theta_i = 1$.

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m} \lambda_j \mathfrak{J}_{\check{d}_{1j}}$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\alpha_{\check{d}_{1j}}^{2}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\alpha_{\check{d}_{1j}}^{2}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\alpha_{\check{d}_{1j}}^{2}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(1+\mathscr{b}_{\check{d}_{1j}}^{2}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(1-\mathscr{b}_{\check{d}_{1j}}^{2}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+\mathscr{b}_{\check{d}_{1j}}^{2}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(1-\mathscr{b}_{\check{d}_{1j}}^{2}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

For m = 1, we get $\lambda_j = 1$.

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{i=1}^{n} \theta_i \mathfrak{J}_{\check{d}_{i1}}$$

$$= \left\langle \frac{\sqrt{2\prod_{i=1}^{n}\left(\alpha_{\check{d}_{i1}}^{2}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{i1}}^{2}\right)^{\theta_i}+\prod_{i=1}^{n}\left(\alpha_{\check{d}_{i1}}^{2}\right)^{\theta_i}}}, \frac{\sqrt{\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{i1}}^{2}\right)^{\theta_i}-\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{i1}}^{2}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{i1}}^{2}\right)^{\theta_i}+\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{i1}}^{2}\right)^{\theta_i}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

So, Eq. (4) true for n = 1, m = 1.

Assume that the equation grasps for n $= \delta_2$, m $= \delta_1 + 1$ and for n $= \delta_2 + 1$, m $= \delta_1$

$$\otimes_{j=1}^{\delta_1+1}\lambda_j\left(\otimes_{i=1}^{\delta_2}\theta_i\mathfrak{I}_{\check{d}_{ij}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$\otimes_{j=1}^{\delta_1+1}\lambda_j\left(\otimes_{i=1}^{\delta_2+1}\theta_i\mathfrak{I}_{\check{d}_{ij}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\beta_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

Now we show the Eq. (4) for m $= \delta_1 + 1$ and n $= \delta_2 + 1$

$$\otimes_{j=1}^{\delta_1+1}\lambda_j\left(\otimes_{i=1}^{\delta_2+1}\theta_i\mathfrak{I}_{\check{d}_{ij}}\right) = \otimes_{j=1}^{\delta_1+1}\lambda_j\left(\otimes_{i=1}^{\delta_2}\theta_i\mathfrak{I}_{\check{d}_{ij}} \otimes \theta_{i+1}\mathfrak{I}_{\check{d}_{(\delta_2+1)j}}\right)$$

$$= \left(\otimes_{j=1}^{\delta_1+1}\otimes_{i=1}^{\delta_2}\theta_i\lambda_j\mathfrak{I}_{\check{d}_{ij}}\right)\left(\otimes_{j=1}^{\delta_1+1}\lambda_j\theta_{i+1}\mathfrak{I}_{\check{d}_{(\delta_2+1)j}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right.$$

$$\otimes\frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(\alpha^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2-\alpha^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\left(\alpha^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}},$$

$$\frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\otimes\frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+b^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\left(1-b^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+b^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\left(1-b^2_{\check{d}_{(\delta_2+1)j}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}\Bigg\rangle$$

$$=\Bigg\langle\frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}},$$

$$\frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-b^2_{\check{d}_{ij}}s\right)^{\theta_i}\right)^{\lambda_j}}}\Bigg\rangle$$

$$=\otimes_{j=1}^{\delta_1+1}\lambda_j\left(\otimes_{i=1}^{\delta_2+1}\theta_i\mathfrak{J}_{\check{d}_{ij}}\right)$$

So, it is true for m $=\delta_1+1$ and n $=\delta_2+1$.

***Example 3.1*** Let $\mathcal{R}=\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4\}$ be a set of experts with the given weight vector $\theta_i=(0.1,\ 0.3,\ 0.3,\ 0.3)^T$. The team of experts is going to describe the attractiveness of a house under-considered set of attributes $\mathring{A}=\{d_1=lawn,\ d_2=security\ system\}$ with their corresponding sub-attributes Lawn $=d_1=\{d_{11}=with\ grass,\ d_{12}=without\ grass\}$ Security system $=d_2=\{d_{21}=guards,\ d_{22}=cameras\}$. Let $\mathring{A}=d_1\times d_2$ be a set of sub-attributes

$\mathring{A}=d_1\times d_2=\{d_{11},\ d_{12}\}\times\{d_{21},\ d_{22}\}=\left\{(d_{11},\ d_{21}),\ (d_{11},\ d_{22}),\ (d_{12},\ d_{21}),\ (d_{12},\ d_{22})\right\}$

$\mathring{A}=\left\{\check{d}_1,\check{d}_2,\ \check{d}_3,\ \check{d}_4\right\}$ represents the set sub-attributes with weights with weight vector $\lambda_j=(0.2,\ 0.2,\ 0.2,\ 0.4)^T$. The supposed rating values for all attributes in the form of PFSNs $\left(\mathfrak{J}_{\check{d}_{ij}},\ \mathring{A}\right)=\left(aij,\ b_{ij}\right)_{4\times 4}$ given as:

$$\left(\mathfrak{J}_{\check{d}_{ij}},\ \text{Å}\right)=\begin{bmatrix}(0.5,\ 0.8) & (0.7,\ 0.5) & (0.4,\ 0.6) & (0.7,\ 0.4)\\ (0.5,\ 0.6) & (0.9,\ 0.1) & (0.3,\ 0.7) & (0.4,\ 0.5)\\ (0.4,\ 0.8) & (0.7,\ 0.5) & (0.4,\ 0.6) & (0.3,\ 0.5)\\ (0.3,\ 0.7) & (0.6,\ 0.5) & (0.5,\ 0.4) & (0.5,\ 0.7)\end{bmatrix}$$

As we know that

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right)=\otimes_{j=1}^{m}\lambda_j\left(\otimes_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{ij}}\right)$$

$$=\left\langle\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}},\right.$$
$$\left.\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle$$

