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Ranked-Set Sampling Based Distribution Free Control Chart with Application in CSTR Process

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ABSTRACT

Nonparametric (distribution-free) control charts have been introduced in recent years when quality characteristics do not follow a specific distribution. When the sample selection is prohibitively expensive, we prefer ranked-set sampling over simple random sampling because ranked set sampling-based control charts outperform simple random sampling-based control charts. In this study, we proposed a nonparametric homogeneously weighted moving average based on the Wilcoxon signed-rank test with ranked set sampling ($NPHWMA_{RSS}$) control chart for detecting shifts in the process location of a continuous and symmetric distribution. Monte Carlo simulations are used to obtain the run length characteristics to evaluate the performance of the proposed $NPHWMA_{RSS}$ control chart. The proposed $NPHWMA_{RSS}$ control chart's performance is compared to that of parametric and nonparametric control charts. These control charts include the exponentially weighted moving average (EWMA) control chart, Wilcoxon signed-rank with simple random sampling based the nonparametric EWMA control chart, the nonparametric EWMA sign control chart, Wilcoxon signed-rank with ranked set sampling-based the nonparametric EWMA control chart, and the homogeneously weighted moving average control charts. The findings show that the proposed $NPHWMA_{RSS}$ control chart performs better than its competitors, particularly for the small shifts. Finally, an example is presented to demonstrate how the proposed scheme can be implemented practically.

KEYWORDS

Nonparametric control charts; run-length; ranked set sampling; Wilcoxon signed-rank

1 Introduction

The statistical process control (SPC) toolkit collects various statistical tools used to monitor the production and service processes. Control charts are one of the most important tools in the SPC toolkit for detecting changes in process parameters (location and/or dispersion). There are numerous control charts for efficient process monitoring in the SPC literature. Shewhart [1] introduced the basic control charts, called the Shewhart control charts. These control charts are also known as memoryless control charts because they only use current process information. Although the Shewhart control charts are simple to design and implement, they are less sensitive for small to moderate shifts detection. The other control charts that are more sensitive than the Shewhart control charts are the Cumulative sum (CUSUM) control chart, introduced by Page [2], and the exponentially weighted moving average (EWMA) control chart suggested by Roberts [3]. The CUSUM and EWMA control charts are also called the memory control chart, as these control charts use both current and previous information about the process.

Generally, it is assumed that the process often follows a specific distribution. When this assumption is violated, the quality of these objects may be affected. Nonparametric (NP) control charts are used in this scenario because they do not require the assumption of specified distribution. In addition, the population variance is unnecessary when employing NP control charts to observe the shifts in the process location. The sign (SN) and Wilcoxon signed-rank (SR) are commonly used NP techniques in the control charts context. For instance, Yang et al. [4] studied the NP EWMA sign (NPEWMA-SN) control chart for monitoring shifts in process location. Similarly, Graham et al. [5] and Graham et al. [6] investigated single observation-based NP EWMA sign and NP EWMA signed-rank control charts to monitor process location, respectively. Likewise, Chakraborty et al. [7] introduced the generally weighted moving average SN (GWMA-SN) control chart to detect the process location shifts. Later, Abid et al. [8] offered the ranked set sampling (RSS) based NP EWMA Wilcoxon signed-rank (NPREWMA-SR) control chart for monitoring shifts in process location efficiently. Moreover, Ali et al. [9] recommended NP control charts to detect location parameter changes. Also, Ali et al. [9] designed the RSS-based NP EWMA SN (NPREWMA-SN) control chart for monitoring shifts in process location.

The CUSUM and EWMA types schemes and modified versions are used for efficient process monitoring. For instance, Lucas et al. [10] offered the combined Shewhart-EWMA (CS-EWMA) control chart to enhance the shift detection ability in the process location. Similarly, Shamma et al. [11] proposed a double EWMA (DEWMA) chart, outperforming the EWMA control chart in early process location shifts detection. Likewise, Zhang et al. [12] introduced the DEWMA chart's run length (RL) features. They suggested that DEWMA and EWMA charts are equally beneficial for detecting large shifts in process parameters. Later, Abbas et al. [13] and Zaman et al. [14] designed the mixed EWMA-CUSUM (MEC) and mixed CUSUM-EWMA (MCE) charts, respectively, to efficiently monitor process location shifts. Also, Anwar et al. [15] suggested auxiliary information-based (AIB) MEC and MCE control charts for monitoring the process location parameter. For other CUSUM and EWMA based studies, the readers are referred to the work of Adegoke et al. [16], Sanusi et al. [17], Haq [18], Zaman et al. [19], Aslam et al. [20], Adeoti et al. [21], Anwar et al. [15], Abbasi et al. [22], Rasheed et al. [23], Rasheed et al. [24], Anwar et al. [25], Liu et al. [26], and Rasheed et al. [27].

Hunter [28] noticed that the EWMA control chart assigns a higher weight to current observations and less weight to prior observations of the process. Later, Abbas [29] proposed the homogeneously weighted moving average (HWMA) control chart for observing shifts in process location. After that, Abid et al. [30] introduced the mixed HWMA-CUSUM (MHC) control chart for monitoring shifts

in the process location. Similarly, Adeoti et al. [31] advocated a hybrid HWMA (HHWMA) control chart to detect the process location shifts. In addition, Abid et al. [32] offered the double HWMA (DHWMA) control chart to observe process location shifts. They showed that the DHWMA control chart has better detection ability than the HWMA control chart. Recently, Anwar et al. [33] suggested an AIB DHWMA control chart for improved process location monitoring.

Simple random sampling (SRS) and perfect RSS are frequently used in SPC to monitor underlying process data. These sampling techniques are commonly used with parametric and nonparametric control charts. However, when actual measurements are difficult to obtain, or the cost of sampling is prohibitively expensive, the RSS should be used instead of SRS. Ranking errors can impact estimator performance and result in ambiguous estimates. Dell et al. [34] studied the impacts of inaccuracy and imperfect RSS on the performance of mean estimators. They claim that the RSS mean estimator remains unbiased when imperfect ranking is used and outperforms the SRS estimator. Besides that, the RSS's efficiency remains higher than that of the imperfect RSS and SRS. Numerous NP HWMA type control charts based on simple random sampling (SRS) are available. Still, according to the author's knowledge, no NP HWMA control chart based on perfect and imperfect RSS is present in the literature. Capitalizing this research gap, this article introduces an NP HWMA SR control chart with perfect and imperfect RSS (NPHWMA$_\text{RSS}$) for monitoring process location shifts for continuous and symmetric distributions. The performance measures, such as average run length (ARL), median run length (MDRL), and standard deviation of run-length (SDRL), are computed under different distributions, including the normal, student's t, contaminated normal (CN), Laplace, and logistic distributions. Based on these performance measures, the proposed NPHWMA$_\text{RSS}$ control chart is compared to the competing control charts, such as EWMA, NPEWMA-SR, NPEWMA-SN, NPREWMA-SR, and HWMA control charts. The comparison indicates the superiority of proposed NPHWMA$_\text{RSS}$ control chart over the competing control charts in small to moderate shift detection.

The remainder of the paper is structured as follows: Section 2 offers the proposed control chart's methodology and design and the competing control chart. Section 3 evaluates the IC and OOC performance of the proposed NPHWMA$_\text{RSS}$ control chart. Section 4 covers a comparative study of the proposed NPHWMA$_\text{RSS}$ chart, whereas Section 5 addresses a real-life application. Lastly, Section 6 provides the concluding remarks of the study.

2 Competing and Proposed Control Charts

This section explains the design structure of the proposed control chart and the competing control charts. The competing control charts include EWMA, NPEWMA-SR, NPEWMA-SN, NPREWMA-SR, and HWMA control charts. More details are included in the following subsections, given as follows.

2.1 Classical EWMA Control Chart

Suppose X is the process characteristic that follows a normal distribution, i.e., $X \sim N(\mu_0 + \delta\sigma_0, \sigma_0)$. Let $\overline{X}_t = \sum_{i=1}^{n} X_{it}/n$ be the t-th sample mean and $S_t = \sqrt{\sum_{i=1}^{n} \left(X_{it} - \overline{X}_t\right)^2 / (n-1)}$ be the t-th sample standard deviation, so for the in-control (IC) situation, $\overline{X}_t \sim N\left(\mu_0, \sigma_0^2/n\right)$ for $t = 1, 2, \ldots$. Roberts [3] introduced the classical EWMA control chart, which effectively identifies small to moderate shifts in process location. The classical EWMA control chart is defined by the plotting statistic given as follows:

$$E_t = \eta \overline{X}_t + (1 - \eta) E_{t-1}, \tag{1}$$

where η is a smoothing constant such that $\eta \in (0, 1]$ and the initial value $E_0 = \mu_0$. The control limits are based on the statistic E_t can be defined as follows:

$$
\left.\begin{aligned}
LCL_{(EWMA)t} &= \mu_0 - L\sigma_0 \sqrt{\frac{\eta}{n(2-\eta)} \left\{1 - (1-\eta)^{2t}\right\}} \\
CL_{(EWMA)t} &= 0 \\
UCL_{(EWMA)t} &= \mu_0 + L\sigma_0 \sqrt{\frac{\eta}{n(2-\eta)} \left\{1 - (1-\eta)^{2t}\right\}}
\end{aligned}\right\},
\tag{2}
$$

The L is the control chart coefficient, and its value is determined so that IC ARL is equal to desired pre-assumed value. The process is considered to IC if $LCL_{(EWMA)t} < E_t < UCL_{(EWMA)t}$; otherwise, the process is declared out of control (OOC).

2.2 NPEWMA-SR Control Chart

Graham et al. [6] suggested the SRS-based NP EWMA SR (NPEWMA-SR) control chart for monitoring shifts in the process location. If R_{tq}^+ is the rank of the absolute deviation of observations from true median M_0, i.e., $\left|X_{tq} - M_0\right|$ for $t = 1, 2, 3, \ldots$ and $q = 1, 2, 3, \ldots, n$, then the Wilcoxon signed rank statistic is denoted as $SR_{(SRS)t}$ and can be defined as follows:

$$
SR_{(SRS)t} = \sum_{q=1}^{n} sig\left(X_{tq} - M_0\right) R_{tq}^+, \text{ where } sig\left(X_{tq} - M_0\right) = \begin{cases} -1 & \text{if } \left(X_{tq} - M_0\right) < 0 \\ 0 & \text{if } \left(X_{tq} - M_0\right) = 0 \\ 1 & \text{if } \left(X_{tq} - M_0\right) > 0 \end{cases}
$$

Bakir [35] and Abbas et al. [36] suggested that $SR_{(SRS)t}$ is linearly associated to signed rank statistic T_n^+ by the relation, that is, $SR_{(SRS)t} = 2T_n^+ - \dfrac{n(n+1)}{2}$. As a result, the mean and variance of $SR_{(SRS)t}$ are, respectively, given as $E\left(SR_{(SRS)t}\right) = 0$ and $var\left(SR_{(SRS)t}\right) = \left(\dfrac{n(n+1)(2n+1)}{6}\right)$. The plotting statistic of the NPEWMA-SR control chart is defined as follows:

$$
E_{SR(SRS)t} = \eta SR_{(SRS)t} + (1-\eta) E_{SR(SRS)t-1},
\tag{3}
$$

where $E_{SR(SRS)0} = \mu_0$. The control limits are based on $E(SR_{(SRS)t})$ and $var(SR_{(SRS)t})$ are as follows:

$$
\left.\begin{aligned}
LCL_{(NPEWMA-SR)t} &= \mu_0 - L\sqrt{\frac{\eta}{2-\eta}\left(\frac{n(n+1)(2n+1)}{6}\right)} \\
CL_{(NPEWMA-SR)t} &= 0 \\
UCL_{(NPEWMA-SR)t} &= \mu_0 + L\sqrt{\frac{\eta}{2-\eta}\left(\frac{n(n+1)(2n+1)}{6}\right)}
\end{aligned}\right\},
\tag{4}
$$

If the plotting statistic $E_{SR(SRS)t} > UCL_{(NPEWMA-SR)t}$ or $E_{SR(SRS)t} < LCL_{(NPEWMA-SR)t}$, the underlying process is considered OOC; conversely, the process is IC.

2.3 NPEWMA-SN Control Chart

Suppose $X_{t1}, X_{t2}, \ldots, X_{tm}$ denotes the tth $(t = 1, 2, 3, \ldots)$ rational sub-groups of independent observations from an undefined continuous distribution of the process characteristic. Let M_0 denotes the true IC median of the process, and $X_{tq} - M_0$ represent all possible differences for $t = 1, 2, 3, \ldots$ and $q = 1, 2, 3, \ldots, n$. In this case, the sign statistic, i.e., $SN_t = sign\left(X_{tq} - M_0\right)$, indicate values of -1, 0, and 1, where the value -1 implies that the difference is smaller than M_0, 0 (zero) if the difference

is equal to M_0 and 1 if the difference is greater than M_0. The mean and variance of the statistic SN_t are $E(SN_t) = n(2p-1)$ and $var(SN_t) = 4np(1-p)$, respectively. Graham et al. [6] introduced the NPEWMA-SN control chart by defining the plotting statistic $E_{(SN)t}$, based on the statistic SN_t given as follows:

$$E_{(SN)t} = \eta SN_t + (1-\eta) E_{(SN)t-1}, \tag{5}$$

where $E_{(SN)0} = \mu_0$. The control limits are based on $E(SN_t)$ and $var(SN_t)$ can be defined as follows:

$$\left. \begin{array}{l} LCL_{(NPEWMA-SN)} = n(2p-1) - L\sqrt{\dfrac{\eta}{2-\eta}(4np(1-p))} \\[3mm] CL_{(NPEWMA-SN)} = n(2p-1) \\[3mm] UCL_{(NPEWMA-SN)} = n(2p-1) - L\sqrt{\dfrac{\eta}{2-\eta}(4np(1-p))} \end{array} \right\}, \tag{6}$$

The underlying process is said to be IC if $LCL_{(NPEWMA-SN)} < E_{(SN)t} < UCL_{(NPEWMA-SN)}$; otherwise, the process under the NPEWMA-SN control chart is considered OOC.

