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Aggregation Operators for Interval-Valued Pythagorean Fuzzy Hypersoft Set with Their Application to Solve MCDM Problem

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ABSTRACT

Experts use Pythagorean fuzzy hypersoft sets (PFHSS) in their investigations to resolve the indeterminate and imprecise information in the decision-making process. Aggregation operators (AOs) perform a leading role in perceptivity among two circulations of prospect and pull out concerns from that perception. In this paper, we extend the concept of PFHSS to interval-valued PFHSS (IVPFHSS), which is the generalized form of interval-valued intuitionistic fuzzy soft set. The IVPFHSS competently deals with uncertain and ambagious information compared to the existing interval-valued Pythagorean fuzzy soft set. It is the most potent method for amplifying fuzzy data in the decision-making (DM) practice. Some operational laws for IVPFHSS have been proposed. Based on offered operational laws, two inventive AOs have been established: interval-valued Pythagorean fuzzy hypersoft weighted average (IVPFHSWA) and interval-valued Pythagorean fuzzy hypersoft weighted geometric (IVPFHSWG) operators with their essential properties. Multi-criteria group decision-making (MCGDM) shows an active part in contracts with the difficulties in industrial enterprise for material selection. But, the prevalent MCGDM approaches consistently carry irreconcilable consequences. Based on the anticipated AOs, a robust MCGDM technique is deliberate for material selection in industrial enterprises to accommodate this shortcoming. A real-world application of the projected MCGDM method for material selection (MS) of cryogenic storing vessels is presented. The impacts show that the intended model is more effective and reliable in handling imprecise data based on IVPFHSS.

KEYWORDS

Interval-valued pythagorean fuzzy hypersoft set; IVPFHSWA operator; IVPFHSWG operator; MCGDM

1 Introduction

MCGDM is deliberated as the most suitable method for a verdict on the adequate alternative from all possible choices, following criteria or attributes. Most decisions are taken when the intentions and confines are usually unspecified or unclear in real-life circumstances. Zadeh offered the idea of the fuzzy set (FS) [1] to overcome such vague and indeterminate facts. It is a fundamental tool to handle the insignificances and hesitations in decision-making (DM). The existing FS cannot deal with the scenarios when the experts consider a membership degree (MD) in intervals form during the DM procedure. Turksen [2] presented the interval-valued FS (IVFS) with fundamental operations. The prevailing FS and IVFS cannot deliver the information about any alternative's non-membership degree (NMD). Atanassov [3] overcame the mentioned above limitations and developed the intuitionistic fuzzy set (IFS). Wang et al. [4] introduced several operations such as Einstein product, Einstein sum, etc., and AOs for IFS. Atanassov [5] prolonged the IFS to an interval-valued intuitionistic fuzzy set (IVIFS) with some basic operations and their properties. Garg et al. [6] protracted the idea of IFS and settled the cubic intuitionistic fuzzy set (CIFS).

The models mentioned above have been well-recognized by the specialists. Still, the existing IFS cannot handle the inappropriate and vague data because it envisions the linear inequality among the MD and NMD. For example, if decision-makers choose MD and NMD 0.6 and 0.7, respectively, then the IFS, as mentioned earlier, cannot deal with it because $0.6 + 0.7 \geq 1$. Yager [7] offered the Pythagorean fuzzy set (PFS) to resolve the inadequacy mentioned above by modifying the elementary state $\kappa + \delta \leq 1$ to $\kappa^2 + \delta^2 \leq 1$. He also established the score and accuracy functions to compute the ranking. Rahman et al. [8] planned Einstein weighted geometric operator for PFS and showed a multi-attribute group decision-making (MAGDM) technique using their planned operator. Zhang et al. [9] developed some basic operational laws and prolonged the approach for order of preference by similarity to ideal solution (TOPSIS) to resolve multi-criteria decision-making (MCDM) problems for PFS. Wei et al. [10] offered the Pythagorean fuzzy power AOs and discussed their important features. Using their presented operators, they also established a DM technique to resolve multi-attribute decision-making (MADM). Wang et al. [11] demonstrated the interaction operational laws for Pythagorean fuzzy numbers (PFNs) and settled power Bonferroni mean operators. IIbahar et al. [12] offered the Pythagorean fuzzy proportional risk assessment technique to assess the professional health risk. Zhang [13] proposed a novel DM approach based on similarity measures to resolve MCGDM problems for the PFS. Peng et al. [14] offered the AOs for interval-valued PFS (IVPFS) and established a DM technique using their planned methodology. Rahman et al. [15] prolonged the weighted geometric aggregation operator for IVPFS and offered a DM technique based on their developed operator.