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{44}}\right)$$

$$=\left\langle\frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}},\right.$$
$$\left.\frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathscr{b}_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle$$

$$=\left\langle=\frac{\sqrt{2\begin{bmatrix}\left\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\right\}^{0.2}\left\{(0.49)^{0.1}(0.81)^{0.3}(0.36)^{0.3}(0.25)^{0.3}\right\}^{0.2}\\ \left\{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\right\}^{0.2}\left\{(0.49)^{0.1}(0.16)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\right\}^{0.4}\end{bmatrix}}}{\sqrt{\begin{matrix}\left\{(1.75)^{0.1}(1.75)^{0.3}(1.84)^{0.3}(1.91)^{0.3}\right\}^{0.2}\left\{(1.51)^{0.1}(1.09)^{0.3}(1.64)^{0.3}(1.75)^{0.3}\right\}^{0.2}\\ \left\{(1.84)^{0.1}(1.91)^{0.3}(1.84)^{0.3}(1.75)^{0.3}\right\}^{0.2}\left\{(1.51)^{0.1}(1.84)^{0.3}(1.91)^{0.3}(1.75)^{0.3}\right\}^{0.4}\\ +\\ \left\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\right\}^{0.2}\left\{(0.49)^{0.1}(0.81)^{0.3}(0.36)^{0.3}(0.25)^{0.3}\right\}^{0.2}\\ \left\{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\right\}^{0.2}\left\{(0.49)^{0.1}(0.16)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\right\}^{0.4}\end{matrix}}},\right.$$

$$\frac{\sqrt{\begin{array}{c}\left\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\right\}^{0.2}\left\{(1.25)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\right\}^{0.2}\\ \left\{(1.36)^{0.1}(1.49)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\right\}^{0.2}\left\{(1.16)^{0.1}(1.25)^{0.3}(1.49)^{0.3}(1.25)^{0.3}\right\}^{0.4}\\ -\\ [\left\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\right\}^{0.2}\left\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\right\}^{0.2}\\ \left\{(0.64)^{0.1}(0.51)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\right\}^{0.2}\left\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\right\}^{0.4}\end{array}}}{\sqrt{\begin{array}{c}\left\{(0.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\right\}^{0.4}\left\{(1.25)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\right\}^{0.2}\\ \left\{(1.36)^{0.1}(1.49)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\right\}^{0.2}\left\{(1.16)^{0.1}(1.25)^{0.3}(1.49)^{0.3}(1.25)^{0.3}\right\}^{0.4}\\ +\\ [\left\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\right\}^{0.2}\left\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\right\}^{0.2}\\ \left\{(0.64)^{0.1}(0.51)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\right\}^{0.2}\left\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\right\}^{0.4}\end{array}}}\right\rangle$$

$$=\left\langle \frac{\sqrt{2\,[(0.4953)\,(0.6938)\,(0.5664)\,(0.3355)]}}{\sqrt{\begin{array}{c}(1.2346)\,(1.1676)\,(1.2208)\,(1.4786)+\\ (0.4953)\,(0.6938)\,(0.5664)\,(0.3355)\end{array}}}\,,\ \frac{\sqrt{\begin{array}{c}(1.1477)\,(1.0651)\,(1.0872)\,(1.1850)-\\ [(0.7841)\,(0.9220)\,(0.9035)\,(0.7079)]\end{array}}}{\sqrt{\begin{array}{c}(1.1477)\,(1.0651)\,(1.0872)\,(1.1850)+\\ [(0.7841)\,(0.9220)\,(0.9035)\,(0.7079)]\end{array}}}\right\rangle$$

$$=\langle 0.2211\ ,\ 0.7392.\rangle\,.$$

***Theorem 3.2*** Let $\mathfrak{J}_{\check{d}_{ij}} = a_{\check{d}_{ij}},\ b_{\check{d}_{ij}}$ be a collection of PFHSNs, then

$$PFHSWG\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right)\ \geq\ \text{PFHSEWG}\ \left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\ \right)$$

where $\theta_i$, $\lambda_j$ signify the weight vectors such as $\theta_i > 0,\ \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0,\ \sum_{j=1}^{n}\lambda_j = 1$.

Proof: As we know that

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-\alpha_{\check{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(\alpha_{\check{d}_{ij}}^{2}\right)}$$

$$\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-\left(\alpha_{\check{d}_{ij}}^{2}\right)\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(\alpha_{\check{d}_{ij}}^{2}\right)}=\sqrt{2}$$

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{2}$$

$$\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}\geq\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\tag{5}$$

Again

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)}$$

$$\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)}=\sqrt{2}$$

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{2}$$

$$\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}\leq\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}} \tag{6}$$

Let PFHSWG $\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \mathfrak{I}_{\check{d}} = \left(a_{\mathfrak{I}_{\check{d}}}, \mathscr{b}_{\mathfrak{I}_{\check{d}}}\right)$ and PFHSEWG $\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \mathfrak{I}_{\check{d}}^{\varepsilon} = \left(a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}, \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}\right)$.

Then, inequalities (5) and (6) can be transformed into the following forms $a_{\mathfrak{I}_{\check{d}}} \leq a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}$ and $\mathscr{b}_{\mathfrak{I}_{\check{d}}} \geq \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}$, respectively.

So, $S\left(\mathfrak{I}_{\check{d}}\right) = a_{\mathfrak{I}_{\check{d}}}{}^2 - \mathscr{b}_{\mathfrak{I}_{\check{d}}}{}^2 \leq a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2 - \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2 = S\left(\mathfrak{I}_{\check{d}}^{\varepsilon}\right)$. Hence, $S\left(\mathfrak{I}_{\check{d}}\right) \leq S\left(\mathfrak{I}_{\check{d}}^{\varepsilon}\right)$

If $S\left(\mathfrak{I}_{\check{d}}\right) < S\left(\mathfrak{I}_{\check{d}}^{\varepsilon}\right)$, then

$$\text{PFHSWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) < \text{PFHSEWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \tag{7}$$

If $S\left(\mathfrak{I}_{\check{d}}\right) = S\left(\mathfrak{I}_{\check{d}}^{\varepsilon}\right)$, then $a_{\mathfrak{I}_{\check{d}}}{}^2 - \mathscr{b}_{\mathfrak{I}_{\check{d}}}{}^2 = a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2 - \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2$, so $a_{\mathfrak{I}_{\check{d}}} = a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}$ and $\mathscr{b}_{\mathfrak{I}_{\check{d}}} = \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}$.