2.4 NPREWMA-SR Control Chart

Abid et al. [8] recommended the RSS-based NP EWMA SR (NPREWMA-SR) control chart for monitoring the process location shifts. When obtaining measurements for a variable of interest is both expensive and complicated, RSS is desirable to SRS; however, ranking these measurements can be accomplished using a extensively available and cost-effective ranking technique.

When measured values are deleterious or costly, but ranking findings is comparatively simple, the RSS is effective [37]. The proposed structure based on RSS is a more efficient alternative to the usual nonparametric control chart relying on Wilcoxon signed rank statistic. RSS involves the selection of a simple random sample of size n from a particular population, followed by the ranking of the findings by experts. The RSS is mostly defined in terms of perfect and imperfect ranking. The perfect and imperfect ranking structures are similar in general. A correlation of one between the two variables is required to achieve perfect ranking in any bivariate normal model where the variable of interest (X) and the ranking variable (Y) are used; otherwise, the ranking will be imperfect. The RSS sampling technique, stating that RSS with sample size n is obtained from a quality variable $X_{tj(h)}$. The suffixes t ($t = 1, 2, 3, \ldots$) used for sample number, j ($j = 1, 2, \ldots, n$) for observations, and h ($h = 1, 2, \ldots, m$) represents the cycle number in RSS approach, respectively. The unit with the lowest rank is selected for measurement. The remaining $n-1$ findings are handed back to a population, and a new sample of n is drawn. The experts rank the findings once more, and the second lowest is chosen, evaluated, and the $n-1$ remaining findings are brought back. We continue the process m times, yielding a ranked set sample with size $r = nm$. In this study, we assume $m = 1$ for a valid assessment of the proposed control chart to the competitors, so $r = nm$, which is $r = n$.

Then, using RSS data from the hth cycle, the unbiased estimator of the population mean, as defined by Takahasi et al. [38], is described as: $\overline{X}_{rss,h} = \dfrac{1}{n}\sum_{t=1}^{n} X_{tj(h)},\ h = 1, 2, \ldots, m$. Then variance of $\overline{X}_{rss,h}$ is given by

$$var\left(\overline{X}_{rss,h}\right) = \frac{1}{n}\sum_{t=1}^{n}\sigma_{tj}^2, \quad \text{where } \sigma_{tj}^2 = E\left[X_{tj} - E\left(X_{tj}\right)\right]^2$$

Assume that the variable of interest X is difficult to measure and order, but that there is a linked variable Y that is correlated with X (Dell et al. [34]). A variable Y can be utilised to obtain the rank

of X in the following way: n^2 bivariate units are chosen from the population and structured into n sets of size n each. The X associated with the smallest Y is determined from the first set of size n. The X associated with the second smallest Y is measured from the second set of size n. This process is repeated until the X associated with the largest Y from the nth set is determined. The cycle is repeated r times until nr Xs are evaluated. It should be noted that the variable X will be ranked with errors, i.e., X_{tj}, is the *tth* judgement order statistic from the *tth* set of size n in the *hth* cycle of size r. This is referred to as imperfect ranked set sampling (IRSS). Suppose that (X, Y) has a bivariate normal distribution and that the regression of X on Y is linear. Then, in the manner of Stokes [39], we can write

$$X = \mu_X + \gamma \frac{\sigma_X}{\sigma_Y}(Y - \mu_Y) + \varepsilon,$$

where Y and ε are independent, and ε has a mean of zero and a variance $\sigma_X^2(1 - \gamma^2)$, where γ is the correlation between X and Y and μ_X, μ_Y, σ_X, σ_Y are the means and standard deviations of the variables X and Y. Let $Y_{tj(h)}$ and $X_{tj(h)}$ represent the *tth* smallest Y value and the corresponding X value obtained from the *tth* set in the *hth* cycle, respectively. The above equation can be written as

$$X_{tj(h)} = \mu_X + \gamma \frac{\sigma_X}{\sigma_Y}(Y_{tj(h)} - \mu_Y) + \varepsilon_{th}$$

For the *hth* cycle, the unbiased estimator of the mean of the variable of interest X with ranking based on the concomitant variable Y, i.e., using the IRSS method, can be written as

$$\overline{X}_{irss,h} = \frac{1}{n}\sum_{t=1}^{n} X_{tj(h)}, \ h = 1, 2, \ldots, m$$

Then variance of $\overline{X}_{irss,h}$ (see Stokes [39]) is given by

$$var\left(\overline{X}_{irss,h}\right) = \frac{\sigma_X^2}{n}\left[(1 - \gamma^2) + \frac{\gamma^2}{n}\sum_{t=1}^{n}\sigma_{Y(tj)}^2\right],$$

where $\sigma_{Y(tj)}^2$ represents the variance of the tth order statistic from a sample of size n from the standard normal distribution.

Different authors, like, Kim et al. [40], Abid et al. [8], and Abbas et al. [36] used the RSS-based Wilcoxon signed-rank statistic to monitor shifts in the process location, defined as follows:

$$SR_{RSS_t} = \sum_{j=1}^{n}\sum_{h=1}^{m} sign\left(X_{tj(h)} - \theta_0\right) R_{tj(h)}^+ \tag{7}$$

where θ_0 is known as the IC median of the process. The mean and variance of SR_{RSS_t} statistic are $E(SR_{RSS_t}) = 0$ and $Var(SR_{RSS_t}) = \left(\frac{r(r+1)(2r+1)}{6}\right)\varpi_0^2$, respectively. The quantity ϖ_0^2 is used to improve the efficiency of the control chart and can be defined as $\varpi_0^2 = 1 - \frac{4}{n}\sum_{j=1}^{n}\left(F_k(0) - \frac{1}{2}\right)^2$. The values of $F_k(0)$ can be obtained by solving the following mathematical expression:

$$F_k(0) = \frac{r!}{(j-1)!(r-j)!}\int_{-\infty}^{0} F(t)^{j-1}(1 - F(t))^{r-j} f(t) dt$$

One may consult [22] for further detail on the RSS approach.

The plotting statistic of the NPREWMA-SR control chart is defined as:

$$E_{(SR_{RSS})_t} = \eta SR_{(RSS)_t} + (1 - \eta)E_{(SR_{RSS})_{t-1}}, \tag{8}$$

The initial value of the plotting statistic $\left(E_{(SR_{RSS})_0}\right)$ is equal to zero. The mean and variance of this statistic are, respectively, defined as $E\left(E_{(SR_{RSS})_t}\right) = 0$ and $Var\left(E_{(SR_{RSS})_t}\right) = \frac{\eta}{2-\eta}\left(1-(1-\eta)^{2t}\right)\left(\frac{r(r+1)(2r+1)}{6}\right)$

ϖ_0^2. For a large value of t, the variance of $E_{(SR_{RSS})_t}$ reduces to $\frac{\eta}{2-\eta}\left(\frac{r(r+1)(2r+1)}{6}\right)\varpi_0^2$. The control limits for the NPREWMA-SR chart are given as follows:

$$\left.\begin{aligned} LCL_{\text{NPREWMA-SR}} &= -L\sqrt{\frac{\eta}{2-\eta}\left(\frac{r(r+1)(2r+1)}{6}\right)\varpi_0^2} \\ CL_{\text{NPREWMA-SR}} &= 0 \\ UCL_{\text{NPREWMA-SR}} &= +L\sqrt{\frac{\eta}{2-\eta}\left(\frac{r(r+1)(2r+1)}{6}\right)\varpi_0^2} \end{aligned}\right\} \tag{9}$$

The process is considered to be IC when $E_{(SR_{RSS})_t} < UCL_{\text{NPREWMA-SR}}$ or $E_{(SR_{RSS})_t} > LCL_{\text{NPREWMA-SR}}$; otherwise, it goes OOC.

2.5 HWMA Control Chart
Abbas [29] designed the HWMA control chart to monitor the process location, which is defined by the plotting statistic, given as follows:

$$H_t = \eta \overline{X}_t + (1-\eta)\overline{\overline{X}}_{t-1}, \tag{10}$$

where $\overline{\overline{X}}_{t-1}$ is the mean of sample average of $t-1$ samples. The mean of the HWMA statistic is $E(H_t) = \mu_0$, while its variance is defined by

$$var(H_t) = \left.\begin{aligned} &\frac{\eta^2\sigma_0^2}{n}, & if\ t=1 \\ &\frac{\eta^2\sigma_0^2}{n} + (1-\eta)^2\frac{\sigma_0^2}{n(t-1)}, & if\ t>1 \end{aligned}\right\},$$

The control limits of the HWMA control chart are defined as

$$LCL_{(HWMA)_t} = \left.\begin{aligned} &\mu_0 - L\sqrt{\frac{\eta^2\sigma_0^2}{n}}, & if\ t=1 \\ &\mu_0 - L\sqrt{\frac{\sigma_0^2}{n}\left\{\eta^2 + \frac{(1-\eta)^2}{(t-1)}\right\}}, & if\ t>1 \end{aligned}\right\}$$

$$LCL_{(HWMA)_t} = \left.\begin{aligned} &\mu_0 + L\sqrt{\frac{\eta^2\sigma_0^2}{n}}, & if\ t=1 \\ &\mu_0 + L\sqrt{\frac{\sigma_0^2}{n}\left\{\eta^2 + \frac{(1-\eta)^2}{(t-1)}\right\}}, & if\ t>1 \end{aligned}\right\} \tag{11}$$

The process is said to IC when $LCL_{(HWMA)_t} < H_t < UCL_{(HWMA)_t}$; elsewhere, it is considered OOC.

2.6 Proposed NPHWMA$_{RSS}$ Control Chart

This subsection explains the methodology and formulation procedure of the proposed RSS-based NP HWMA control chart for process location shifts. The proposed control chart is labeled as the NPHWMA$_{RSS}$ control chart. The plotting statistic of the proposed NPHWMA$_{RSS}$ control chart is denoted by NPH_t and defined as follows:

$$NPH_t = \eta SR_{(RSS)t} + (1 - \eta) \overline{SR}_{(RSS)t-1}, \tag{12}$$

The starting value of the statistic NPH_t is equal to the IC mean of the process, i.e., $NPH_0 = \mu_0$. The μ_0 should be specified value. The mean of the statistic NPH_t is $E(NPH_t) = \mu_0$, whereas its variance is given as

$$var(NPH_t) = \begin{cases} \eta^2 \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2, & \text{if } t = 1 \\ \left(\eta^2 + \dfrac{(1-\eta)^2}{t-1} \right) \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2, & \text{if } t > 1 \end{cases},$$

As a result, the control limits of the NPHWMA$_{RSS}$ chart are given as follows:

$$LCL_{(\text{NPHWMA}_{RSS})t} = \begin{cases} \mu_0 - L \sqrt{\eta^2 \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2}, & \text{if } t = 1 \\ \mu_0 - L \sqrt{\left(\eta^2 + \dfrac{(1-\eta)^2}{t-1} \right) \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2}, & \text{if } t > 1 \end{cases}$$

$$CL_{(\text{NPHWMA}_{RSS})t} = \mu_0 \tag{13}$$

$$UCL_{(\text{NPHWMA}_{RSS})t} = \begin{cases} \mu_0 + L \sqrt{\eta^2 \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2}, & \text{if } t = 1 \\ \mu_0 + L \sqrt{\left(\eta^2 + \dfrac{(1-\eta)^2}{t-1} \right) \left(\dfrac{r(r+1)(2r+1)}{6} \right) \varpi_0^2}, & \text{if } t > 1 \end{cases}$$

To detect the shifts in the process location, the proposed NPHWMA$_{RSS}$ control chart is constructed by plotting the statistic NPH_t against the control limits defined by Eq. (13). If $NPH_t > UCL_{(\text{NPHWMA}_{RSS})t}$ or $NPH_t < LCL_{(\text{NPHWMA}_{RSS})t}$, the process is considered OOC; otherwise, it is deemed IC.