All of the above techniques have broad applications, but these theories have some limitations on parametric chemistry due to their ineffectiveness. Molodtsov [16] introduced the soft sets (SS) theory and defined some basic operations with their features to handle the misperception and haziness. Maji et al. [17] extended the theory of SS and developed many basic and binary operations for SS. Maji et al. [18] developed the fuzzy soft set with some desirable properties by merging two existing notions, FS and SS. Maji et al. [19] protracted the intuitionistic fuzzy soft set (IFSS) and some important operations with their essential properties. Arora et al. [20] presented the AOs for IFSS and planned a DM technique based on their developed operators. Jiang et al. [21] introduced the interval-valued IFSS (IVIFSS) and discussed its basic properties. Zulqarnain et al. [22] planned the TOPSIS technique based on the correlation coefficient (CC) for IVIFSS to resolve MADM problems. Peng et al. [23] anticipated the Pythagorean fuzzy soft sets (PFSS) by merging two prevailing theories, PFS and SS. Zulqarnain et al. [24] presented some operational laws for PFSS and prolonged the AOs and interaction AOs for PFSS. Zulqarnain et al. [25] developed the operational interaction laws for

PFSS and protracted the interaction AOs based on established operational laws. They also established the DM methodologies using their developed AOs and interaction AOs with their application in green supplier chain management. Zulqarnain et al. [26] prolonged the Einstein-ordered operational laws for PFSS and introduced the Einstein-ordered weighted ordered geometric AO for PFSS. They also established a MAGDM technique to solve complex real-life problems. Zulqarnain et al. [27] protracted the Einstein-ordered weighted ordered average AO for PFSS and offered a DM technique based on their developed operator. Zulqarnain et al. [28] settled the TOPSIS method for PFSS using correlation coefficient and developed the MADM approach to resolve DM obstacles. Zulqarnain et al. [29] prolonged the AOs for IVPFSS and presented a MAGDM approach to solving real-life difficulties.

Samarandche [30] proposed the idea of hypersoft set (HSS), which penetrates multiple sub-attributes in the parameter function f, which is a characteristic of the cartesian product with the n attribute. Samarandche HSS is the most suitable theory compared to SS and other existing concepts because it handles the multiple sub-attributes of the considered parameters. Several HSS extensions and their DM methods have been proposed. Rahman et al. [31] developed the DM techniques based on similarity measures for the possibility IFHSS. Zulqarnain et al. [32] extended the notion of IFHSS to PFHSS with fundamental operations and their properties. Rahman et al. [33] established a DM methodology for neutrosophic HSS. Saeed et al. [34] utilized the neutrosophic hypersoft mapping to diagnose the brain tumor. Zulqarnain et al. [35] extended the TOPSIS method based on the correlation coefficient for IFHSS and used it to resolve MADM complications. Zulqarnain et al. [36] expanded the AOs under the IFHSS environment and developed a DM approach based on their presented AOs. Zulqarnain et al. [37] developed the correlation-based TOPSIS approach for PFHSS and utilized their established technique to select the most appropriate face mask. PFHSS is a hybrid intellectual structure of PFSS. An enhanced sorting process fascinates investigators to crack baffling and inadequate information. Rendering to the investigation outcomes, PFHSS plays a vital role in decision-making by collecting numerous sources into a single value. The existing AOs for PFHSS cannot cope with the situation when the information of any multi-sub attribute is given in the form of intervals. To overcome the shortcomings mentioned above, we merged the IVPFS and hypersoft set (HSS) and introduced IVPFHSS, a novel hybrid structure to cope with uncertain problems. Therefore, to inspire the current research of IVPFHSS, we will state AOs based on rough data. The core objectives of the present study are given as follows:

- The IVPFHSS capably contracts the multifaceted concerns seeing the multi sub-attributes of the deliberated parameters in the DM procedure. To preserve this benefit in concentration, we prolong the PFHSS to IVPFHSS and establish the AOs for IVPFHSS.
- The AOs for IVPFHSS are well-known attractive estimate AOs. It has been observed that the prevalent AOs aspect is unresponsive to scratch the precise finding over the DM process in some situations. To overcome these specific complications, these AOs necessary to be revised. We determine innovative operational laws for interval-valued Pythagorean fuzzy hypersoft numbers (IVPFHSNs).
- Interval-valued Pythagorean fuzzy hypersoft weighted average and geometric operators have been introduced with their necessary properties using developed operational laws.
- A novel algorithm based on the planned operators to resolve the DM problem is established to resolve MCGDM issues under the IVPFHSS scenario.
- Material selection is an imperative feature of manufacturing as it realizes the concrete conditions for all ingredients. MS is an arduous but significant step in professional development. The

manufacturer's efficiency, productivity, and eccentric will suffer due to the absence of material selections.

- A comparative analysis of advanced MCGDM technique and current methods has been presented to consider utility and superiority.

The organization of this paper is assumed as follows: the second section of this paper involves some basic notions that support us in developing the structure of the subsequent study. In Section 3, some novel operational laws for IVPFHSN have been projected. Also, in the same section, IVPFHSWA and IVPFHSWG operators have been introduced based on our developed operators' basic properties. In Section 4, an MCGDM approach has been constructed based on the proposed AOs. In the same section, a numerical example has been discussed to confirm the pragmatism of the established technique for material selection in the manufacturing industry. Furthermore, a brief comparative analysis has been delivered to confirm the competency of the developed approach in Section 5.

2 Preliminaries

This section contains some basic definitions that will structure the following work.

Definition 2.1. [16] Let U and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively. Let $\mathcal{P}(U)$ be the power set of U and $A \subseteq \mathbb{N}$. A pair (Ω, A) is called a SS over U, and its mapping is expressed as follows:

$\Omega\colon A \to \mathcal{P}(U)$

Also, it can be defined as follows:

$$(\boldsymbol{\Omega}, \boldsymbol{A}) = \{\boldsymbol{\Omega}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{U}) : \boldsymbol{t} \in \mathbb{N}, \boldsymbol{\Omega}(\boldsymbol{t}) = \boldsymbol{\emptyset}\ \boldsymbol{if}\ \boldsymbol{t} \notin \boldsymbol{A}\}$$

Definition 2.2. [30] Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3, \ldots, t_n\}$, $(n \geq 1)$ and T_i represented the set of attributes and their corresponding sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1, 2, 3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathrm{N}$. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \dddot{A})$ is known as HSS and defined as follows:

$\Omega\colon T_1 \times T_2 \times T_3 \times \ldots; \times T_n = \dddot{A} \to \mathcal{P}(U)$.

It is also defined as

$$(\Omega, \dddot{A}) = \left\{\check{d}, \Omega_{\dddot{A}}\left(\check{d}\right) : \check{d} \in \dddot{A}, \Omega_{\dddot{A}}\left(\check{d}\right) \in \mathcal{P}(U)\right\}.$$

Definition 2.3. [14] U be a universe of discourse, and A be any subset of U. Then, the interval-valued Pythagorean fuzzy set (IVPFS) A over U is defined as

$$A = \left\{\left(x, \left(\left[\kappa_A^l(t), \kappa_A^u(t)\right], \left[\delta_A^l(t), \delta_A^u(t)\right]\right)\right) | t \in U\right\}$$

where, $\left[\kappa_A^l(t), \kappa_A^u(t)\right]$ and $\left[\delta_A^l(t), \delta_A^u(t)\right]$ represent the MD and NMD intervals, respectively. Also, $\kappa_A^l(t), \kappa_A^u(t), \delta_A^l(t), \delta_A^u(t) \in [0, 1]$. and satisfied the subsequent condition $0 \leq \left(\kappa_A^u(t)\right)^2 + \left(\delta_A^u(t)\right)^2 \leq 1$.