Then, $A\left(\mathfrak{I}_{\check{d}}\right) = a_{\mathfrak{I}_{\check{d}}}{}^2 + \mathscr{b}_{\mathfrak{I}_{\check{d}}}{}^2 = a_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2 + \mathscr{b}_{\mathfrak{I}_{\check{d}}^{\varepsilon}}{}^2 = A\left(\mathfrak{I}_{\check{d}}^{\varepsilon}\right)$. Thus,

$$\text{PFHSWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \text{PFHSEWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \tag{8}$$

From inequalities (7) and (8), we get

$$\text{PFHSWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \leq \text{PFHSEWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right).$$

***Example 3.2*** Using the data given in Example 3.1

$$PFHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{44}}\right) = \left\langle \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(a_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathscr{b}_{\check{d}_{ij}}{}^2\right)^{\Omega_i}\right)^{\gamma_j}}\right\rangle$$

$\text{PFHSWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{44}}\right)$

$$= \left\langle \frac{\left(\begin{matrix}\left\{(0.5)^{0.1}(0.5)^{0.3}(0.4)^{0.3}(0.3)^{0.3}\right\}^{0.2}\left\{(0.7)^{0.1}(0.9)^{0.3}(0.7)^{0.3}(0.6)^{0.3}\right\}^{0.2}\\ \left\{(0.4)^{0.1}(0.3)^{0.3}(0.4)^{0.3}(0.5)^{0.3}\right\}^{0.2}\left\{(0.7)^{0.1}(0.4)^{0.3}(0.3)^{0.3}(0.5)^{0.3}\right\}^{0.4}\end{matrix}\right),}{\sqrt{1-\left[\begin{matrix}\left[\left\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\right\}^{0.2}\left\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\right\}^{0.2}\right.\\ \left\{(0.64)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.84)^{0.3}\right\}^{0.2}\left\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\right\}^{0.4}\end{matrix}\right]}} \right\rangle$$

$$= \left\langle ((0.8330)\,(0.9365)\,(0.8293)\,(0.7033))\ ,\ \sqrt{1-[(0.7841)\,(0.9220)\,(0.9035)\,(0.7079)]}\right\rangle$$

$$= \langle 0.4549\ ,\ 0.7332\rangle$$

Hence, from Examples 3.1 and 3.2, it is proved that

$\text{PFHSWG}\ \left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right) \le \text{PFHSEWG}\ \left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right).$

### *3.2 Properties of PFHSEWA Operator*

***Idempotency 3.2.1*** If $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{J}_{\check{d}_k} = \left(a_{\check{d}_{ij}},\ b_{\check{d}_{ij}}\right)\forall i,\ j$, then $\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}_k}$

Proof: As we know that

$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right)$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\left(\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}, \frac{\sqrt{\left(\left(1+b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}-\left(\left(1-b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1+b_{\check{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(1-b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\alpha_{\check{d}_{ij}}^{2}}}{\sqrt{\left(2-\alpha_{\check{d}_{ij}}^{2}\right)+\left(\alpha_{\check{d}_{ij}}^{2}\right)}}\ ,\ \frac{\sqrt{\left(1+b_{\check{d}_{ij}}^{2}\right)-\left(1-b_{\check{d}_{ij}}^{2}\right)}}{\sqrt{\left(1+b_{\check{d}_{ij}}^{2}\right)+\left(1-b_{\check{d}_{ij}}^{2}\right)}} \right\rangle = \left\langle a_{\check{d}_{ij}}\ ,\ b_{\check{d}_{ij}}\right\rangle = \mathfrak{J}_{\check{d}_k}.$$

***Boundedness 3.2.2*** Let $\mathfrak{J}_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}},\ b_{\check{d}_{ij}}\right)$ be a collection PFHSNs and $\mathfrak{J}_{min} = min\left(\mathfrak{J}_{\check{d}_{ij}}\right)$, $\mathfrak{J}_{max} = max\left(\mathfrak{J}_{\check{d}_{ij}}\right)$. Then, $\mathfrak{J}_{min} \le \text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}},\ \mathfrak{J}_{\check{d}_{12}},\ldots,\mathfrak{J}_{\check{d}_{nm}}\right) \le \mathfrak{J}_{max}$

Proof: Let $f(x) = \sqrt{\frac{2-x^2}{x^2}}$, $x \in [0,\ 1]$, then $\frac{d}{dx}(f(x)) = \frac{-2}{x^3}\sqrt{\frac{x^2}{2-x^2}} < 0$. So, $f(x)$ is a non-increasing function on $[0,\ 1]$. As $a_{\check{d}_{ij}min} \le a_{\check{d}_{ij}} \le a_{\check{d}_{ij}max}\forall i,j$. Then, $f\left(a_{\check{d}_{ij}max}\right) \le f\left(a_{\check{d}_{ij}}\right) \le f\left(a_{\check{d}_{ij}min}\right)$. So, $\sqrt{\frac{2-a_{\check{d}_{ij}max}^2}{a_{\check{d}_{ij}max}^2}} \le \sqrt{\frac{2-a_{\check{d}_{ij}}^2}{a_{\check{d}_{ij}}^2}} \le \sqrt{\frac{2-a_{\check{d}_{ij}min}^2}{a_{\check{d}_{ij}min}^2}}$.

Let $\theta_i$ and $\lambda_j$ signify the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}max}}{a^2_{\check{d}_{ij}max}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}min}}{a^2_{\check{d}_{ij}min}}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{2-a^2_{\check{d}_{ij}max}}{a^2_{\check{d}_{ij}max}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\left(\frac{2-a^2_{\check{d}_{ij}min}}{a^2_{\check{d}_{ij}min}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\frac{2-a^2_{\check{d}_{ij}max}}{a^2_{\check{d}_{ij}max}}\right)} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\frac{2-a^2_{\check{d}_{ij}min}}{a^2_{\check{d}_{ij}min}}\right)}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{2-a^2_{\check{d}_{ij}max}}{a^2_{\check{d}_{ij}max}}\right)} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{1+\left(\frac{2-a^2_{\check{d}_{ij}min}}{a^2_{\check{d}_{ij}min}}\right)}$$

$$\Leftrightarrow \sqrt{\frac{2}{a^2_{\check{d}_{ij}max}}} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\frac{2}{a^2_{\check{d}_{ij}min}}}$$

$$\Leftrightarrow \sqrt{\frac{a^2_{\check{d}_{ij}min}}{2}} \le \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}} \le \sqrt{\frac{a^2_{\check{d}_{ij}max}}{2}}$$

$$\Leftrightarrow a_{\check{d}_{ij}min} \le \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a^2_{\check{d}_{ij}}}{a^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}} \le a_{\check{d}_{ij}max}$$

$$\Leftrightarrow a_{\check{d}_{ij}min} \le \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-a^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \le a_{\check{d}_{ij}max} \tag{9}$$

Again, let $g(y) = \sqrt{\frac{1-y^2}{1+y^2}}$, $y \in [0,\ 1]$. Then, $\frac{d}{dy}(g(y)) = \frac{-2y}{(1+y^2)^2}\sqrt{\frac{1+y^2}{1-y^2}} < 0$. So, $g(y)$ is decreasing function on $[0,\ 1]$. Thus, $b_{\check{d}_{ij}min} \le b_{\check{d}_{ij}} \le b_{\check{d}_{ij}max} \forall\ i,j$. So, $g\left(b_{\check{d}_{ij}max}\right) \le g\left(b_{\check{d}_{ij}}\right) \le g\left(b_{\check{d}_{ij}min}\right) \forall\ i,j$.