3 Performance Evaluation

The IC and OOC performances of the proposed NPHWMA$_{RSS}$ control chart for monitoring shifts in process location are presented in this section. Generally, we study the RL distribution and its associated features to assess the behavior of the control chart. The average run length (ARL) is the basic metric to evaluate control chart performance. It can be defined as the average number of sample points plotted on the control chart before the control chart shows an OOC signal [41]. Mathematically it is defined as $ARL = \sum_{i=1}^{N} RL_i / N$, where N is the size of simulation runs (see Almanjahie et al. [42]). For the IC process, the control chart should have large ARL$_0$ to avoid the false alarm, while for the OOC process the ARL$_1$ should be small for early shifts detection. For better performance of the control chart, it is necessary that the ARL$_1$ should be smaller than the other charts at a fixed value of ARL$_0$. In this study, the ARL$_0$ is set on 500, with sample sizes n of 5 and 10. To investigate the performance behavior of the proposed NPHWMA$_{RSS}$ chart, various values of $\eta \in (0.05, 0.10, 0.25,$

0.50) and $\delta \in$ (0, 0.025, 0.05, 0.075, 0.10, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 2.50, 3.00, 5.00) are used. Sometimes, the researcher may want to examine the chart's performance across the entire range of shifts, i.e., $\delta_{min} < \delta < \delta_{max}$. In this case, a variety of performance measures, such as extra quadratic loss (EQL), performance comparison index (PCI), and relative average run length (RARL) can be used. Mathematically, these terms are defined as follows:

(i) $EQL = (\delta_{max} - \delta_{min})^{-1} \int_{\delta_{min}}^{\delta_{max}} \sigma^2 ARL(\delta) \, d\delta$, where $ARL(\delta)$ is the ARL at a specific shift (δ) and δ_{min} and δ_{max} represents the minimum and maximum shift values, respectively.

(ii) $PCI = EQL/EQL_{benchmark}$. The PCI value for the benchmark chart is 1, and the PCI value for all other charts is greater than 1.

(iii) $RARL = (\delta_{max} - \delta_{min})^{-1} \int_{\delta_{min}}^{\delta_{max}} \dfrac{ARL(\delta)}{ARL_{benchmark}(\delta)} d\delta$, where δ_{max} and δ_{min} are the maximum and minimum shift values, respectively. Also, $ARL(\delta)$ and $ARL_{benchmark}(\delta)$ are the ARL values of a given control chart and the benchmark control chart at shift (δ), respectively. A benchmark chart has a lower ARL at a particular point. The RARL value for the benchmark chart is always one. If RARL is greater than one, the benchmark chart is assumed to be superior to the competing chart. It is important to note that the EQL and RARL measures are calculated by numerical integration (trapezoidal) for our study.

Subsection 3.1 offers the IC performance of the proposed NPHWMA$_{RSS}$ control chart, while the OOC performance of the proposed control chart is provided in Subsection 3.2.

3.1 IC Performance

This subsection addresses the design and implementation of the proposed NPHWMA$_{RSS}$ control chart is given in Subsection 3.1.1. Similarly, Subsection 3.1.2 covers the IC robustness of the proposed NPHWMA$_{RSS}$ control chart.

3.1.1 Design and Implementation

The design of the proposed NPHWMA$_{RSS}$ control chart is based on the sample size n, smoothing parameter, η, and control chart width L. The combinations of n and η are used to determine the L value so that the ARL$_0$ is approximately equal to its desired value. Further, the extensive Monte Carlo simulations are used to implement the proposed NPHWMA$_{RSS}$ control chart. An algorithm is developed in statistical software R to approximate the run length distribution. The RL characteristics are computed with the replicate of size 50,000. The simulation algorithm consists of the steps given as follows:

(i)   Generate a random sample from the underlying distribution.

(ii) Specify the design parameters, i.e., η and L.

(iii) Draw a sample from any distribution used in this study.

(iv) Compute the plotting statistic NPH_t in Eq. (12).

(v)   Find $LCL_{(NPHWMA_{RSS})_t}$ and $UCL_{(NPHWMA_{RSS})_t}$ from Eq. (13).

(vi) Plot NPH_t against $LCL_{(NPHWMA_{RSS})_t}$ and $UCL_{(NPHWMA_{RSS})_t}$ along t.

(vii) If $NPH_t > UCL_{(NPHWMA_{RSS})_t}$ or $NPH_t < LCL_{(NPHWMA_{RSS})_t}$, then stop the simulation and calculate the sequence order RL. For instance, at $t = 235$, if $NPH_t > UCL_{(NPHWMA_{RSS})_t}$ or $NPH_t < LCL_{(NPHWMA_{RSS})_t}$, record 235 as a first RL.

(viii) Repeat Steps (ii) to (vii) 50,000 times and calculate the ARL. Check whether it is desired ARL_0; otherwise, adjust constants accordingly in step (ii) and repeat from (ii) to (viii) steps until the desired ARL_0 obtained.

(ix) To compute ARL_1 values, draw a shifted sample from the underlying distribution, and repeat Steps (ii) to (ix).

3.1.2 IC Robustness

This Subsection highlights the proposed $NPHWMA_{RSS}$ chart's IC robustness when shifts are reflected in the process location. The various distributions considered for this study, for instance, standard normal distribution, i.e., $N(0, 1)$, double exponential or Laplace distribution, i.e., $DE\left(0, \frac{1}{\sqrt{2}}\right)$, heavy tail student's t distribution, i.e., $t(\upsilon)$, logistic distribution i.e., $\left(0, \frac{\sqrt{3}}{\pi}\right)$, contaminated normal (CN) distribution, which is the mixture of $N(0, \sigma_0^2)$ and $N(0, \sigma_1^2)$ (see Table 1). USING NP CONTROL CHARTS, the IC RL distribution and related properties are identical for all continuous and symmetric distributions. Table 2 demonstrates the IC RL characteristics of the proposed $NPHWMA_{RSS}$ control chart for monitoring location shifts. These characteristics are computed under normal and non-normal continuous symmetric distributions. The distributions mentioned above are reparametrized to have zero mean/median and unit variance for performance comparison. The analysis indicates that for a specified value of η, the IC RL remains the same for all considered distributions.

Table 1: PDFs of the continuous distributions used for this study

	Distributions	PDF
(i)	Standard Normal	$f(X) = \left(\dfrac{e^{(-X^2/2)}}{\sqrt{2\pi}}\right)$, where $M_0 = 0$ and $\sigma^2 = 1$
(ii)	Student's t_ν	$f(X) = \left(\Gamma\left((\nu+1)/2\right) / \left(\Gamma(\nu/2)\sqrt{\nu\pi}\right)\right)$ $\left(1 + \left(X^2/\nu\right)\right)^{((\nu+1)/2)}$, where $M_0 = 0$ and $\sigma^2 = (\nu/(\nu-2))$ and $\nu = 4,\ 8$ are taken
(iii)	Logistic	$f(X) = e^{(-\pi X)/\sqrt{3}} / \left(\left(\sqrt{3}/\pi\right)\left(1 + e^{(-\pi X)/\sqrt{3}}\right)^2\right)$, where $M_0 = 0$ and $\sigma^2 = \dfrac{3}{\pi^2}$
(iv)	Laplace	$f(X) = (1/2)\, e^{-|X|}$, where $M_0 = 0$ and $\sigma^2 = \dfrac{1}{2}$
(v)	Contaminated Normal (CN)	$f(X) = \left(0.95 e^{(-X^2/2)}/\sqrt{2\pi}\right) + \left(0.05 e^{-\left(X^2/2\sigma_0^2\right)}/\sigma_0\sqrt{2\pi}\right)$, where $M_0 = 0$ and $\sigma^2 = 0.95 + 0.05\sigma_0^2$

Note: where $X \in \mathscr{R}$.

Table 2: RL characteristics of the proposed NPHWMA$_{RSS}$ control chart under different distributions with nominal ARL$_0$ = 500 and n = 10

(η, L)	Distr.	Metrics	0	0.025	0.05	0.075	0.1	0.25	0.5	0.75	1	1.5	2	2.5	3	5
(0.05, 2.011)	Normal	ARL	502.18	126.47	47.44	24.83	15.55	3.86	1.56	1.04	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	450.00	108.00	41.00	22.00	14.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	369.20	93.57	32.86	16.66	10.06	1.91	0.87	0.24	0.03	0.00	0.00	0.00	0.00	0.00
	CN	ARL	498.95	131.67	49.92	26.01	16.50	4.06	1.65	1.06	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	447.00	114.00	44.00	23.00	15.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	354.98	96.62	35.19	17.65	10.85	2.00	0.93	0.31	0.04	0.00	0.00	0.00	0.00	0.00
	Laplace	ARL	503.02	89.97	31.42	16.34	10.24	2.93	1.31	1.03	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	447.00	78.00	28.00	14.00	9.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	357.84	66.10	21.28	10.81	6.45	1.46	0.69	0.20	0.07	0.01	0.00	0.00	0.00	0.00
	Logistic	ARL	502.13	115.25	42.13	22.17	13.63	3.57	1.45	1.03	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	447.00	99.00	36.00	19.00	12.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	357.84	83.49	29.56	14.54	8.81	1.75	0.81	0.23	0.03	0.00	0.00	0.00	0.00	0.00
	t(4)	ARL	503.11	153.20	59.65	31.34	19.66	4.77	1.97	1.22	1.03	1.00	1.00	1.00	1.00	1.00
		MDRL	449.00	130.00	52.00	27.00	17.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	361.52	115.04	42.01	21.93	13.15	2.47	1.07	0.59	0.23	0.02	0.03	0.00	0.00	0.00
	t(8)	ARL	500.36	138.20	53.18	28.35	17.56	4.24	1.74	1.10	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	437.00	115.50	46.00	25.00	16.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	369.84	105.14	36.77	19.05	11.45	2.15	0.98	0.40	0.08	0.00	0.00	0.00	0.00	0.00
(0.10, 2.268)	Normal	ARL	501.08	141.18	53.78	28.36	17.91	4.33	1.69	1.06	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	416.00	116.00	47.00	25.00	16.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	409.43	104.61	35.48	17.45	10.47	2.03	0.92	0.30	0.04	0.00	0.00	0.00	0.00	0.00
	CN	ARL	498.89	188.39	84.69	47.51	31.26	7.05	2.63	1.35	1.10	1.02	1.00	1.00	1.00	1.00
		MDRL	401.00	123.00	49.00	27.00	17.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	392.85	106.58	37.05	18.45	11.47	2.12	0.97	0.36	0.06	0.00	0.00	0.00	0.00	0.00
	Laplace	ARL	499.36	100.25	35.70	18.93	11.97	3.22	1.42	1.05	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	409.00	84.00	32.00	17.00	11.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	401.09	72.33	22.59	11.48	6.81	1.50	0.76	0.27	0.08	0.00	0.00	0.00	0.00	0.00
	Logistic	ARL	500.14	129.12	47.66	25.23	15.93	3.92	1.58	1.05	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	409.00	108.00	41.00	22.00	14.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	401.09	94.77	31.64	15.46	9.24	1.78	0.86	0.25	0.06	0.00	0.00	0.00	0.00	0.00
	t(4)	ARL	498.01	169.35	66.27	36.09	22.57	5.34	2.19	1.28	1.05	1.00	1.00	1.00	1.00	1.00
		MDRL	398.50	141.00	56.00	32.00	20.00	5.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	395.19	124.85	45.22	22.77	13.46	2.65	1.12	0.63	0.27	0.03	0.01	0.00	0.00	0.00
	t(8)	ARL	499.21	153.55	60.16	32.02	20.27	4.77	1.92	1.15	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	400.50	125.00	52.00	29.00	18.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	400.94	119.02	39.91	19.71	11.99	2.23	1.02	0.46	0.14	0.00	0.00	0.00	0.00	0.00
(0.25, 2.378)	Normal	ARL	499.24	199.51	70.75	34.06	20.14	4.25	1.69	1.08	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	352.00	147.00	55.00	28.00	17.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	482.13	185.61	58.78	25.64	13.68	2.04	0.77	0.28	0.06	0.00	0.00	0.00	0.00	0.00
	CN	ARL	498.56	208.84	75.15	36.41	21.90	4.54	1.79	1.11	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	349.00	156.00	58.00	30.00	18.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	483.77	187.55	62.66	27.27	15.57	2.17	0.80	0.32	0.08	0.00	0.00	0.00	0.00	0.00
	Laplace	ARL	501.64	142.64	44.83	21.23	12.98	3.12	1.40	1.06	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	361.00	105.00	36.00	18.00	11.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	486.33	130.14	34.08	14.51	8.05	1.41	0.62	0.24	0.08	0.00	0.00	0.00	0.00	0.00

(Continued)

Table 2 (continued)