Definition 2.4 [29] U be a universe of discourse and $\mathbb{N}$ be a set of attributes. Then a pair $(\Omega, \mathbb{N})$ is called an interval-valued Pythagorean fuzzy soft set (IVPFSS) over U. Its mapping can be expressed as

$\Omega\colon \mathbb{N} \to \wp K^U$

where $\wp K^U$ represents the collection of interval-valued Pythagorean fuzzy subsets of the universe of discourse U.

$(\Omega, \mathbb{N}) = \left\{x, (\left[\kappa_A^l(t), \kappa_A^u(t)\right], \left[\delta_A^l(t), \delta_A^u(t)\right]) | t \in A\right\}$

where, $\left[\kappa_A^l(t), \kappa_A^u(t)\right], \left[\delta_A^l(t), \delta_A^u(t)\right]$ represent the MD and NMD intervals, respectively. Also, $\kappa_A^l(t), \kappa_A^u(t), \delta_A^l(t), \delta_A^u(t) \in [0, 1]$. And satisfied the subsequent condition $0 \leq \left(\kappa_A^u(t)\right)^2 + \left(\delta_A^u(t)\right)^2 \leq 1$ and $A \subset \mathbb{N}$.

Definition 2.5. [30] Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3, \ldots, t_n\}$,(n $\geq$ 1) and T_i represented the set of attributes and their corresponding sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathrm{N}$. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \dddot{A})$ is known as IFHSS and defined as follows:

$\Omega : T_1 \times T_2 \times T_3 \times \ldots; \times T_n = \dddot{A} \rightarrow IFS^U$.

It is also defined as $(\Omega, \dddot{A}) = \left\{\left(\check{d}, \Omega_{\dddot{A}}\left(\check{d}\right)\right) : \check{d} \in \dddot{A}, \Omega_{\dddot{A}}\left(\check{d}\right) \in IFS^U \in [0, 1]\right\}$, where $\Omega_{\dddot{A}}\left(\check{d}\right) = \left\{\left\langle\zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\right\rangle : \zeta \in U\right\}$, where $\kappa_{\Omega(\check{d})}(\zeta)$ and $\delta_{\Omega(\check{d})}(\zeta)$ represents the membership and non-membership values, respectively, such as $\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \in [0, 1]$, and $0 \leq \kappa_{\Omega(\check{d})}(\zeta) + \delta_{\Omega(\check{d})}(\zeta) \leq 1$.

Definition 2.6 [32] Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3, \ldots, t_n\}$,(n $\geq$ 1) and T_i represented the set of attributes and their corresponding sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathrm{N}$. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \dddot{A})$ is known as PFHSS and defined as follows:

$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} \rightarrow PFS^U$.

It is also defined as $(\Omega, \dddot{A}) = \left\{\left(\check{d}, \Omega_{\dddot{A}}\left(\check{d}\right)\right) : \check{d} \in \dddot{A}, \Omega_{\dddot{A}}\left(\check{d}\right) \in PFS^U \in [0, 1]\right\}$, where $\Omega_{\dddot{A}}\left(\check{d}\right) = \left\{\left\langle\zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\right\rangle : \zeta \in U\right\}$, where $\kappa_{\Omega(\check{d})}(\zeta)$ and $\delta_{\Omega(\check{d})}(\zeta)$ represents the MD and NMD, respectively, such as $\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \in [0, 1]$, and $0 \leq \left(\kappa_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\delta_{\Omega(\check{d})}(\zeta)\right)^2 \leq 1$.

The PFHSN is stated as $\mathcal{F} = \left\{\left(\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\right)\right\}$.

3 Aggregation Operators for Interval Valued Pythagorean Fuzzy Hypersoft Sets

In this section, we will extend the idea of IVPFSS to interval-valued Pythagorean fuzzy hypersoft sets (IVPFHSS) with some fundamental notions and introduce the operational laws for IVPFHSNs. We propose interval-valued Pythagorean fuzzy hypersoft weighted average (IVPFHSWA) and interval-valued Pythagorean fuzzy hypersoft geometric (IVPFHSWG) operators using the developed operational laws.