$$\Rightarrow \sqrt{\frac{1-b^2_{\check{d}_{ij}max}}{1+b^2_{\check{d}_{ij}max}}} \le \sqrt{\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}} \le \sqrt{\frac{1-b^2_{\check{d}_{ij}min}}{1+b^2_{\check{d}_{ij}min}}}$$

Let $\theta_i$ and $\lambda_j$ symbolize the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}max}}{1+b^2_{\check{d}_{ij}max}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}min}}{1+b^2_{\check{d}_{ij}min}}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{1-b^2_{\check{d}_{ij}max}}{1+b^2_{\check{d}_{ij}max}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\left(\frac{1-b^2_{\check{d}_{ij}min}}{1+b^2_{\check{d}_{ij}min}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\frac{1-b^2_{\check{d}_{ij}max}}{1+b^2_{\check{d}_{ij}max}}\right)} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\frac{1-b^2_{\check{d}_{ij}min}}{1+b^2_{\check{d}_{ij}min}}\right)}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{1-b^2_{\check{d}_{ij}max}}{1+b^2_{\check{d}_{ij}max}}\right)} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{1+\left(\frac{1-b^2_{\check{d}_{ij}min}}{1+b^2_{\check{d}_{ij}min}}\right)}$$

$$\Leftrightarrow \sqrt{\frac{2}{1+b^2_{\check{d}_{ij}max}}} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\frac{2}{1+b^2_{\check{d}_{ij}min}}}$$

$$\Leftrightarrow \sqrt{\frac{1+b^2_{\check{d}_{ij}min}}{2}} \le \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}} \le \sqrt{\frac{1+b^2_{\check{d}_{ij}max}}{2}}$$

$$\Leftrightarrow \sqrt{1+b^2_{\check{d}_{ij}min}} \le \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}} \le \sqrt{1+b^2_{\check{d}_{ij}max}}$$

$$\Leftrightarrow b_{\check{d}_{ij}min} \le \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-b^2_{\check{d}_{ij}}}{1+b^2_{\check{d}_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}-1} \le b_{\check{d}_{ij}max}$$

$$\Leftrightarrow b_{\check{d}_{ij}min} \le \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \le b_{\check{d}_{ij}max} \tag{10}$$

Let PFHSEWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}_k}$, then inequalities (9) and (10) can be written as $a_{\check{d}_{ij}min} \leq a_{\check{d}_{ij}} \leq a_{\check{d}_{ij}max}$ and $b_{\check{d}_{ij}max} \leq b_{\check{d}_{ij}} \leq b_{\check{d}_{ij}min}$. Thus, $S\left(\mathfrak{J}_{\check{d}_k}\right) = a_{\check{d}_{ij}}{}^2 - b_{\check{d}_{ij}}{}^2 \leq a^2_{\check{d}_{ij}\max} - b^2_{\check{d}_{ij}\min} = S\left(\mathfrak{J}_{\check{d}_k max}\right)$ and $S\left(\mathfrak{J}_{\check{d}_k}\right) = a_{\check{d}_{ij}}{}^2 - b_{\check{d}_{ij}}{}^2 \geq a^2_{\check{d}_{ij}\min} - b^2_{\check{d}_{ij}\max} = S\left(\mathfrak{J}_{\check{d}_k min}\right)$.

If $S\left(\mathfrak{J}_{\check{d}_k}\right) < S\left(\mathfrak{J}_{\check{d}_k max}\right)$ and $S\left(\mathfrak{J}_{\check{d}_k}\right) > S\left(\mathfrak{J}_{\check{d}_k min}\right)$. Then, we have

$$\mathfrak{J}_{\check{d}_k min} < \text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{ij}}\right) < \mathfrak{J}_{\check{d}_k max} \tag{11}$$

If $S\left(\mathfrak{J}_{\check{d}_k}\right) = S\left(\mathfrak{J}_{\check{d}_k max}\right)$, then we have $a_{\check{d}_{ij}}{}^2 = a_{\check{d}_{ij}max}{}^2$ and $b_{\check{d}_{ij}}{}^2 = b_{\check{d}_{ij}max}{}^2$. Thus, $S\left(\mathfrak{J}_{\check{d}_k}\right) = a_{\check{d}_{ij}}{}^2 - b_{\check{d}_{ij}}{}^2 = a_{\check{d}_{ij}max}{}^2 - b_{\check{d}_{ij}max}{}^2 = S\left(\mathfrak{J}_{\check{d}_k max}\right)$. Therefore,

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}_k max} \tag{12}$$

If $S\left(\mathfrak{J}_{\check{d}_k}\right) = S\left(\mathfrak{J}_{\check{d}_k min}\right)$. Then, we have $a_{\check{d}_{ij}}{}^2 - b_{\check{d}_{ij}}{}^2 = a_{\check{d}_{ij}min}{}^2 - b_{\check{d}_{ij}min}{}^2 \Rightarrow a_{\check{d}_{ij}}{}^2 = a_{\check{d}_{ij}min}{}^2$ and $b_{\check{d}_{ij}}^2 = b^2_{\check{d}_{ij}min}$. Thus, $A\left(\mathfrak{J}_{\check{d}_k}\right) = a_{\check{d}_{ij}}{}^2 + b_{\check{d}_{ij}}{}^2 = a_{\check{d}_{ij}min}{}^2 + b_{\check{d}_{ij}min}{}^2 = A\left(\mathfrak{J}_{\check{d}_k min}\right)$. Therefore,

$$\text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}_k min} \tag{13}$$

So proved that

$$\mathfrak{J}_{\check{d}_k min} \leq \text{PFHSEWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}_{\check{d}_k max}$$

***Homogeneity 3.2.3*** Prove that PFHSEWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \partial$ PFHSEWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right)$ for $\partial > 0$.

Proof: Let $\mathfrak{J}_{\check{d}_{ij}}$ be a PFHSN and $\partial$ is a positive number, then by

$$\partial\mathfrak{J}_{\check{d}_{ij}} = \left\langle \frac{\sqrt{\left(1 + a_{\check{d}_k}{}^2\right)^{\partial} - \left(1 - a_{\check{d}_k}{}^2\right)^{\partial}}}{\sqrt{\left(1 + a_{\check{d}_k}{}^2\right)^{\partial} + \left(1 - a_{\check{d}_k}{}^2\right)^{\partial}}}, \frac{\sqrt{2\left(b_{\check{d}_k}{}^2\right)^{\partial}}}{\sqrt{\left(2 - b_{\check{d}_k}{}^2\right)^{\partial} + \left(b_{\check{d}_k}{}^2\right)^{\partial}}} \right\rangle$$

So,

$$\text{PFHSEWG}\left(\partial\mathfrak{J}_{\check{d}_{11}}, \partial\mathfrak{J}_{\check{d}_{12}}, \ldots, \partial\mathfrak{J}_{\check{d}_{nm}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2 - \alpha^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + b^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - b^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + b^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - b^2_{\check{d}_{ij}}\right)^{\partial\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}, \right.$$

$$\left. \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}} \right\rangle$$

$$= \partial\ \mathrm{PFHSEWG}\ \left(\partial\mathfrak{J}_{\check{d}_{11}},\ \partial\mathfrak{J}_{\check{d}_{12}}, \ldots, \partial\mathfrak{J}_{\check{d}_{nm}}\right)$$

## 4 Multi-Criteria Decision Making Approach for PFHSEWG Operator

This section proposes a DM method to address the difficulties of MCDM based on the planned PFHSEWG operator with a numerical example.