(η, L)	Distr.	Metrics	0	0.025	0.05	0.075	0.1	0.25	0.5	0.75	1	1.5	2	2.5	3	5
	Logistic	ARL	501.43	186.92	61.84	29.98	17.59	3.82	1.56	1.07	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	361.00	137.00	48.00	24.00	15.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	486.33	171.10	50.74	22.06	11.73	1.79	0.71	0.25	0.06	0.00	0.00	0.00	0.00	0.00
	t(4)	ARL	498.26	234.05	90.27	44.46	26.26	5.45	2.09	1.29	1.06	1.00	1.00	1.00	1.00	1.00
		MDRL	348.00	169.00	68.00	35.00	22.00	5.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	474.73	218.25	77.94	34.85	18.99	2.83	0.94	0.53	0.24	0.05	0.02	0.00	0.00	0.00
	t(8)	ARL	501.34	218.60	80.07	39.05	23.30	4.76	1.87	1.17	1.02	1.00	1.00	1.00	1.00	1.00
		MDRL	363.00	156.00	61.00	32.00	19.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	501.18	202.78	68.13	29.45	16.61	2.36	0.85	0.41	0.13	0.00	0.00	0.00	0.00	0.00
(0.50, 2.374)	Normal	ARL	499.38	296.64	141.90	68.53	36.74	4.65	1.57	1.08	1.00	1.00	1.00	1.00	1.00	1.00
		MDRL	325.00	211.00	99.00	48.00	27.00	4.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	509.63	287.35	139.94	66.88	32.77	2.95	0.64	0.27	0.05	0.00	0.00	0.00	0.00	0.00
	CN	ARL	498.40	303.84	152.45	73.77	40.33	4.96	1.65	1.11	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	325.50	220.00	105.50	52.00	30.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	517.20	287.35	152.63	71.42	36.83	3.31	0.68	0.31	0.11	0.00	0.00	0.00	0.00	0.00
	Laplace	ARL	498.13	247.80	88.85	38.50	19.86	3.13	1.35	1.06	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	342.00	168.00	63.00	29.00	15.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	513.24	251.42	85.06	34.86	16.72	1.75	0.53	0.23	0.08	0.00	0.00	0.00	0.00	0.00
	Logistic	ARL	500.77	289.61	126.37	59.33	30.43	4.07	1.49	1.06	1.01	1.00	1.00	1.00	1.00	1.00
		MDRL	337.00	198.00	92.00	42.00	23.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	529.29	288.03	119.37	55.71	26.59	2.52	0.60	0.23	0.07	0.02	0.00	0.00	0.00	0.00
	t(4)	ARL	499.39	320.80	173.31	89.89	49.88	6.34	1.96	1.27	1.06	1.00	1.00	1.00	1.00	1.00
		MDRL	332.50	228.00	120.00	65.00	36.00	5.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	511.33	310.38	177.69	84.51	46.13	4.46	0.87	0.47	0.23	0.05	0.00	0.00	0.00	0.00
	t(8)	ARL	501.75	308.78	164.26	78.51	43.25	5.42	1.75	1.16	1.02	1.00	1.00	1.00	1.00	1.00
		MDRL	339.50	221.00	112.00	57.00	31.00	5.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		SDRL	534.91	302.47	166.64	73.80	39.90	3.55	0.75	0.37	0.14	0.00	0.00	0.00	0.00	0.00

3.2 OOC Performance

The design parameters of the proposed control chart used to determine OOC efficiency are displayed in Table 2. In order to investigate the OOC performance of the proposed NPHWMA_RSS control chart, numerous values of $\delta \in$ (0.025, 0.05, 0.075, 0.10, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 2.50, 3.00, 5.00) are practiced. The ARL_1 measures are used to compare the OOC performance of the proposed and competing control charts.

Table 2, and Figs. 1–3 provide the following outcomes:

(i)   The proposed chart's IC RL distribution looks similar to all distributions examined in this study (see Table 2).

(ii) As the smoothing parameter values increase, the performance of the proposed $NPHWMA_{RSS}$ control chart deteriorates. (see Fig. 1).

(iii) As the sample size increases, the shift detection ability of the proposed $NPHWMA_{RSS}$ conrol chart enhances. (see Fig. 2).

(iv) The proposed $NPHWMA_{RSS}$ control chart gains better performance under the Laplace distribution than the other distributions (see Fig. 3).

(v) The proposed chart's ARL_1 values increase as η increases at a certain size of the shift. For instance, under normal distribution at $\eta = 0.25$, $n = 10$, and $\delta = 0.025$, the $ARL_1 = 199.51$, whereas when $\eta = 0.50$, $n = 10$, and $\delta = 0.025$, the $ARL_1 = 296.64$ (see Table 2).

(vi) The RL distribution of the proposed $NPHWMA_{RSS}$ control chart is positively skewed, as $ARL > MRDL$ (see Table 2).

(vii) The efficiency of the proposed control chart with perfect RSS outperforms the efficiency of the proposed control chart with imperfect RSS. For example, with perfect RSS under normal distribution at $\eta = 0.10$, $n = 10$, and $\delta = 0.025$, 0.05, 0.075, 0.1, 0.25 the ARL measures are 141.18, 53.78, 28.36, 17.91, 4.33 whereas for imperfect RSS at $\eta = 0.05$, $n = 10$, $\gamma = 0.30$, $\delta = 0.025$, 0.05, 0.075, 0.1, 0.25 the ARL measures are 286.98, 130.60, 75.72, 52.80, 12.71 (see Table 3). Similarly, with perfect RSS under t-distribution at $\eta = 0.10$, $n = 10$, and $\delta = 0.025$, 0.05, 0.075 the ARL measures are 153.55, 60.16, 32.02 whereas for imperfect RSS at $\eta = 0.10$, $n = 10$, $\delta = 0.025$, 0.05, 0.075 and $\gamma = 0.30$ the ARL measures are 311.25, 155.76, 88.89 (see Table 4).

(viii) The ARL_1 values of the proposed $NPHWMA_{RSS}$ chart are smaller than competing charts with different shift sizes in process location (see Fig. 4).
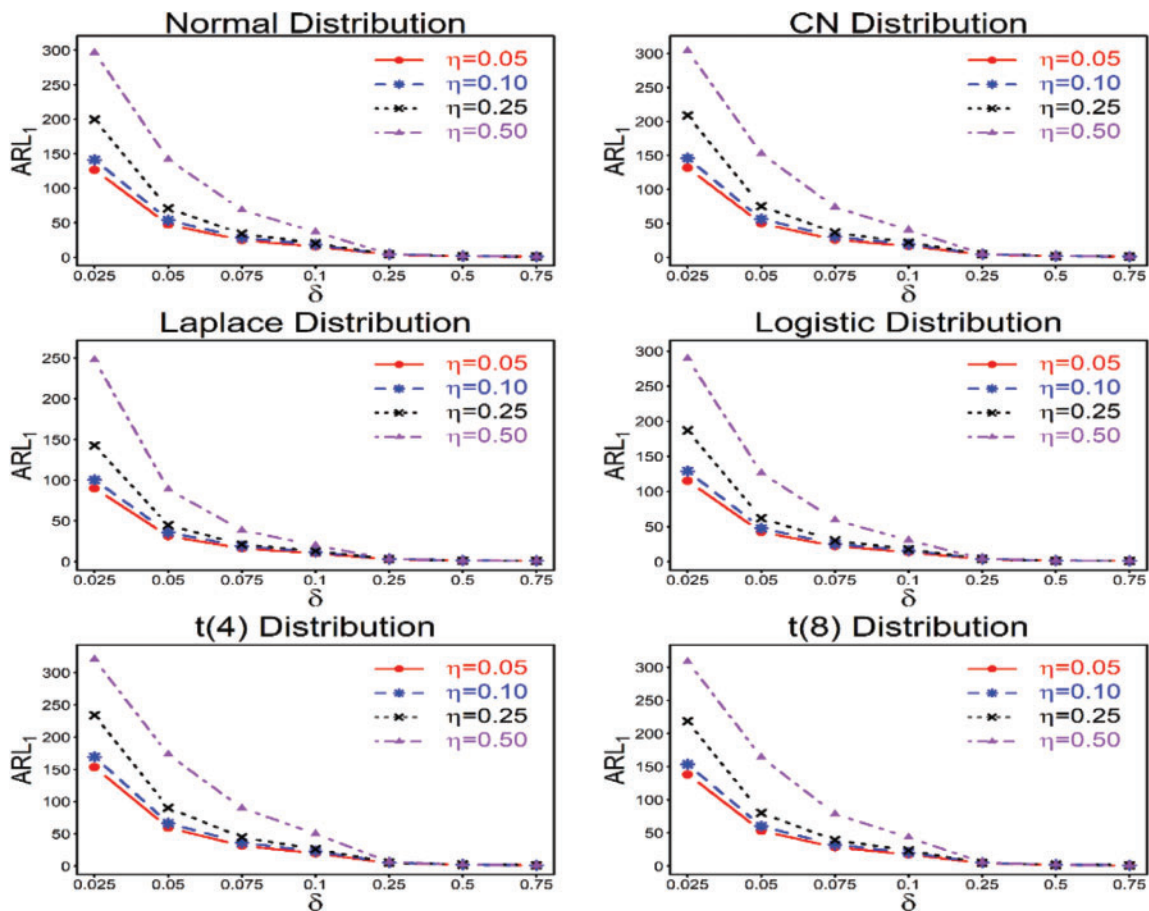

Figure 1: ARL characteristics of the proposed $NPHWMA_{RSS}$ control chart for different values of η when $n = 10$ and $ARL_0 = 500$
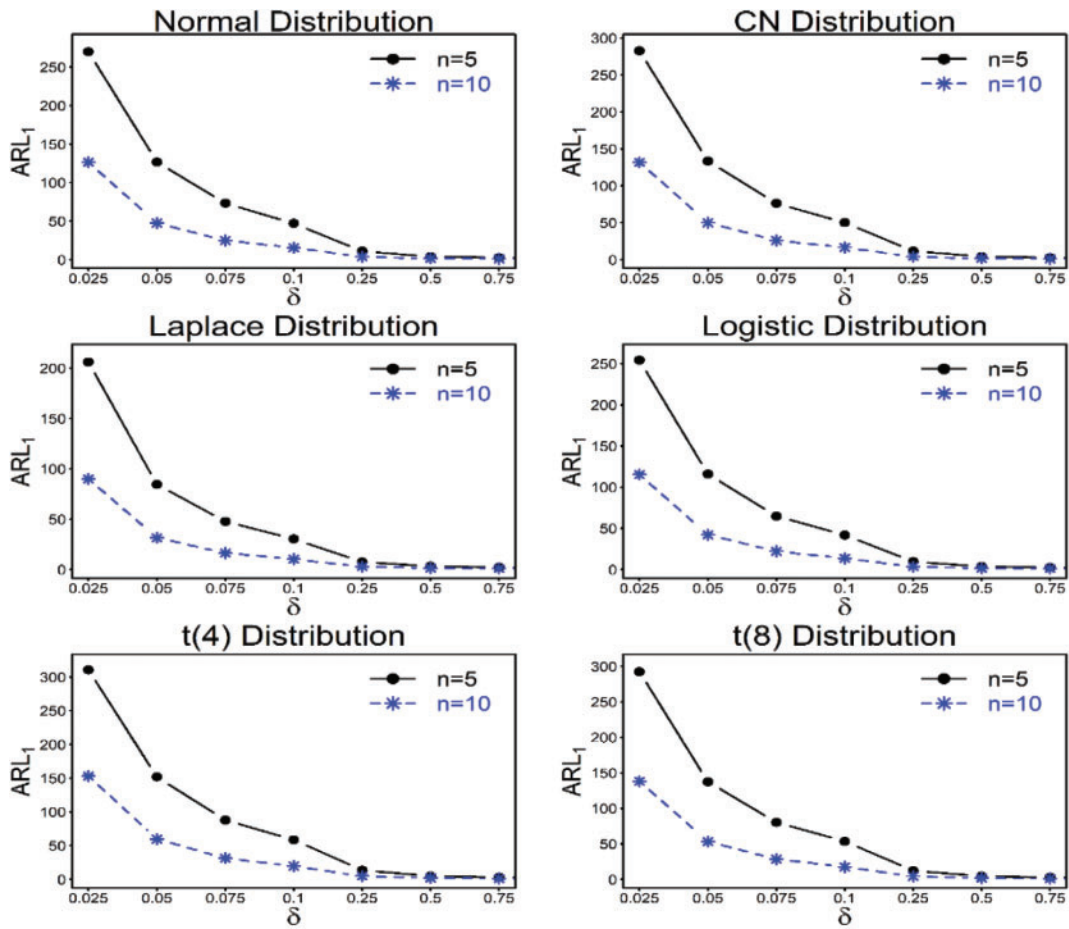
Figure 2: ARL characteristics of the proposed NPHWMA$_{RSS}$ control chart when $n = 5, 10$ and ARL$_0 = 500$
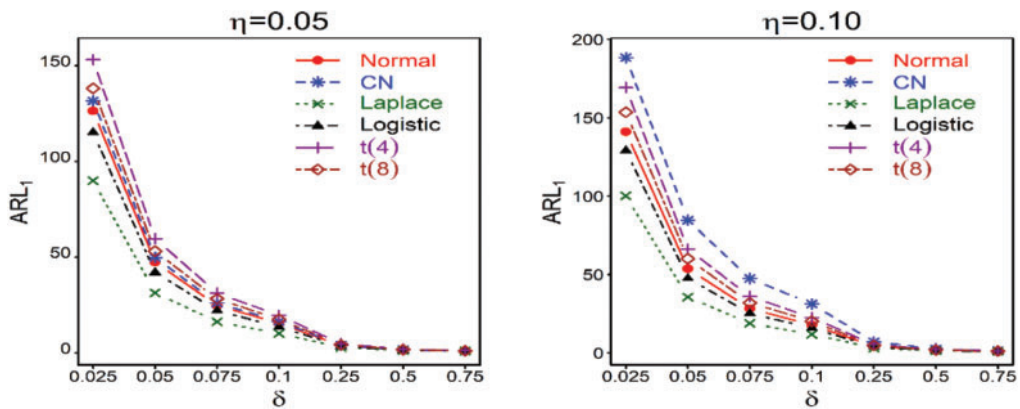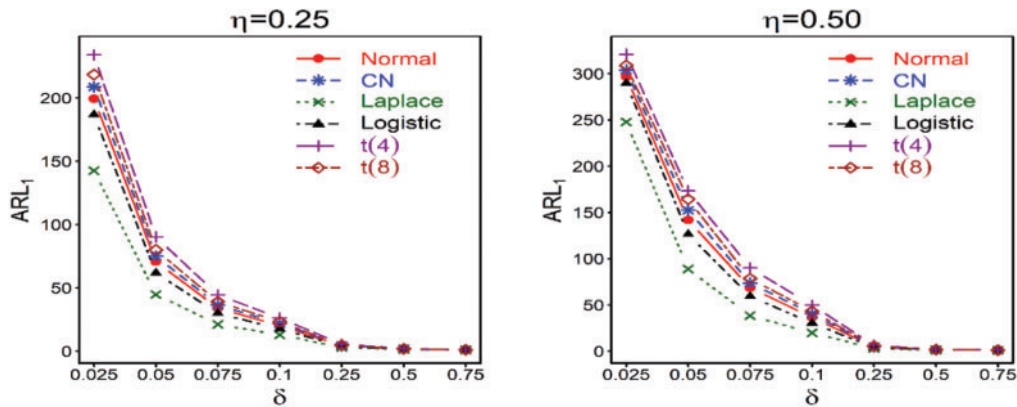