Definition 3.1. Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3, \ldots, t_n\}$,(n $\geq$ 1) and T_i represented the set of attributes and their corresponding sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \epsilon$ {1,2,3 ... n}. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in$ N. Then the pair ($\mathcal{F}$, $T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \dddot{A})$ is known as IVPFHSS and defined as follows:

$\Omega : T_1 \times T_2 \times T_3 \times \ldots \times T_n = \dddot{A} \to IVPFHS^U$.

It is also defined as

$(\Omega, \dddot{A}) = \left\{\left(\check{d}, \Omega_{\dddot{A}}\left(\check{d}\right)\right) : \check{d} \in \dddot{A}, \Omega_{\dddot{A}}\left(\check{d}\right) \in IVPFS^U \in [0,1]\right\}$, where $\Omega_{\dddot{A}}\left(\check{d}\right) = \left\{\left\langle \zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\right\rangle : \zeta \in U\right\}$, and $\kappa_{\Omega(\check{d})}(\zeta) = \left[\kappa^l_{\Omega(\check{d})}(\zeta), \kappa^u_{\Omega(\check{d})}(\zeta)\right]$, $\delta_{\Omega(\check{d})}(\zeta) = \left[\delta^l_{\Omega(\check{d})}(\zeta), \delta^u_{\Omega(\check{d})}(\zeta)\right]$, where $\kappa_{\Omega(\check{d})}(\zeta)$ and $\delta_{\Omega(\check{d})}(\zeta)$ represents the membership and non-membership intervals, respectively, such as $\kappa^l_{\Omega(\check{d})}(\zeta), \kappa^u_{\Omega(\check{d})}(\zeta), \delta^l_{\Omega(\check{d})}(\zeta), \delta^u_{\Omega(\check{d})}(\zeta) \in [0,1]$, and $0 \leq \left(\kappa^u_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\delta^u_{\Omega(\check{d})}(\zeta)\right)^2 \leq 1$.

The IVPFHSN can be stated as $\mathcal{F} = \left(\left[\kappa^l_{\Omega(\check{d})}(\zeta), \kappa^u_{\Omega(\check{d})}(\zeta)\right], \left[\delta^l_{\Omega(\check{d})}(\zeta), \delta^u_{\Omega(\check{d})}(\zeta)\right]\right)$.

The score and accuracy functions have been presented to compute the alternative ranking for IVPFHSS can be stated as, if $\mathcal{F} = \left(\left[\kappa^l_{\Omega(\check{d})}(\zeta), \kappa^u_{\Omega(\check{d})}(\zeta)\right], \left[\delta^l_{\Omega(\check{d})}(\zeta), \delta^u_{\Omega(\check{d})}(\zeta)\right]\right)$ be an IVPFHSN. Then

$$S(\mathcal{F}) = \frac{\left(\kappa^l_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\kappa^u_{\Omega(\check{d})}(\zeta)\right)^2 - \left(\delta^l_{\Omega(\check{d})}(\zeta)\right)^2 - \left(\delta^u_{\Omega(\check{d})}(\zeta)\right)^2}{2}$$

And

$$A(\mathcal{F}) = \frac{\left(\kappa^l_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\kappa^u_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\delta^l_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\delta^u_{\Omega(\check{d})}(\zeta)\right)^2}{2}$$

Definition 3.2. Let $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right], \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]\right)$, $\mathcal{F}_{\check{d}_{11}} = \left(\left[\kappa^l_{\check{d}_{11}}, \kappa^u_{\check{d}_{11}}\right], \left[\delta^l_{\check{d}_{11}}, \delta^u_{\check{d}_{11}}\right]\right)$, and

$\mathcal{F}_{\check{d}_{12}} = \left(\left[\kappa^l_{\check{d}_{12}}, \kappa^u_{\check{d}_{12}}\right], \left[\delta^l_{\check{d}_{12}}, \delta^u_{\check{d}_{12}}\right]\right)$ be three IVPFHSNs and β be a positive real number, and by algebraic norms, we have