### *4.1 Proposed Approach*

Consider $\mathfrak{H} = \left\{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\right\}$ be a set of $s$ alternatives $O = \{O_1, O_2, O_3, \ldots, O_r\}$ be a set of $r$ experts. The weights of experts are given as $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ such that $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$. Let $\mathfrak{L} = \{d_1, d_2, \ldots, d_m\}$ expressed the set of attributes with their corresponding multi sub-attributes such as $\mathfrak{L}' = \left\{\left(d_{1\rho} \times d_{2\rho} \times \ldots \times d_{m\rho}\right) \text{ for all } \rho \in \{1, 2, \ldots, t\}\right\}$ with weights $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ such that $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and can be indicated as $\mathfrak{L}' = \left\{\check{d}_\partial : \partial \in \{1, 2, \ldots, m\}\right\}$. Experts $\{\kappa^i : i = 1, 2, \ldots, n\}$ evaluate the alternatives $\left\{\mathfrak{H}^{(z)} : z = 1, 2, \ldots, s\right\}$ in PFHSNs form $\left(\mathfrak{H}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\alpha_{\check{d}_{ij}}, \mathscr{b}_{\check{d}_{ij}}\right)_{n*m}$, under the preferred sub-attributes $\left\{\check{d}_\partial : \partial = 1, 2, \ldots, k\right\}$. Where $0 \le \alpha_{\check{d}_{ij}}, \mathscr{b}_{\check{d}_{ij}} \le 1$ and $0 \le \left(\alpha_{\check{d}_{ij}}\right)^2 + \left(\mathscr{b}_{\check{d}_{ij}}\right)^2 \le 1$ for all $i, k$. Experts deliver their estimations for each alternative in the form of PFHSNs $\mathcal{L}_k$ and a step-by-step algorithm to attain the supreme alternative is given in the following.

Step 1: Obtain decision matrices for each alternative in the form of PFHSNs $F = \left(\mathfrak{J}_{\check{d}_{ij}}\right)_{n*m}$.

$$\left(\mathfrak{H}^{(z)}_{\check{d}_{ik}}, \mathfrak{L}'\right)_{n\times\partial} = \begin{array}{c} O_1 \\ O_2 \\ \vdots \\ O_n \end{array} \begin{pmatrix} \left(\alpha^{(z)}_{\check{d}_{11}}, \mathscr{b}^{(z)}_{\check{d}_{11}}\right) & \left(\alpha^{(z)}_{\check{d}_{12}}, \mathscr{b}^{(z)}_{\check{d}_{12}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{1\partial}}, \mathscr{b}^{(z)}_{\check{d}_{1\partial}}\right) \\ \left(\alpha^{(z)}_{\check{d}_{21}}, \mathscr{b}^{(z)}_{\check{d}_{21}}\right) & \left(\alpha^{(z)}_{\check{d}_{22}}, \mathscr{b}^{(z)}_{\check{d}_{22}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{2\partial}}, \mathscr{b}^{(z)}_{\check{d}_{2\partial}}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\alpha^{(z)}_{\check{d}_{n1}}, \mathscr{b}^{(z)}_{\check{d}_{n1}}\right) & \left(\alpha^{(z)}_{\check{d}_{n2}}, \mathscr{b}^{(z)}_{\check{d}_{n2}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{n\partial}}, \mathscr{b}^{(z)}_{\check{d}_{n\partial}}\right) \end{pmatrix}$$

Step 2: Convert the cost type attributes to benefit type using the normalization rule.

$$M_{\check{d}_{ij}} = \begin{cases} \mathfrak{J}^c_{\check{d}_{ij}} = \left(\mathscr{b}_{\check{d}_{ij}}, \alpha_{\check{d}_{ij}}\right) & \textit{cost type parameter} \\ \mathfrak{J}_{\check{d}_{ij}} = \left(\alpha_{\check{d}_{ij}}, \mathscr{b}_{\check{d}_{ij}}\right) & \textit{benefit type parameter} \end{cases}$$

Step 3: Use the settled PFHSEWG operator to collective the PFHSNs $\mathfrak{J}_{\check{a}_{ij}}$ for each alternative $\mathfrak{H} = \left\{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\right\}$ into the decision matrix $\mathcal{L}_k$.

Step 4: Use Eq. (1) to calculate the scores for all alternatives.

Step 5: Pick the alternative with the highest score and check the ranking.

### *4.2 Numerical Example*

In this section, a practical MCDM problem comprises a decisive adequate material selection model to confirm that the conventional approach is pertinent and reasonable.

***Case Study 4.2.1*** According to the Diplomatic Board on Climate Variation, extreme ecological humiliation results from social accomplishments [45]. The climate variation has substantial ecological significance, containing the extermination of animal classes [46]. Lesser farming production [47]. Extra thrilling Meteorological conditions configurations [48], and humanoid movement [49]. Have Increasing momentum to moderate universal greenhouse gas discharges to alleviate climate variation corridors. For example, France recently approved a prerequisite of 40% Condense greenhouse gas discharges by 2030 paralleled 1990 [50]. Still, the routine of carbon gasses is not the solitary fabricator of greenhouse gases. The environmental protection agency released 76% of fossil fuel interpretation of all anthropogenic releases in the United States [51]. It can be realistically contingent that an extensive decrease in Greenhouse gas radiation means less usage of fossil fuels. But, this is not an informal assignment since the invention. What is formed from hydrocarbons is an energy transporter and the key energy cause. To have a substantial impression on decarbonization, it would be included in a globally friendly way. In 2017, fossil fuels accounted for extra than 85% of global energy production [52].