Figure 3: (Continued)

Figure 3: ARL characteristics of the proposed NPHWMA$_{RSS}$ control chart under various distributions when $n = 10$ and $ARL_0 = 500$

Table 3: RL characteristics of the proposed control chart with imperfect ranked set sampling under normal distribution at $n = 10$ and $ARL_0 = 500$

η, L	γ	Metrics	0	0.025	0.03	0.05	0.075	0.1	0.15	0.2	0.25	0.5	0.75	1	1.5	2
														δ		
		ARL	503.22	286.98	240.57	130.60	75.72	52.80	28.50	18.16	12.71	4.68	3.01	2.25	1.42	1.06
	0.30	MDRL	392.00	223.00	180.00	103.00	64.00	44.00	25.00	16.00	11.00	4.00	3.00	2.00	1.00	1.00
		SDRL	468.43	250.51	209.98	105.67	55.86	37.14	18.60	11.16	7.52	2.04	1.11	0.93	0.67	0.26
		ARL	499.82	269.98	227.40	130.60	73.83	47.87	26.11	17.12	12.19	4.47	2.86	2.11	1.28	1.03
	0.50	MDRL	395.50	216.00	187.50	106.00	62.50	40.00	23.00	15.00	11.00	4.00	3.00	2.00	1.00	1.00
0.10, 0.146		SDRL	414.40	220.74	184.91	99.76	51.47	34.81	16.79	10.76	7.07	1.85	1.06	0.94	0.57	0.19
		ARL	498.53	242.88	218.82	114.88	65.66	42.44	22.21	14.35	10.09	3.85	2.41	1.71	1.09	1.01
	0.70	MDRL	396.50	196.00	171.00	97.50	56.00	37.00	20.00	12.00	9.00	4.00	3.00	1.00	1.00	1.00
		SDRL	405.83	196.22	181.43	84.26	45.16	26.44	13.61	8.83	5.57	1.50	1.01	0.83	0.33	0.08
		ARL	501.68	204.92	165.07	81.86	45.38	28.37	15.48	10.11	6.96	2.73	1.61	1.13	1.00	1.00
	0.90	MDRL	404.00	164.00	133.50	69.00	39.00	25.00	14.00	9.00	6.00	3.00	1.00	1.00	1.00	1.00
		SDRL	385.02	157.71	128.65	58.18	29.52	17.93	8.70	5.47	3.50	1.18	0.83	0.42	0.04	0.00
		ARL	498.74	331.22	294.20	162.40	98.42	65.62	30.66	19.25	13.07	4.53	2.76	2.11	1.52	1.21
	0.30	MDRL	329.50	226.00	197.00	118.00	74.00	51.50	25.00	16.00	11.00	4.00	3.00	2.00	2.00	1.00
		SDRL	502.90	336.37	309.30	159.22	87.72	55.53	23.43	13.28	8.08	2.18	1.02	0.73	0.53	0.41
		ARL	500.42	327.91	297.51	163.35	97.75	62.07	29.80	18.34	12.17	4.23	2.69	1.97	1.32	1.06
	0.50	MDRL	357.50	242.00	209.00	121.00	73.00	49.00	24.00	15.00	10.00	4.00	3.00	2.00	1.00	1.00
0.25, 0.153		SDRL	489.09	296.59	282.42	144.38	86.65	49.57	22.84	12.51	7.51	1.94	0.98	0.74	0.49	0.23
		ARL	501.32	331.07	287.17	166.33	83.97	54.15	25.91	15.60	10.69	3.71	2.29	1.69	1.14	1.01
	0.70	MDRL	347.00	226.00	215.00	124.00	65.00	42.00	21.00	13.00	9.00	3.00	2.00	2.00	1.00	1.00
		SDRL	520.56	333.88	269.23	151.53	69.94	42.46	18.43	9.92	6.11	1.58	0.91	0.68	0.35	0.10
		ARL	502.33	280.96	231.97	121.18	62.80	35.46	17.70	10.17	7.12	2.65	1.62	1.15	1.01	1.00
	0.90	MDRL	359.50	212.00	164.00	93.50	49.00	29.00	15.00	9.00	6.00	3.00	2.00	1.00	1.00	1.00
		SDRL	501.46	256.05	223.18	105.46	49.46	27.38	11.55	6.29	3.84	1.07	0.68	0.36	0.08	0.00

Note: γ is used to symbolise the correlation.

Table 4: RL characteristics of the proposed control chart with imperfect ranked set sampling under t distribution at $n = 10$ and $\text{ARL}_0 = 500$

η, L	γ	Metrics	0	0.025	0.03	0.05	0.075	0.1	0.15	0.2	0.25	0.5	0.75	1	1.5	2
															δ	
	0.30	ARL	501.62	311.25	249.58	155.76	88.89	60.13	33.29	20.60	14.63	5.29	3.25	2.48	1.66	1.20
		MDRL	405.00	231.00	188.00	121.00	71.00	51.00	28.00	18.00	13.00	5.00	3.00	3.00	1.00	1.00
		SDRL	464.47	285.31	220.21	132.93	69.64	42.24	22.97	12.13	8.87	2.45	1.22	1.00	0.82	0.51
	0.50	ARL	503.04	290.08	255.91	146.17	81.45	56.94	29.88	19.57	13.61	4.94	3.09	2.31	1.48	1.14
		MDRL	375.00	222.50	201.50	117.00	65.50	48.00	27.00	17.00	12.00	4.50	3.00	3.00	1.00	1.00
0.10, 0.145		SDRL	453.48	249.66	215.80	114.61	62.75	38.17	18.64	12.20	8.02	2.24	1.21	0.94	0.74	0.42
	0.70	ARL	499.23	273.43	238.61	130.92	74.66	49.04	25.39	16.39	11.74	4.22	2.71	1.97	1.25	1.05
		MDRL	406.00	219.50	189.00	107.00	62.00	42.00	23.00	15.00	11.00	4.00	3.00	2.00	1.00	1.00
		SDRL	417.80	221.03	197.59	97.67	54.35	32.58	15.90	9.83	6.56	1.76	1.13	0.94	0.58	0.26
	0.90	ARL	502.42	233.03	177.10	97.43	51.95	33.06	17.59	11.17	7.87	3.03	1.94	1.28	1.01	1.00
		MDRL	396.00	186.50	146.00	85.00	45.00	29.00	15.00	10.00	7.00	3.00	2.00	1.00	1.00	1.00
		SDRL	414.70	180.12	136.98	67.50	34.19	20.71	10.46	6.25	4.15	1.35	0.98	0.59	0.12	0.05
	0.30	ARL	508.06	362.98	312.46	194.71	113.83	71.56	38.21	21.27	14.64	4.97	3.08	2.30	1.65	1.36
		MDRL	353.00	239.50	220.00	140.00	86.00	56.00	30.00	18.00	13.00	4.00	3.00	2.00	2.00	1.00
		SDRL	507.05	383.37	305.19	186.42	103.06	59.99	30.62	14.95	9.34	2.34	1.16	0.83	0.60	0.49
	0.50	ARL	498.66	373.70	326.01	201.62	107.64	70.25	33.08	20.59	13.94	4.74	2.93	2.17	1.47	1.19
		MDRL	350.00	271.50	243.00	145.00	80.00	56.00	25.00	17.00	12.00	4.00	3.00	2.00	1.00	1.00
0.25, 0.1575		SDRL	490.07	364.82	308.20	197.58	95.47	57.40	26.85	14.77	9.07	2.28	1.11	0.84	0.58	0.41
	0.70	ARL	502.46	361.77	320.90	180.75	102.58	65.83	30.23	18.75	12.55	4.18	2.53	1.87	1.24	1.06
		MDRL	347.50	270.50	227.50	135.00	77.00	50.00	24.00	16.00	11.00	4.00	3.00	2.00	1.00	1.00
		SDRL	467.10	334.58	306.67	169.06	88.28	52.84	22.00	13.36	7.52	1.88	0.96	0.78	0.46	0.25
	0.90	ARL	499.95	302.75	249.43	133.41	68.85	42.11	20.06	11.83	8.24	2.98	1.78	1.31	1.02	1.01
		MDRL	356.50	210.00	184.00	98.00	54.00	33.00	17.00	10.00	7.00	3.00	2.00	1.00	1.00	1.00
		SDRL	478.56	292.42	235.64	121.67	54.66	33.21	13.54	7.31	4.66	1.21	0.79	0.53	0.15	0.11

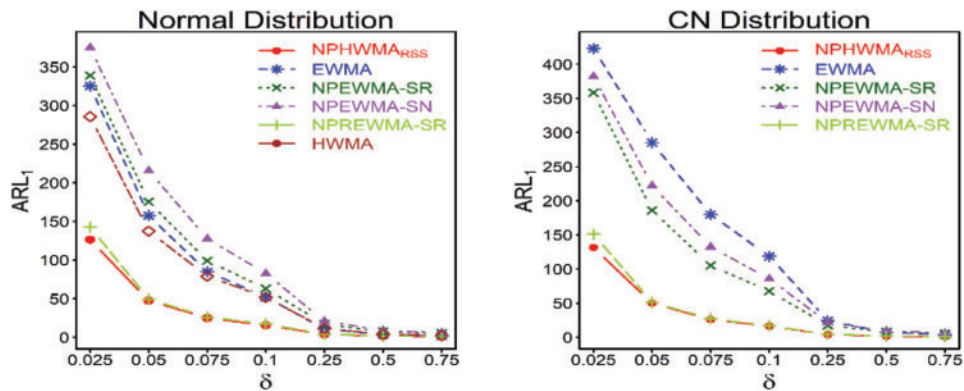Note: γ is used to symbolise the correlation.
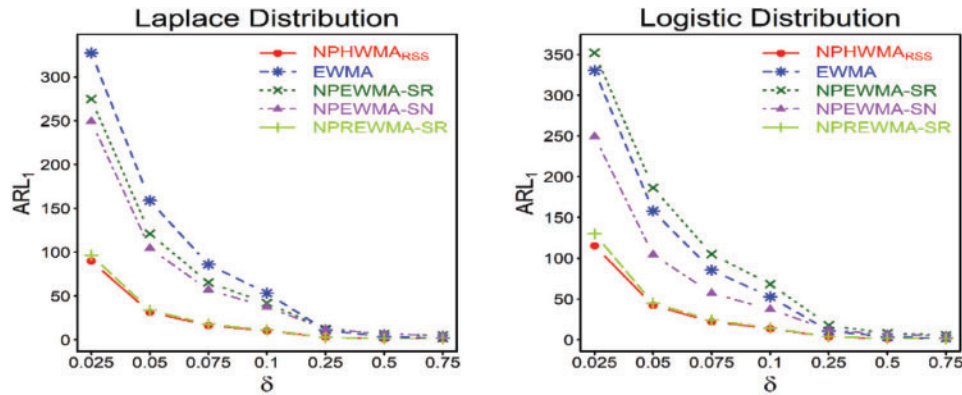

Figure 4: (Continued)

Figure 4: ARL meaures of the proposed NPHWMA$_{RSS}$ and competing control charts under various distributions when $n = 10$ and ARL$_0 = 500$

4 Comparative Study

This section compares the proposed chart against the competing charts, including EWMA, NPEWMA-SR, NPEWMA-SN, NPREWMA-SR, and HWMA control charts. The comparisons are given in the following subsection.

4.1 Proposed vs. EWMA Control Chart

The ARL$_1$ comparison for the proposed NPHWMA$_{RSS}$ control chart against the EWMA control chart indicates that the proposed control chart outperforms the EWMA control chart. For example, under normal distribution, when $\eta = 0.05$, $n = 10$ and $\delta = 0.025, 0.05, 0.075, 0.10, 0.25, 0.50, 0.75$, the ARL$_1$ values of the proposed NPHWMA$_{RSS}$ control chart, i.e., 126.47, 47.44, 24.83, 15.55, 3.86, 1.56, 1.04, are smaller than the ARL$_1$ values of the EWMA control chart, i.e., 325.27, 157.51, 84.87, 52.28, 10.93, 3.49, 1.91 (see Tables 2 and 5). Similarly, when we consider Laplace distribution for comparison, we observed the same behavior as the proposed NPHWMA$_{RSS}$ control chart.