1. $\mathcal{F}_{\check{d}_{11}} \oplus \mathcal{F}_{\check{d}_{12}} = \left(\left[\sqrt{\kappa^{l\ 2}_{\check{d}_{11}} + \kappa^{l\ 2}_{\check{d}_{12}} - \kappa^{l\ 2}_{\check{d}_{11}} \kappa^{l\ 2}_{\check{d}_{12}}}, \sqrt{\kappa^{u\ 2}_{\check{d}_{11}} + \kappa^{u\ 2}_{\check{d}_{12}} - \kappa^{u\ 2}_{\check{d}_{11}} \kappa^{u\ 2}_{\check{d}_{12}}}\right], \left[\delta^l_{\check{d}_{11}} \delta^l_{\check{d}_{12}}, \delta^u_{\check{d}_{11}} \delta^u_{\check{d}_{12}}\right]\right)$
2. $\mathcal{F}_{\check{d}_{11}} \otimes \mathcal{F}_{\check{d}_{12}} = \left(\left[\kappa^l_{\check{d}_{11}} \kappa^l_{\check{d}_{12}}, \kappa^u_{\check{d}_{11}} \kappa^u_{\check{d}_{12}}\right], \left[\sqrt{\delta^{l\ 2}_{\check{d}_{11}} + \delta^{l\ 2}_{\check{d}_{12}} - \delta^{l\ 2}_{\check{d}_{11}} \delta^{l\ 2}_{\check{d}_{12}}}, \sqrt{\delta^{u\ 2}_{\check{d}_{11}} + \delta^{u\ 2}_{\check{d}_{12}} - \delta^{u\ 2}_{\check{d}_{11}} \delta^{u\ 2}_{\check{d}_{12}}}\right]\right)$
3. $\beta \mathcal{F}_{\check{d}_k} =$
$\left(\left[\sqrt{1 - \left(1 - \kappa^{l\ 2}_{\check{d}_k}\right)^\beta}, \sqrt{1 - \left(1 - \kappa^{u\ 2}_{\check{d}_k}\right)^\beta}\right], \left[\delta^{l\ \beta}_{\check{d}_k}, \delta^{u\ \beta}_{\check{d}_k}\right]\right) = \left(\sqrt{1 - \left(1 - \left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right]^2\right)^\beta}, \left[\delta^{l\ \beta}_{\check{d}_k}, \delta^{u\ \beta}_{\check{d}_k}\right]\right)$
4. $\mathcal{F}^\beta_{\check{d}_k} =$
$\left(\left[\kappa^{l\ \beta}_{\check{d}_k}, \kappa^{u\ \beta}_{\check{d}_k}\right], \left[\sqrt{1 - \left(1 - \delta^{l\ 2}_{\check{d}_k}\right)^\beta}, \sqrt{1 - \left(1 - \delta^{u\ 2}_{\check{d}_k}\right)^\beta}\right]\right) = \left(\left[\kappa^{l\ \beta}_{\check{d}_k}, \kappa^{u\ \beta}_{\check{d}_k}\right], \sqrt{1 - \left(1 - \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]^2\right)^\beta}\right)$.

Definition 3.3. Let $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right], \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]\right)$ be a collection of IVPFHSNs, and ω_i and ν_j are the weight vector for experts and multi sub-parameters, respectively, with given conditions $\omega_i > 0, \sum_{i=1}^n \omega_i = 1; \nu_j > 0, \sum_{j=1}^m \nu_j = 1$. Then, the IVPFHSWA operator is defined as IVPFHSWA: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFHSWA}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \oplus_{j=1}^m \nu_j \left(\oplus_{i=1}^n \omega_i \mathcal{F}_{\check{d}_{ij}}\right)$$

Theorem 3.1. Let $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be a collection of IVPFHSNs, where ($i = 1, 2, 3 \ldots, n$ and $j = 1, 2, 3, \ldots, m$), and the aggregated value is also an IVPFHSN, such as

$$\text{IVPFHSWA}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right)$$

$$= \left(\sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^m \left(\prod_{i=1}^n \left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

ω_i and ν_j show the expert's and multi sub-attributes weights, respectively, such as $\omega_i > 0, \sum_{i=1}^n \omega_i = 1, \nu_j > 0, \sum_{j=1}^m \nu_j = 1$.