Consequently, energy scarcities resolve instantaneously if the world completely alters to a hydrogen budget that eradicates fossil fuel feasting. This component delivers significant tasks in verdict an appropriate power source. Though, this investigation will not insurance this issue. As mortality is impending, the 'end of low-priced oil' eras," with complete compromise in science and power engineering that essential discover new energy exporters. Severe reduction procedure across nations exposed hydrogen will be the eventual optimal. Hydrogen, conceivable as complementary energy in cars, influences industrial innovations such as hydrogen fuel cells to deliver manufacturing deprived of producing any $CO_2$ involuntary transmission authority and straight fuel for internal burning engines. One of its impelling features is the propensity for hydrogen fabrications in the variation of feedstocks it produces. Since there is virtually no abundant hydrogen in wildlife. The single choice is to proclamation it from the organic bond of other grains. There are two conducts to produce hydrogen, amongst other belongings: furious hydrocarbons or cracking water. Condensation fermentation is used to disrupt depressed hydrocarbons. Water excruciating can be completely straight in compelling circumstances, temperature, or energy use. A new way to produce hydrogen from water is to burn coal in the attendance of water suspension.

It kinds intelligence to renovate fossil discarded energies such as natural gas is earliest transformed into hydrogen. In conclusion, while fossil fuels develop excessively and prospective unlawfully for worldwide warming, renewable, most important energy will originate into the depiction for financial or environmental causes. In expressions of power, the recently formed hydrogen fuel is dissimilar to the frequently used ones in gratified weight and volume. This hydrogen is frivolously associated with its energy capability is the top prominent feature. The energy content of hydrogen per kilogram is 120 MJ. Hydrogen has a little volumetric energy compactness related to its exceptional gravimetric density. The compactness of hydrogen is committed by its accumulation state. Unfluctuating densities up to 700 bar are not massive sufficient belongings of hydrocarbons similar to gasoline and diesel. Only fluid hydrogen can influence a reasonable amount, still less than a quarter of the amount of gasoline. So, hydrogen containers for motorized solicitations will conquer more than fluid hydrocarbons formerly used containers [53]. Cryogenic storing containers are also recognized as cryogenic holding vessels. Dewar flask is, in fact, a double-walled super-insulator container. It vehicles fluid oxygen, nitrogen, hydrogen, helium, and argon, temperatures $<110\,\mathrm{K}/163°\mathrm{C}$. Fluid hydrogen has been familiar as a more significant energy cause. Since water is impartial a surplus gas, it's unbelievably non-toxic ecological security when rehabilitated to power. Constituents used in cryogenic container enterprises are contingent on protection and budget [54]. Essentially the exertion of cryogenic vessels is security apprehensions and enterprise conditions. In perspective, short temperature embrittlement can be designated as follows:

Fracture toughness: The steaming point of melted nitrogen is around −196°C, whereas the steaming point of liquefied nitrogen is around. The temperature of hydrogen gas is approximate −253°C. The substantial cannot find ductility and converts hard. So, the considerable requirement is robust and sufficient to endure Inelastic crack. Face-centered cube metal webs are suitable since they are impervious to low temperatures. All nickel-copper compounds, aluminum, its compounds, and austenitic stainless steels contain an extra 7% nickel to construct a storing cryogenic vessel [55]. Heat transfer: heat transient over low-temperature container barriers are principally conductive. Constituents with low, warm air conductivity are chosen. Thermal stress: due to slight temperature, interior barriers contract, instigating thermal straining. So, constituents with slight thermal conductivity are suitable. Thermal diffusivity: in practice, the collective thermal isolation is ridiculous. The material must be selected in such a tactic that it can disperse heat as rapidly as conceivable.

Material assortment in any manufacturing arena is a very significant enterprise phase. Manufacturing enterprise is prepared by enactment, budget, ecological compassion objectives, And commonly inadequate by the material. The most acceptable product strategy selects the best appropriate material design criteria by providing an extreme presentation at the lowermost probable budget. Material selection is By seeing numerous contradictory DM procedures. AO shows a vital part in DM. The existing Einstein AO has originated as a DM procedure in this circumstance. These AOs must be modernized to talk about these definite concerns. We intend some novel operations and escorting AO for aggregating innumerable PFHSN. Our projected ideal outclasses other models. Conferring to the clarification stated above and DM perception, all structures can be categorized. The case study was shown in a motorized portions engineering corporation in Malaysia, and a motorized constituent, cryogenic storing, accompanied the study. As part of applying the concept of sustainability, companies must choose suitable materials for produced parts. It focuses first on cryogenic storage containers and then on other factors input of weights for gathering parameters and materials from DM. PFHSN theory and proposed AO are used to overcome complexity and indecision human judgment. MS with three remember the essential pillar of sustainability: materials must be reasonable, ecologically pleasant, and beneficial to humanity.

The most imperative aspects (parameters) to consider when selecting a substantial dashboard DM. Choose the procedure starts with an initial screening of material used for dashboards, captivating into justification structures intrinsic to the application. In the screening process, identify the fabrics that may be appropriate. It is serious about deciding the material that can be used initial MS for the instrument board process. Four materials are selected, subsequently examining the abilities: $\mathfrak{H}^1 = $ Ti–6Al–4V, $\mathfrak{H}^2 = $ SS301–FH, $\mathfrak{H}^3 = $ 70Cu–30Zn, and $\mathfrak{H}^4 = $ Inconel 718. The attribute of material selection is given as follows: $\mathfrak{L} = \{d_1 = $ Specific gravity $=$ attaining data around the meditation of resolutions of numerous materials, $d_2 = $ Toughness index, $d_3 = $ Yield stress, $d_4 = $ Easily accessible$\}$. The corresponding subattributes of the considered parameters, Specific gravity = attaining data around the meditation of resolutions of numerous materials $= d_1 = \{d_{11} =$ assess corporal variations, $d_{12} =$ govern the degree of regularity among tasters}, Toughness index $= d_2 = \{d_{21} =$ Charpy V − Notch Impact Energy, $d_{22} =$ Plane StrainFracture Toughness}, Yield stress $= d_3 = \{d_{31} =$ forging, $d_{32} =$ *rolling or pressing*}, Easily accessible $= d_4 = \{d_{41} =$ Easily accessible}. Let $\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4$ be a set of sub-attributes

$$\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4 = \{d_{11},\ d_{12}\} \times \{d_{21},\ d_{22}\} \times \{d_{31},\ d_{32}\} \times \{d_{41}\}$$

$$= \left\{ \begin{array}{l} (d_{11},\ d_{21},\ d_{31},\ d_{41}),\ (d_{11},\ d_{21},\ d_{32},\ d_{41}),\ (d_{11},\ d_{22},\ d_{31},\ d_{41}),\ (d_{11},\ d_{22},\ d_{32},\ d_{41}), \\ (d_{12},\ d_{21},\ d_{31},\ d_{41}),\ (d_{12},\ d_{21},\ d_{32},\ d_{41}),\ (d_{12},\ d_{22},\ d_{31},\ d_{41}),\ (d_{12},\ d_{22},\ d_{32},\ d_{41}) \end{array} \right\},$$

$\mathfrak{L}' = \left\{\check{d}_1, \check{d}_2,\ \check{d}_3,\ \check{d}_4,\ \check{d}_5,\ \check{d}_6,\ \check{d}_7,\ \check{d}_8\right\}$ be a set of all sub-attributes with weights $(0.12,\ 0.18,\ 0.1,\ 0.15,\ 0.05,\ 0.22,\ 0.08,\ 0.1)^T$. Let $\{O_1,\ O_2,\ O_3\}$ be a group of experts with weights $(0.143, 0.514, 0.343)^T$. Experts provided their preference for alternatives in PFHSNs form to judge the best alternative.