Table 5: ARL characteristic of the EWMA control chart under different distributions with nominal ARL$_0 = 500$ and n = 10

Distr.	(η, L)	0	0.025	0.03	0.05	0.075	0.1	0.15	0.2	0.25	0.5	0.75	1	1.5	2
Normal	0.05. 2.641	499.91	325.27	278.92	157.51	84.87	52.28	25.79	15.86	10.93	3.49	1.91	1.34	1.02	1.00
t(4)	0.05, 2.662	502.82	334.16	288.38	164.12	88.65	54.36	26.78	16.37	11.11	3.53	1.92	1.33	1.02	1.00
CN	0.05, 4.162	499.14	423.12	395.08	285.25	180.06	118.71	60.67	36.16	24.37	7.43	3.80	2.42	1.35	1.07
Laplace	0.05, 2.646	498.03	327.27	283.82	158.87	85.97	53.20	26.23	15.94	11.00	3.50	1.91	1.33	1.02	1.00
Logistic	0.05, 2.645	499.62	330.21	281.76	157.97	85.42	52.68	25.97	15.92	10.87	3.47	1.90	1.33	1.02	1.00

For instance, with the same control chart properties, the proposed NPHWMA$_{RSS}$ control chart provides the ARL$_1$ values of 89.97, 31.42, 16.34, 10.24, 2.93, 1.31, 1.03, while the EWMA control chart delivers the ARL$_1$ values of 327.27, 158.87, 85.97, 53.20, 11.00, 3.50, 1.91, respectively (see Tables 2 and 5). Fig. 4 depicts the efficiency of the proposed control chart over the EWMA control chart. Likewise, the proposed NPHWMA$_{RSS}$ control chart outperforms the EWMA control chart in terms of overall performance. As an illustration, in the case of normal distribution at $\eta = 0.05$ the EQL and RARL values of the proposed control chart are 1.41 *and* 1.00, while the EQL and

RARL values of EWMA control chart are 1.67, 1.19 (see Table 6). These findings show that the proposed NPHWMA$_{RSS}$ control chart performs better than the EWMA control chart in detecting process location shifts.

Table 6: Overall performance measures of the proposed and competing control charts

Distributions		Proposed	EWMA	NPEWMA-SR	NPEWMA-SN	NPREWMA-SR	HWMA
Normal	EQL	1.41	1.67	5.86	5.42	1.55	1.76
	RARL	1.00	1.19	4.16	3.85	1.20	1.25
	PCI	1.00	1.69	4.35	4.59	1.10	1.84
CN	EQL	1.41	2.48	5.91	5.55	1.51	–
	RARL	1.00	1.75	4.19	3.93	1.20	–
	PCI	1.00	3.06	4.37	4.61	1.19	–
Laplace	EQL	1.39	1.67	5.74	5.09	1.41	–
	RARL	1.00	1.20	4.13	3.66	1.11	–
	PCI	1.00	1.98	4.30	4.04	1.21	–
Logistic	EQL	1.40	1.67	5.80	5.26	1.52	–
	RARL	1.00	1.19	4.15	3.75	1.31	–
	PCI	1.00	1.77	4.32	4.38	1.27	–
t(4)	EQL	1.45	1.68	6.21	6.03	1.61	–
	RARL	1.00	1.16	4.30	4.17	1.13	–
	PCI	1.00	1.50	4.41	4.54	1.16	–

4.2 Proposed vs. NPEWMA-SR Control Chart

In comparison with the NPEWMA-SR control chart, the proposed NPHWMA$_{RSS}$ control chart indicates superior performance for all combinations of δ and η. For instance, in case of logistic distribution, when $n = 10$, $\eta = 0.05$, and $\delta = 0.025, 0.05, 0.075, 0.10, 0.25, 0.50, 0.75$, the proposed NPHWMA$_{RSS}$ control chart yields ARL$_1$ values of 115.25, 42.13, 22.17, 13.63, 3.57, 1.45, 1.03, while the NPEWMA-SR control charts has the ARL$_1$ values of 322.05, 158.72, 89.31, 55.32, 15.24, 7.19, 5.20, respectively (see Tables 2 and 7).

Table 7: ARL characteristic of the NPEWMA-SR control chart under different distributions with nominal ARL$_0$ = 500 and n = 10

		δ													
Distr.	(η, L)	0	0.025	0.03	0.05	0.075	0.1	0.15	0.2	0.25	0.5	0.75	1	1.5	2
Normal	0.05, 2.610	500.56	338.91	299.55	175.39	99.14	63.12	33.83	22.34	16.86	7.67	5.38	4.45	4.01	4.00
t(4)	0.05, 2.610	500.01	369.44	329.69	209.40	125.02	79.43	41.95	27.49	20.37	9.01	6.20	5.06	4.19	4.02
CN	0.05, 2.610	501.81	358.41	307.45	186.11	105.25	67.67	35.78	23.63	17.56	7.95	5.55	4.58	4.02	4.00
Laplace	0.05, 2.610	502.08	274.75	232.61	121.05	65.47	42.01	23.70	16.45	12.56	6.54	4.98	4.34	4.02	4.00
Logistic	0.05, 2.610	497.80	322.05	283.01	158.72	89.31	55.32	30.47	20.32	15.23	7.19	5.20	4.39	4.01	4.00

Similarly, when we examine $t_{(4)}$ distribution, with $n = 10$, $\eta = 0.05$, and $\delta = 0.025, 0.10, 0.25$, the ARL$_1$ values of the proposed NPHWMA$_{RSS}$ and the NPEWMA-SR control chart are 153.20, 19.66, 4.77, and 369.44, 79.43, and 16.86, respectively (see Tables 2 and 7). Table 6 shows that the PCI

performance outperforms the NPEWMA-SR control chart. For example, the proposed control chart has a PCI value of 1.00, whereas the NPEWMA-SR control chart has a PCI value of 4.37 under the CN distribution (see Table 6). The efficiency of the proposed NPHWMA$_{RSS}$ control chart to the NPEWMA-SR control chart can also be observed in Fig. 4.

4.3 Proposed vs. NPEWMA-SN Control Chart

The ARL$_1$ comparison reveals that the proposed NPHWMA$_{RSS}$ control chart outperforms the NPEWMA-SN control chart for all choices of η and δ (see Tables 2 and 8). In detail, for CN distribution, with $\eta = 0.05$, $\delta = 0.075$, 0.10, 0.25, the ARL$_1$ values for the proposed NPHWMA$_{RSS}$ control chart are 26.01, 16.50, 4.06, while for the NPEWMA-SN control chart these values are 132.56, 85.70, 21.61, respectively (see Tables 2 and 8). Similarly, under normal distribution, at $\eta = 5\%$, a 2.5% increase in process location parameter reduces the ARL by 24.76% for the NPEWMA-SN control chart, while the NPHWMA$_{RSS}$ reduces ARL by 74.81% (see Tables 2 and 8). Also, Fig. 4 also shows the superiority of the NPHWMA$_{RSS}$ control chart over the NPEWMA-SN control chart. Table 6 shows that the proposed control chart outperforms the NPEWMA-SN control chart in EQL and PCI values under different distributions.

Table 8: ARL characteristic of the NPEWMA-SN control chart under different distributions with nominal ARL$_0$ = 500 and n = 10

Distr.	(η, L)	0	0.025	0.03	0.05	0.075	0.1	0.15	0.2	0.25	0.5	0.75	1	1.5	2
Normal	0.05, 2.612	498.65	375.18	337.56	215.79	127.74	82.71	44.10	28.69	20.91	9.00	6.03	4.78	3.65	3.15
t(4)	0.05, 2.612	499.56	384.51	353.97	230.05	139.92	91.23	48.60	31.31	22.83	9.81	6.57	5.23	4.08	3.56
CN	0.05, 2.612	498.13	382.16	340.34	222.10	132.56	85.70	45.88	29.61	21.61	9.26	6.19	4.89	3.74	3.23
Laplace	0.05, 2.612	499.22	249.33	207.01	104.53	57.02	37.48	21.85	15.56	12.19	6.57	5.01	4.29	3.58	3.22
Logistic	0.05, 2.612	497.04	352.00	309.11	186.43	105.09	68.34	36.09	23.92	17.70	8.01	5.55	4.53	3.59	3.18

4.4 Proposed vs. NPREWMA-SR Control Chart

The proposed NPHWMA$_{RSS}$ control chart performs better than the NPREWMA-SR control chart. For instance, at $n = 10$, $\eta = 0.05$, $\delta = 0.025$, the NPREWMA-SR control chart with Laplace distribution triggers the OOC signal after 94.56 observations, whereas the proposed NPHWMA$_{RSS}$ control chart produces the OOC signal after 89.97 observations (see Tables 2 and 9). Similarly, in the case of CN distribution, when $n = 10$, $\eta = 0.50$ and $\delta = 0.025$, 0.05, 0.25, 0.50, then the ARL$_1$ values for the proposed NPHWMA$_{RSS}$ control charts are 131.67, 49.92, 4.06, 1.65, whereas the ARL$_1$ values for the NPREWMA-SR control charts are 149.06, 51.18, 4.67, and 1.89, respectively (see Tables 2 and 9 and Fig. 4). Furthermore, under $t_{(8)}$ distribution, at $n = 10$, $\eta = 0.05$, and $\delta = 0.025$, 0.05, the OOC ARL for the proposed NPHWMA$_{RSS}$ and the NPREWMA-SR control chart are 138.20, 53.18, and 152.17, 54.38, respectively. It is also observed that the proposed control chart outperforms the NPREWMA-SR control chart in terms of overall performance. For example, under Logistic distribution, the RARL value of the proposed control chart is 1.00, while the RARL value of the NPREWMA-SR control chart is 1.31 (see Table 6).

Table 9: RL characteristics of the NPREWMA-SR control chart under different distributions at $\eta = 0.05$, $L = 2.01$, $n = 10$ and $ARL_0 = 500$

Distr.	Metrics	δ													
		0	0.025	0.05	0.075	0.1	0.25	0.5	0.75	1	1.5	2	2.5	3	5
Normal	ARL	498.93	140.45	49.45	26.08	17.38	4.19	1.84	1.09	1.01	1.00	1.00	1.00	1.00	1.00
	MDRL	340.00	99.00	37.00	19.00	12.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	506.56	134.53	38.65	17.05	10.00	1.76	0.52	0.16	0.00	0.00	0.00	0.00	0.00	0.00
t(4)	ARL	498.21	172.26	61.22	32.50	20.02	5.97	2.02	1.32	1.06	1.01	1.00	1.00	1.00	1.00
	MDRL	336.00	120.00	47.00	24.00	15.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	512.96	168.00	51.86	23.07	12.97	2.29	0.71	0.33	0.14	0.00	0.01	0.00	0.00	0.00
t(8)	ARL	499.17	152.17	54.38	30.46	18.88	5.21	1.87	1.19	1.02	1.00	1.00	1.00	1.00	1.00
	MDRL	343.00	107.00	41.00	22.00	14.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	509.61	150.91	44.02	19.86	11.16	2.04	0.61	0.24	0.07	0.00	0.00	0.00	0.00	0.00
CN	ARL	499.32	149.06	51.18	27.41	17.04	4.67	1.89	1.13	1.04	1.02	1.00	1.00	1.00	1.00
	MDRL	336.50	107.00	38.00	20.00	13.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	504.65	141.93	41.37	18.19	10.84	1.91	0.55	0.19	0.04	0.00	0.00	0.00	0.00	0.00
Laplace	ARL	501.56	94.56	33.72	17.80	11.09	3.01	1.40	1.10	1.04	1.01	1.00	1.00	1.00	1.00
	MDRL	353.50	69.00	24.00	12.00	8.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	514.79	87.82	23.45	10.66	6.10	1.23	0.41	0.15	0.04	0.00	0.00	0.00	0.00	0.00
Logistic	ARL	501.32	126.99	44.83	24.29	14.34	3.98	1.67	1.05	1.01	1.00	1.00	1.00	1.00	1.00
	MDRL	353.50	91.00	33.00	17.00	10.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	514.79	124.04	32.53	14.82	8.67	1.59	0.47	0.16	0.02	0.00	0.00	0.00	0.00	0.00

4.5 Proposed vs. HWMA Control Chart

The proposed $NPHWMA_{RSS}$ control chart outperformed the HWMA control chart in terms of early shift detection in the process location. For example, investigating the normal distribution with $n = 10$, $\eta = 0.10$, and $\delta = 0.025$, 0.05, 0.075, the ARL_1 values for the proposed $NPHWMA_{RSS}$ control chart are observed as; 141.18, 53.78, 28.36, whereas, the ARL_1 values for the HWMA control charts are reported as; 307.44, 152.37, 87.46, respectively (see Tables 2 and 10). Similarly, under normal distribution, at $n = 10$, and $\eta = 25\%$ reduces the ARL by 124.65% in the HWMA control chart, with a 7.5% increase in the process location parameter, whereas the $NPHWMA_{RSS}$ control chart reduces ARL by 34.06% (see Tables 2 and 10). The proposed control chart performs better than the HWMA control chart in terms of overall performance. For example, under normal distribution, the PCI value of the proposed control chart is 1.00, whereas the PCI value of the HWMA control chart is 1.84 (see Table 6). In addition, Fig. 3 illustrates that the proposed $NPHWMA_{RSS}$ control chart outperforms the HWMA control chart.