***PFHSEWG Operator 4.2.2***

Step 1: According to the expert's opinion, Pythagorean fuzzy hypersoft decision matrices for all alternatives are given in Tables 1–4.

**Table 1:** PFHS decision matrix for $\mathfrak{H}^1$

| | $\check{d}_1$ | $\check{d}_2$ | $\check{d}_3$ | $\check{d}_4$ | $\check{d}_5$ | $\check{d}_6$ | $\check{d}_7$ | $\check{d}_8$ |
|---|---|---|---|---|---|---|---|---|
| $O_1$ | (.3, .8) | (.7, .3) | (.6, .7) | (.5, .4) | (.2, .4) | (.4, .6) | (.5, .8) | (.9, .3) |
| $O_2$ | (.7, .6) | (.3, .4) | (.6, .5) | (.3, .9) | (.5, .4) | (.4, .6) | (.7, .5) | (.4, .8) |
| $O_3$ | (.5, .7) | (.8, .5) | (.7, .4) | (.4, .3) | (.4, .9) | (.2, .4) | (.8, .4) | (.7, .5) |

**Table 2:** PFHS decision matrix for $\mathfrak{H}^2$

| | $\check{d}_1$ | $\check{d}_2$ | $\check{d}_3$ | $\check{d}_4$ | $\check{d}_5$ | $\check{d}_6$ | $\check{d}_7$ | $\check{d}_8$ |
|---|---|---|---|---|---|---|---|---|
| $O_1$ | (.6, .7) | (.4, .6) | (.3, .4) | (.9, .2) | (.3, .8) | (.2, .4) | (.7, .5) | (.4, .5) |
| $O_2$ | (.8, .5) | (.7, .4) | (.9, .2) | (.7, .4) | (.4, .5) | (.9, .3) | (.2, .7) | (.3, .8) |
| $O_3$ | (.8, .5) | (.7, .4) | (.8, .5) | (.5, .2) | (.5, .7) | (.7, .5) | (.7, .6) | (.6, .4) |

**Table 3:** PFHS decision matrix for $\mathfrak{H}^3$

| | $\check{d}_1$ | $\check{d}_2$ | $\check{d}_3$ | $\check{d}_4$ | $\check{d}_5$ | $\check{d}_6$ | $\check{d}_7$ | $\check{d}_8$ |
|---|---|---|---|---|---|---|---|---|
| $O_1$ | (.7, .3) | (.2, .5) | (.1, .6) | (.3, .4) | (.4, .6) | (.8, .4) | (.6, .7) | (.2, .5) |
| $O_2$ | (.3, .7) | (.4, .5) | (.4, .8) | (.3, .4) | (.6, .7) | (.3, .4) | (.9, .2) | (.7, .2) |
| $O_3$ | (.6, .8) | (.4, .5) | (.6, .5) | (.6, .4) | (.7, .5) | (.8, .4) | (.5, .8) | (.4, .5) |

**Table 4:** PFHS decision matrix for $\mathfrak{H}^4$

| | $\check{d}_1$ | $\check{d}_2$ | $\check{d}_3$ | $\check{d}_4$ | $\check{d}_5$ | $\check{d}_6$ | $\check{d}_7$ | $\check{d}_8$ |
|---|---|---|---|---|---|---|---|---|
| $O_1$ | (.8, .4) | (.2, .9) | (.2, .4) | (.4, .6) | (.6, .5) | (.5, .6) | (.4, .5) | (.8, .3) |

(Continued)

**Table 4 (continued)**

| | $\check{d}_1$ | $\check{d}_2$ | $\check{d}_3$ | $\check{d}_4$ | $\check{d}_5$ | $\check{d}_6$ | $\check{d}_7$ | $\check{d}_8$ |
|---|---|---|---|---|---|---|---|---|
| $O_2$ | (.5, .4) | (.7, .6) | (.9, .3) | (.8, .5) | (.9, .2) | (.2, .4) | (.4, .6) | (.6, .5) |
| $O_3$ | (.5, .7) | (.9, .3) | (.3, .5) | (.5, .7) | (.3, .5) | (.8, .5) | (.7, .5) | (.2, .5) |

Step 2: All parameters are of the same type. So, no need to normalize.

Step 3: Apply the proposed PFHSEWG operator to the obtained data (Tables 1–4), and obtain the expert's estimations such as follows:

$\mathcal{L}_1 = \langle 0.4551,\ 0.5997\rangle$ , $\mathcal{L}_2 = \langle 0.6186,\ 0.4829\rangle$ , $\mathcal{L}_3 = \langle 0.5186,\ 0.5298\rangle$ , and $\mathcal{L}_4 = \langle 0.5234,\ 0.5241\rangle$ .

Step 4: Use Eq. (1), $\mathbb{S} = {a_{\mathcal{F}(\check{d}_{ij})}}^2 - {b_{\mathcal{F}(\check{d}_{ij})}}^2$ to compute the score values for all alternatives.

$\mathbb{S}(\mathcal{H}_1) = -0.1525424, \mathbb{S}(\mathcal{H}_2) = 0.149473, \mathbb{S}(\mathcal{H}_3) = -0.011742, \mathbb{S}(\mathcal{H}_4) = -0.000733.$

Step 5: Compute the ranking of the alternatives $\mathbb{S}(\mathcal{H}_2) > \mathbb{S}(\mathcal{H}_4) > \mathbb{S}(\mathcal{H}_3) > \mathbb{S}(\mathcal{H}_1)$ . So, $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$.

Since the material estimation surprises at the theoretic phase through the enactment stage of the plan, there is extra scope to area the appropriateness of the particular materials. Face-centered cube materials are used at small temperatures of −163°C. Austenitic steel $\mathfrak{H}^2 =$ SS301-FH grades first. This is reliable by utilizing previous inquiries and real-world exercises. Austenitic steels are still typically used in liquefied nitrogen or hydrogen storing vessels [55].

## 5 Comparative Studies

To demonstrate the efficiency of the anticipated approach, a comparison with some standing methods under the IFS, IFSS, IFHSS, PFS, PFSS, and proposed PFHSS model.

### *5.1 Superiority of the Proposed Method*

The planned methodology is competent and realistic; we have established an innovative MCDM model under the PFHSS setting over the PFHSEWG operator. Our projecting model is more talented than prevalent methods and can produce the most delicate significance in MCDM problems. The collective model is multipurpose and familiar, adapting to budding volatility, engagement, and productivity. Different models have specific ranking procedures, so there is an immediate difference between the rankings of the proposed techniques to be feasible according to their assumptions. This scientific study and evaluation conclude that results obtained from existing methods are unpredictable compared to hybrid structures. Furthermore, many hybrid FS, IFSS, IFHSS, and PFSS become uncommon in PFHSS due to some fortunate circumstances. It is easy to combine incomplete and uncertain facts in DM techniques. They were mixing inaccurate and insecure data in the DM process. Thus, our intended methodology will be more skilled, imperative, superior, and restored than various hybrid-structured FS. Table 5 below presents the feature analysis of the proposed method and some existing models.

**Table 5:** Feature analysis of different models with a proposed model

| | Fuzzy information | Aggregated parameters information | Einstein aggregated parameters information | Multi sub-attributes information of each attribute |
|---|---|---|---|---|
| IFEWG [7] | ✓ | × | ✓ | × |
| IFWG [56] | ✓ | × | × | × |
| IFSWG [30] | ✓ | ✓ | × | × |
| IFHSWG [41] | ✓ | ✓ | × | ✓ |
| PFSWG [35] | ✓ | ✓ | × | × |
| PFEWG [10] | ✓ | ✓ | ✓ | × |
| PFSEOWG [37] | ✓ | ✓ | ✓ | × |
| PFHSWG [43] | ✓ | ✓ | × | ✓ |
| Proposed operator | ✓ | ✓ | ✓ | ✓ |

### *5.2 Comparative Analysis*

To endorse the usefulness of the planned technique, we compare the attained outcomes with some state-of-the-arts in the PFSS setting are concise in Table 6. In this work, an innovative aggregation operator, the PFHSEWG operator, is projected to fuse suggestive information, and then a score function is utilized to assess the organization of alternatives. The PFHSS is the most generalized form of PFSS because it deals with the multi-sub attributes of the considered parameters. Wang et al. [7] presented some geometric AOs under the IFS setting, ut these AOs cannot deal with the parametrized and sub-parametrized values of the alternatives. Arora et al. [30] prolonged the Pythagorean fuzzy soft weighted geometric operation, which competently accommodated the alternatives' parametrized values. But, it also fails to deal with the sub-parametrized values of the alternatives. Wei et al. [12] developed PFWG unable to handle the parametrized values of the alternatives. Rahman et al. [10] competently deal with the Einstein aggregation value of the alternative but cannot take the parametrization values of the alternatives. Zulqarnain et al. [35] proposed that aggregation operators based on algebraic norms cannot cope with the multi sub-attributes of the considered parameters. On the other hand, our developed model effectively deals with the alternatives' multi- sub-attributes. Zulqarnain et al. [37,57] protracted Einstein weighted and Einstein ordered weighted geometric AOs under PFSS environment are unable to deal with the multi sub-attributes of the alternatives. Zulqarnain et al. [41] introduced the intuitionistic fuzzy hypersoft weighted geometric operator, which handles the sub-parametrized values of the alternatives. Siddique et al. [43] developed the DM technique for PFHSNs using their established laws that cannot accommodate the Einstein aggregated values of the alternatives. Meanwhile, our established approach competently deals with parametrized values of the alternatives and delivers better information than existing techniques. This work recommends innovative Einstein AO, such as PFHSEWG, to integrate the evaluation materials and then use the score function to calculate the substitute score. Therefore, it is inevitable that, based on the above facts, the plan operator in this work is more influential, consistent, and effective.

It is also an appropriate tool for dealing with contemptible inaccuracies and misrepresentation in DM plans. The advantage of expecting skill and associated dealings compared to existing methods is to avoid inspirations based on abominations. Hence, it is a proper tool for integrating erroneous and vague data in DM.

**Table 6:** Comparison of a proposed operator with some existing operators

| Authors | AO | Alternatives ranking | Optimal material |
|---|---|---|---|
| Wang et al. [7] | IFWG | $\mathfrak{H}^2 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^3$ | $\mathfrak{H}^2$ |
| Wang et al. [7] | IFOWG | $\mathfrak{H}^2 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^3$ | $\mathfrak{H}^2$ |
| Arora et al. [30] | IFSWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |
| Wang et al. [7] | IFEWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^3$ | $\mathfrak{H}^2$ |
| Wei et al. [12] | PFWG | $\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4$ | $\mathfrak{H}^2$ |
| Rahman et al. [10] | PFEWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |
| Zulqarnain et al. [35] | PFSWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |
| Zulqarnain et al. [57] | PFSEWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |
| Zulqarnain et al. [37] | PFSEOWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |
| Zulqarnain et al. [41] | IFHSWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^3$ | $\mathfrak{H}^2$ |
| Siddique et al. [43] | PFHSWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^3$ | $\mathfrak{H}^2$ |
| Proposed | PFHSEWG | $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$ | $\mathfrak{H}^2$ |

## 6 Conclusion

In engineering, the subtle stability of designing is impartial; authentic materials and manufacture comprise wide-ranging matters. Mathematical modeling in manufacturing enterprise establishments utilizes all capitals while combining design objectives under financial, superior, and security constraints. Questions must be defined for the most acceptable decision, conferring to judgment necessities. In actual DM, the assessment of alternative facts delivered by the expert is habitually imprecise, rough, and unpredictable; thus, PFHSNs can be used to conduct this indeterminate info. The core goal of this research is to use Einstein's norms to develop some operational laws for PFHSS. Then, a new operator, such as PFHSEWG, developed according to the designed operational laws. Some fundamental properties have been presented using our developed PFHSEWG operator. Also, a DM approach is established to address MCDM problems based on the endorsed operator. To certify the robustness of the settled approach, we provide an inclusive mathematical illustration for MS in the manufacturing industry. A comparative analysis has been presented to ensure the practicality of the planned model. Lastly, based on the outcomes attained, it is determined that the technique projected in this study is the most practical and effective way to solve the problem of MCDM. In the future, several other hybrid AOs for PFHSS will be introduced with their decision-making techniques. Furthermore, the developed AOs can be extended to T-spherical fuzzy hypersoft, and q-rung orthopair fuzzy hypersoft settings with decision-making approaches.

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