Table 10: RL characteristics of the HWMA control chart under normal distribution at $n = 10$ and $ARL_0 = 500$

η, L	Metrics	δ													
		0	0.025	0.05	0.075	0.10	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	5.00
0.05, 2.608	ARL	498.76	285.77	137.48	79.21	51.71	11.72	4.09	2.37	1.53	1.02	1.00	1.00	1.00	1.00
	MDRL	435.00	240.00	116.00	68.00	45.00	10.00	4.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	370.24	222.41	103.36	58.08	37.20	7.51	2.12	1.27	0.89	0.20	0.01	0.00	0.00	0.00
0.10, 2.94	ARL	502.65	307.44	152.37	87.46	58.69	13.76	4.56	2.64	1.72	1.04	1.00	1.00	1.00	1.00
	MDRL	410.00	243.00	126.00	74.00	51.00	12.00	4.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	413.84	252.31	113.91	62.28	39.55	7.95	2.22	1.27	0.94	0.25	0.02	0.00	0.00	0.00

Table 10 (continued)

		δ													
η, L	Metrics	0	0.025	0.05	0.075	0.10	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	5.00
0.25, 3.08	ARL	501.05	377.29	216.60	124.65	77.13	14.77	4.54	2.50	1.66	1.05	1.00	1.00	1.00	1.00
	MDRL	356.00	268.00	156.00	94.00	60.00	13.00	4.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	495.12	364.10	203.26	108.69	64.43	9.58	2.29	1.15	0.80	0.23	0.03	0.00	0.00	0.00
0.50, 3.09	ARL	500.51	451.12	334.17	229.57	156.70	24.59	5.05	2.39	1.56	1.05	1.00	1.00	1.00	1.00
	MDRL	350.00	316.00	232.00	162.00	110.00	19.00	4.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	498.82	445.85	328.76	225.15	153.35	21.26	3.34	1.22	0.67	0.23	0.03	0.00	0.00	0.00

5 Illustrative Example Using Real Data

To demonstrate the applicability of the proposed $\text{NPHWMA}_{\text{RSS}}$ control chart practically, a real-life example is provided in this section. The example used the real-life data of the non-isothermal continuous stirred tank reactor (CSTR) process. This real-life data are considered by numerous authors in their studies, for instance, Lucas et al. [43], Yoon et al. [44], Shi et al. [45], Ridwan et al. [46], and Adegoke et al. [47], etc. The CSTR process has nine variables, one of which we select as the variable of interest (X), representing the output temperature, and this variable is used in real-life application with parameters $\mu_X = 369.88$ and $\sigma_X^2 = 0.32$. The data initially consists of 1000 observations, with the first 500 occurring when the process was in an IC condition. The estimation is carried out with the help of the mentioned parameters, and the control limits are obtained. In order to use the proposed control chart and the existing (NPREWMA-SR) control chart in practise, the variable of interest (X), is used. We used the RSS approach to generate 40 paired observations of size $n = 5$ and $m = 1$ from a normal distribution. After the 24*th* observation, a shift in the process mean is introduced. The parameters of the proposed and NPREWMA-SR control charts used for real-life analysis are $L = 2.011$, $\eta = 0.05$, $\text{ARL}_0 = 500$ and $L = 2.01$, $\eta = 0.05$, $\text{ARL}_0 = 500$, respectively. Fig. 5 indicates that the proposed $\text{NPHWMA}_{\text{RSS}}$ control chart triggers the first OOC signal at sample number 27, while the NPREWMA-SR control chart detects the first OOC point at sample number 29. Similarly, the proposed $\text{NPHWMA}_{\text{RSS}}$ control chart detects overall 14 OOC points, whereas the NPREWMA-SR control chart detects 12 OOC points (see Table 11 and Fig. 5).

Proposed
NPREWMA-SR
Plotting Statistic
10
5
0
-5
-10
0
10
20
30
40
Sample Number

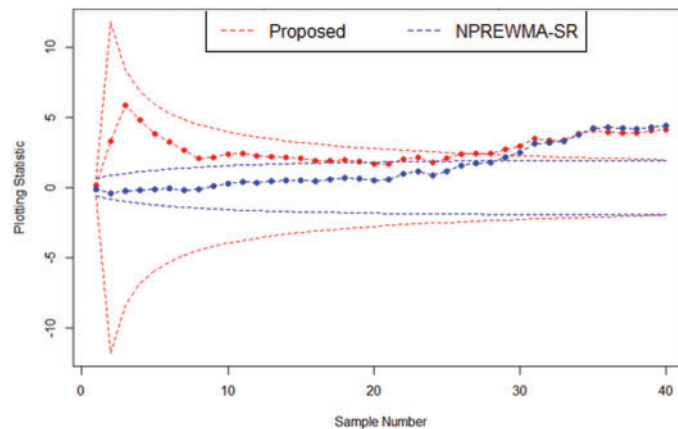

Figure 5: Real-life application of the proposed $\text{NPHWMA}_{\text{RSS}}$ control charts against the NPREWMA-SR control charts

Table 11: Application of the proposed *vs.* NPREWMA-SR control chart

Sample #	Proposed			NPREWMA-SR		
	Statistic	LCL	UCL	Statistic	LCL	UCL
1	0.15	−0.62	0.62	−0.15	−0.61	0.61
2	3.3	−11.82	11.82	−0.39	−0.84	0.84
3	5.85	−8.37	8.37	−0.22	−1.01	1.01
4	4.8	−6.84	6.84	−0.16	−1.14	1.14
5	3.85	−5.93	5.93	−0.1	−1.24	1.24
6	3.28	−5.31	5.31	−0.05	−1.33	1.33
7	2.7	−4.86	4.86	−0.2	−1.4	1.4
8	2.09	−4.5	4.5	−0.14	−1.46	1.46
9	2.15	−4.22	4.22	0.12	−1.52	1.52
10	2.37	−3.98	3.98	0.26	−1.57	1.57
11	2.43	−3.78	3.78	0.4	−1.61	1.61
12	2.28	−3.61	3.61	0.33	−1.65	1.65
13	2.21	−3.46	3.46	0.46	−1.68	1.68
14	2.17	−3.33	3.33	0.49	−1.71	1.71
15	2.09	−3.21	3.21	0.52	−1.73	1.73
16	1.91	−3.11	3.11	0.44	−1.76	1.76
17	1.93	−3.02	3.02	0.57	−1.78	1.78
18	1.99	−2.93	2.93	0.69	−1.8	1.8
19	1.85	−2.85	2.85	0.61	−1.81	1.81
20	1.7	−2.78	2.78	0.53	−1.83	1.83
21	1.67	−2.71	2.71	0.55	−1.84	1.84
22	2.03	−2.65	2.65	0.97	−1.85	1.85
23	2.15	−2.59	2.59	1.17	−1.86	1.86
24	1.77	−2.54	2.54	0.86	−1.87	1.87
25	2.09	−2.49	2.49	1.17	−1.88	1.88
26	2.39	−2.44	2.44	1.56	−1.89	1.89
27	2.44	−2.4	2.4	1.73	−1.9	1.9
28	2.44	−2.35	2.35	1.8	−1.9	1.9
29	2.76	−2.32	2.32	2.16	−1.91	1.91
30	2.97	−2.28	2.28	2.5	−1.91	1.91
31	3.47	−2.24	2.24	3.13	−1.92	1.92
32	3.35	−2.21	2.21	3.22	−1.92	1.92
33	3.4	−2.18	2.18	3.31	−1.92	1.92
34	3.85	−2.15	2.15	3.79	−1.93	1.93
35	4.11	−2.12	2.12	4.25	−1.93	1.93
36	3.97	−2.09	2.09	4.29	−1.93	1.93
37	3.9	−2.06	2.06	4.23	−1.94	1.94
38	3.87	−2.04	2.04	4.16	−1.94	1.94
39	4.05	−2.01	2.01	4.31	−1.94	1.94
40	4.13	−1.99	1.99	4.44	−1.94	1.94

6 Summary, Conclusions, and Future Recommendations

The normality assumption for the process is essential for parametric control charts to monitor process parameters (location and/or dispersion) shifts. If the process under consideration does not follow a normal distribution, the nonparametric (NP) control charts are used. In addition, the homogeneously weighted moving average (HWMA) control chart is introduced for improved process location monitoring. Moreover, the ranked set sampling (RSS) technique outperforms simple random sampling (SRS). This study proposed the NP HWMA control chart with the RSS scheme, which results in an NP HWMA Wilcoxon signed-rank chart under the RSS technique (NPHWMA$_{RSS}$), which is used to improve the shifts detecting ability in the process location. The performance of the proposed chart is examined in terms of ARL, MDRL, and SDRL. The results revealed that the proposed control chart performs better than the competing charts like EWMA, NPEWMA-SR, NPEWMA-SN, NPREWMA-SR, and HWMA. Moreover, a real-life application is also offered to show the proposed control chart's applicability in practice. The proposed study can be extended to the multivariate scenario.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. Shewhart, W. A. (1930). Economic control of quality manufactured product. *Bell Labs Technical Journal, 9(2),* 364–389.
2. Page, E. S. (1954). Continuous inspection schemes. *Biometrika, 41,* 100–115. DOI 10.1093/biomet/41.1-2.100.
3. Roberts, S. (1959). Control chart tests based on geometric moving averages. *Technometrics, 1(3),* 239–250. DOI 10.1080/00401706.1959.10489860.
4. Yang, S. F., Lin, J. S., Cheng, S. W. (2011). A new nonparametric EWMA sign control chart. *Expert Systems with Applications, 38(5),* 6239–6243. DOI 10.1016/j.eswa.2010.11.044.
5. Graham, M., Chakraborti, S., Human, S. (2011). A nonparametric EWMA sign chart for location based on individual measurements. *Quality Engineering, 23,* 227–241. DOI 10.1080/08982112.2011.575745.
6. Graham, M., Chakraborti, S., Human, S. (2011). A nonparametric exponentially weighted moving average signed-rank chart for monitoring location. *Computational Statistics & Data Analysis, 55,* 2490–2503. DOI 10.1016/j.csda.2011.02.013.
7. Chakraborty, N., Chakraborti, S., Human, S. W., Balakrishnan, N. (2016). A generally weighted moving average signed-rank control chart. *Quality and Reliability Engineering International, 32(8),* 2835–2845. DOI 10.1002/qre.1968.
8. Abid, M., Nazir, H. Z., Riaz, M., Lin, Z. (2017). An efficient nonparametric EWMA Wilcoxon signed-rank chart for monitoring location. *Quality and Reliability Engineering International, 33(3),* 669–685. DOI 10.1002/qre.2048.

9. Ali, S., Abbas, Z., Nazir, H. Z., Riaz, M., Zhang, X. et al. (2020). On designing non-parametric EWMA sign chart under ranked set sampling scheme with application to industrial process. *Mathematics, 8(9),* 1497. DOI 10.3390/math8091497.

10. Lucas, J. M., Saccucci, M. S. (1990). Exponentially weighted moving average control schemes: Properties and enhancements. *Technometrics, 32(1),* 1–12. DOI 10.1080/00401706.1990.10484583.

11. Shamma, S. E., Shamma, A. K. (1992). Development and evaluation of control charts using double exponentially weighted moving averages. *International Journal of Quality and Reliability Management, 9(6),* 18–25. DOI 10.1108/02656719210018570.

12. Zhang, L., Chen, G. (2005). An extended EWMA mean chart. *Quality Technology and Quantitative Management, 2,* 39–52. DOI 10.1080/16843703.2005.11673088.

13. Abbas, N., Riaz, M., Does, R. J. M. M. (2012). Mixed exponentially weighted moving average–cumulative sum charts for process monitoring. *Quality and Reliability Engineering International, 29(3),* 345–356. DOI 10.1002/qre.1385.

14. Zaman, B., Riaz, M., Abbas, N., Does, R. J. M. M. (2015). Mixed cumulative sum–exponentially weighted moving average control charts: An efficient way of monitoring process location. *Quality and Reliability Engineering International, 31,* 1407–1421. DOI 10.1002/qre.1678.

15. Anwar, S. M., Aslam, M., Riaz, M., Zaman, B. (2020). On mixed memory control charts based on auxiliary information for efficient process monitoring. *Quality and Reliability Engineering International, 36(6),* 1949–1968. DOI 10.1002/qre.2667.

16. Adegoke, N. A., Riaz, M., Sanusi, R. A., Smith, A. N., Pawley, M. D. (2017). EWMA-type scheme for monitoring location parameter using auxiliary information. *Computers & Industrial Engineering, 114,* 114–119. DOI 10.1016/j.cie.2017.10.013.

17. Sanusi, R. A., Riaz, M., Abbas, N. (2017). Combined shewhart CUSUM charts using auxiliary variable. *Computers & Industrial Engineering, 105,* 329–337. DOI 10.1016/j.cie.2017.01.018.

18. Haq, A. (2018). A new adaptive EWMA control chart using auxiliary information for monitoring the process mean. *Communications in Statistics-Theory and Methods, 47(19),* 4840–4858. DOI 10.1080/03610926.2018.1448417.

19. Zaman, B., Lee, M. H., Riaz, M., Abujiya, M. R. (2019). An adaptive approach to EWMA dispersion chart using Huber and Tukey functions. *Quality and Reliability Engineering International, 35(6),* 1542–1581. DOI 10.1002/qre.2460.

20. Aslam, M., Anwar, S. M. (2020). An improved Bayesian modified-EWMA location chart and Its applications in mechanical and sport industry. *PLoS One, 15(2),* e0229422. DOI 10.1371/journal.pone.0229422.

21. Adeoti, O. A., Malela-Majika, J. C. (2020). Double exponentially weighted moving average control chart with supplementary runs-rules. *Quality Technology & Quantitative Management, 17(2),* 149–172. DOI 10.1080/16843703.2018.1560603.

22. Abbasi, S., Haq, A. (2019). Optimal CUSUM and adaptive CUSUM charts with auxiliary information for process mean. *Journal of Statistical Computation and Simulation, 89(2),* 337–361. DOI 10.1080/00949655.2018.1548619.

23. Rasheed, Z., Zhang, H., Anwar, S. M., Zaman, B. (2021). Homogeneously mixed memory charts with application in the substrate production process. *Mathematical Problems in Engineering, 2021,* 2582210. DOI 10.1155/2021/2582210.

24. Rasheed, Z., Zhang, H., Arslan, M., Zaman, B., Anwar, S. M. et al. (2021b). An efficient robust nonparametric triple EWMA Wilcoxon signed-rank control chart for process location. *Mathematical Problems in Engineering,* 2570198. DOI 10.1155/2021/2570198.

25. Anwar, S. M., Aslam, M., Zaman, B., Riaz, M. (2021). Mixed memory control chart based on auxiliary information for simultaneously monitoring of process parameters: An application in glass field. *Computers & Industrial Engineering, 156,* 107284. DOI 10.1016/j.cie.2021.107284.

26. Liu, X., Khan, M., Rasheed, Z., Anwar, S. M., Arslan, M. (2022). New hybrid EWMA charts for efficient process dispersion monitoring with application in automobile industry. *Computer Modeling in Engineering & Sciences, 131(2),* 1171–1195. DOI 10.32604/cmes.2022.019199.

27. Rasheed, Z., Khan, M., Abiodun, N. L., Anwar, S. M., Khalaf, G. et al. (2022). Improved nonparametric control chart based on ranked set sampling with application of chemical data modelling. *Mathematical Problems in Engineering, 2022,* 7350204. DOI 10.1155/2022/7350204.

28. Hunter, J. S. (1986). The exponentially weighted moving average. *Journal of Quality Technology, 18(1),* 203–210. DOI 10.1080/00224065.1986.11979014.

29. Abbas, N. (2018). Homogeneously weighted moving average control chart with an application in substrate manufacturing process. *Computers & Industrial Engineering, 120,* 460–470. DOI 10.1016/j.cie.2018.05.009.

30. Abid, M., Mei, S., Nazir, H. Z., Riaz, M., Hussain, S. (2020). A mixed HWMA-CUSUM mean chart with an application to manufacturing process. *Quality and Reliability Engineering International, 37(2),* 618–631.

31. Adeoti, O. A., Koleoso, S. O. (2020). A hybrid homogeneously weighted moving average control chart for process monitoring. *Quality and Reliability Engineering International, 36(6),* 2170–2186. DOI 10.1002/qre.2690.

32. Abid, M., Shabbir, A., Nazir, H. Z., Sherwani, R. A. K., Riaz, M. (2020). A double homogeneously weighted moving average control chart for monitoring of the process mean. *Quality and Reliability Engineering International, 36(5),* 1513–1527. DOI 10.1002/qre.2641.

33. Anwar, S. M., Aslam, M., Zaman, B., Riaz, M. (2021). An enhanced double homogeneously weighted moving average control chart to monitor process location with application in automobile field. *Quality and Reliability Engineering International, 38(1),* 174–194. DOI 10.1002/qre.2966.

34. Dell, T. R., Clutter, J. L. (1972). Ranked set sampling theory with order statistics background. *Biometrics, 28(2),* 545–555. DOI 10.2307/2556166.

35. Bakir, S. T. (2004). A distribution-free Shewhart quality control chart based on signed-ranks. *Quality Engineering, 16(4),* 613–623. DOI 10.1081/QEN-120038022.

36. Abbas, Z., Nazir, H. Z., Abid, M., Akhtar, N., Riaz, M. (2020). Enhanced nonparametric control charts under simple and ranked set sampling schemes. *Transactions of the Institute of Measurement and Control, 42(14),* 2744–2759. DOI 10.1177/0142331220931977.

37. Mclntyre, G. A. (1952). A method for unbiased selective sampling using ranked sets. *Australian Journal of Agricultural Research, 3(4),* 385–390. DOI 10.1071/AR9520385.

38. Takahasi, K., Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering. *Annals of the Institute of Statistical Mathematics, 20(1),* 1–31. DOI 10.1007/BF02911622.

39. Stokes, S. L. (1977). Ranked set sampling with concomitant variables. *Communications in Statistics-Theory and Methods, 6(12),* 1207–1211. DOI 10.1080/03610927708827563.

40. Kim, D. A., Kim, Y. C. (1996). Wilcoxon signed rank test using ranked-set sampling. *Korean Journal Computational and Applied Mathematics, 3(2),* 235–243. DOI 10.1007/BF03008904.

41. Montgomery, D. C. (2009). *Introduction to statistical quality control.* New York: John Willey & Sons.

42. Almanjahie, I. M., Dar, J. G., Al-Omari, A. I., Mir, A. (2021). Quantile version of Mathai-Haubold entropy of order statistics. *Computer Modeling in Engineering & Sciences, 128(3),* 907–925. DOI 10.32604/cmes.2021.014896.

43. Lucas, J. M., Crosier, R. B. (1982). Fast initial response for CUSUM quality-control schemes: Give your CUSUM a head start. *Technometrics, 24(3),* 199–205. DOI 10.1080/00401706.1982.10487759.

44. Yoon, S., MacGregor, J. (2001). Fault dignosis with multivariate statistical models part I: Using steady state fault signatures. *Journal of Process Control, 11,* 387–400. DOI 10.1016/S0959-1524(00)00008-1.

45. Shi, X., Lv, Y., Fei, Z., Liang, J. (2013). A multivariable statistical process monitoring method based on multiscale analysis and principal curves. *International Journal of Innovative Computing, Information and Control, 9,* 1781–1800.

46. Ridwan, A. S., Muhammad, R., Nurudeen, A. A., Min, X. (2017). An EWMA monitoring scheme with a single auxiliary variable for industrial processes. *Computers & Industrial Engineering, 114,* 1–10. DOI 10.1016/j.cie.2017.10.001.

47. Adegoke, N. A., Smith, A. N., Anderson, M. J., Sanusi, R. A., Pawley, M. D. (2019). Efficient homogeneously weighted moving average chart for monitoring process mean using an auxiliary variable. *IEEE Access, 7,* 94021–94032. DOI 10.1109/Access.6287639.

Appendix A

According to Kim et al. [40], for an in control process $E(SR_{(RSS)t}) = 0$ and $V(SR_{(RSS)t}) = \left(\dfrac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2$.

A.1.

$$E\left(\overline{SR}_{(RSS)t} \right) = E\left(\frac{1}{t} \sum_{k=1}^{t} SR_{(RSS)k} \right) = \frac{1}{t} \left[E\left(SR_{(RSS)1} \right) + E\left(SR_{(RSS)2} \right) + \cdots + E\left(SR_{(RSS)t} \right) \right]$$

$$= \frac{1}{t} \left[0 + 0 + \cdots + 0 \right] = \frac{1}{t} \left[0 \right] = 0$$

A.2.

$$V\left(\overline{SR}_{(RSS)t-1} \right) = V\left(\frac{1}{t-1} \sum_{k=1}^{t-1} SR_{(RSS)k} + 2 \sum_{k<j}^{t-1} Cov\left(SR_{(RSS)k}, SR_{(RSS)j} \right) \right)$$

$SR_{(RSS)k}$ and $SR_{(RSS)j}$ are the values from a random sample, so these values are independent of each other. Hence, $Covar\left(SR_{(RSS)k}, SR_{(RSS)j} \right) = 0$

$$V\left(\overline{SR}_{(RSS)t-1} \right) = \frac{1}{(t-1)^2} \left[V\left(SR_{(RSS)1} \right) + V\left(SR_{(RSS)2} \right) + \cdots + V\left(SR_{(RSS)t-1} \right) \right]$$

$$= \frac{1}{(t-1)^2} \left[\left(\frac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2 + \left(\frac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2 + \cdots + \left(\frac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2 \right]$$

$$= \frac{1}{(t-1)^2} \left[(t-1) \left(\frac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2 \right] = \frac{1}{t-1} \left(\frac{r\,(r+1)\,(2r+1)}{6} \right) \varpi_0^2$$

A.3.

For an in control process $Covar\left(SR_{(RSS)t}, \overline{SR}_{(RSS)t-1}\right) = 0$

$$Covar\left(SR_{(RSS)t}, \overline{SR}_{(RSS)t-1}\right) = Covar\left(SR_{(RSS)t}, \frac{SR_{(RSS)1} + SR_{(RSS)2} + \cdots + SR_{(RSS)t-1}}{t-1}\right)$$

$$= Covar\left(SR_{(RSS)t}, \frac{SR_{(RSS)1}}{t-1}\right) + Covar\left(SR_{(RSS)t}, \frac{SR_{(RSS)2}}{t-1}\right) + \cdots + Covar\left(SR_{(RSS)t}, \frac{SR_{(RSS)t-1}}{t-1}\right)$$

$$= \frac{1}{t-1} Covar\left(SR_{(RSS)t}, SR_{(RSS)1}\right) + \frac{1}{t-1} Covar\left(SR_{(RSS)t}, SR_{(RSS)2}\right) + \cdots + \frac{1}{t-1} Covar\left(SR_{(RSS)t}, SR_{(RSS)t-1}\right)$$

$$= \frac{1}{t-1}\left[Covar\left(SR_{(RSS)t}, SR_{(RSS)1}\right) + Covar\left(SR_{(RSS)t}, SR_{(RSS)2}\right) + \cdots + Covar\left(SR_{(RSS)t}, SR_{(RSS)t-1}\right)\right]$$

$SR_{(RSS)1}$, $SR_{(RSS)2}$, ..., $SR_{(RSS)t-1}$ denotes the first $(t-1)$ samples and all these samples are independent of the next (i.e., t^{th}\right) sample, this implies that

$$Covar\left(SR_{(RSS)t}, SR_{(RSS)1}\right) = Covar\left(SR_{(RSS)t}, SR_{(RSS)2}\right) = \cdots = Covar\left(SR_{(RSS)t}, SR_{(RSS)t-1}\right) = 0$$

Hence

$$Covar(SR_{(RSS)t}, \overline{SR}_{(RSS)t-1}) = \frac{1}{t-1}[0 + 0 + \cdots + 0] = 0$$

Appendix B

B.1.

The mean of NPH_t, i.e., $E\left(NPH_t\right)$ for an in control process is derived as

$$E(NPH_t) = E\left(\eta SR_{(RSS)t} + (1-\eta)\overline{SR}_{(RSS)t-1}\right)$$

$$= \eta E\left(SR_{(RSS)t}\right) + (1-\eta) E\left(\overline{SR}_{(RSS)t-1}\right) = \eta \cdot 0 + (1-\eta) \cdot 0 = 0$$

$$E\left(NPH_t\right) = 0$$

B.2.

The variance of NPH_t i.e., $V\left(NPH_t\right)$ for an in control process is derived as

$$V(NPH_t) = V\left(\eta SR_{(RSS)t} + (1-\eta)\overline{SR}_{(RSS)t-1}\right)$$

$$= \eta^2 V\left(SR_{(RSS)t}\right) + (1-\eta)^2 V\left(\overline{SR}_{(RSS)t-1}\right) + \eta(1-\eta) Covar\left(SR_{(RSS)t}, \overline{SR}_{(RSS)t-1}\right)$$

$$V\left(NPH_t\right) = \eta^2 V\left(SR_{(RSS)t}\right) + (1-\eta)^2 V\left(\overline{SR}_{(RSS)t-1}\right)$$

If $t = 1$ then

$$V(NPH_1) = \eta^2 V(SR_{(RSS)1}) + (1-\eta)^2 V(\overline{SR}_{(RSS)0})$$

As $\overline{SR}_{(RSS)0} = 0$, so

$$V(NPH_1) = \eta^2 V(SR_{(RSS)1}) = \eta^2\left(\frac{r(r+1)(2r+1)}{6}\right)\varpi_0^2$$

If $t > 1$ then

$$V\left(NPH_t\right) = \eta^2 V\left(SR_{(RSS)t}\right) + (1 - \eta)^2 V\left(\overline{SR}_{(RSS)t-1}\right)$$

$$= \eta^2 \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2 + (1 - \eta)^2 \frac{1}{t - 1} \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2$$

$$= \left(\eta^2 + \frac{(1 - \eta)^2}{t - 1}\right) \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2$$

$$= \eta^2 \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2 + (1 - \eta)^2 \frac{1}{t - 1} \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2$$

$$= \left(\eta^2 + \frac{(1 - \eta)^2}{t - 1}\right) \left(\frac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2$$

Hence

$$V\left(NPH_t\right) = \begin{cases} \eta^2 \left(\dfrac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2, & \text{if } t = 1 \\ \left(\eta^2 + \dfrac{(1 - \eta)^2}{t - 1}\right) \left(\dfrac{r\left(r + 1\right)\left(2r + 1\right)}{6}\right) \varpi_0^2, & \text{if } t > 1 \end{cases}$$