Proof. The above presented IVPFHSWA operator can be proved by using the principle of mathematical induction:

For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\text{IVPFHSWA}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \oplus_{j=1}^m \nu_j \mathcal{F}_{\check{d}_{1j}}$$

$$\text{IVPFHSWA}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \left(\sqrt{1 - \prod_{j=1}^m \left(1 - \left[\kappa^l_{\check{d}_{1j}}, \kappa^u_{\check{d}_{1j}}\right]^2\right)^{\nu_j}}, \prod_{j=1}^m \left(\left[\delta^l_{\check{d}_{1j}}, \delta^u_{\check{d}_{1j}}\right]\right)^{\nu_j}\right)$$

$$= \left(\sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^m \left(\prod_{i=1}^1 \left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right).$$

For $m = 1$, we get $\nu_1 = 1$. Then, we have

$$\text{IVPFHSWA}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \oplus_{i=1}^n \omega_i \mathcal{F}_{\check{d}_{i1}}$$

$$= \left(\sqrt{1 - \prod_{i=1}^n \left(1 - \left[\kappa^l_{\check{d}_{i1}}, \kappa^u_{\check{d}_{i1}}\right]^2\right)^{\omega_i}}, \prod_{i=1}^n \left(\left[\delta^l_{\check{d}_{i1}}, \delta^u_{\check{d}_{i1}}\right]\right)^{\omega_i}\right)$$

$$= \left(\sqrt{1 - \prod_{j=1}^1 \left(\prod_{i=1}^n \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^1 \left(\prod_{i=1}^n \left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

So, the above theorem is proved for $n = 1$ and $m = 1$.

Assume that for $m = \alpha_1 + 1, n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$, the above theorem holds. Such as

$$\oplus_{j=1}^{\alpha_1+1} \nu_j \left(\oplus_{i=1}^{\alpha_2} \omega_i \mathcal{F}_{\check{d}_{ij}}\right) = \left(\sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$\oplus_{j=1}^{\alpha_1}\nu_j\left(\oplus_{i=1}^{\alpha_2+1}\omega_i\mathcal{F}_{\check{d}_{ij}}\right)=\left(\sqrt{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\kappa_{\check{d}_{ij}}^{l},\kappa_{\check{d}_{ij}}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\delta_{\check{d}_{ij}}^{l},\delta_{\check{d}_{ij}}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

For $m=\alpha_1+1$ and $n=\alpha_2+1$, we have

$$\oplus_{j=1}^{\alpha_1+1}\nu_j\left(\oplus_{i=1}^{\alpha_2+1}\omega_i\mathcal{F}_{\check{d}_{ij}}\right)=\oplus_{j=1}^{\alpha_1+1}\nu_j\left(\oplus_{i=1}^{\alpha_2}\omega_i\mathcal{F}_{\check{d}_{ij}}\oplus\omega_{\alpha_2+1}\mathcal{F}_{\check{d}_{(\alpha_2+1)j}}\right)$$

$$=\oplus_{j=1}^{\alpha_1+1}\oplus_{i=1}^{\alpha_2}\nu_j\omega_i\mathcal{F}_{\check{d}_{ij}}\oplus_{j=1}^{\alpha_1+1}\nu_j\omega_{\alpha_2+1}\mathcal{F}_{\check{d}_{(\alpha_2+1)j}}$$

$$=\left(\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\kappa_{\check{d}_{ij}}^{l},\kappa_{\check{d}_{ij}}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\oplus\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\left(1-\left[\kappa_{\check{d}_{(\alpha_2+1)j}}^{l},\kappa_{\check{d}_{(\alpha_2+1)j}}^{u}\right]^2\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}},\right.$$

$$\left.\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(\left[\delta_{\check{d}_{ij}}^{l},\delta_{\check{d}_{ij}}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\oplus\prod_{j=1}^{\alpha_1+1}\left(\left(\left[\delta_{\check{d}_{(\alpha_2+1)j}}^{l},\delta_{\check{d}_{(\alpha_2+1)j}}^{u}\right]\right)^{\omega(\alpha_2+1)}\right)^{\nu_j}\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\kappa_{\check{d}_{ij}}^{l},\kappa_{\check{d}_{ij}}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\delta_{\check{d}_{ij}}^{l},\delta_{\check{d}_{ij}}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

Hence, it holds for $m=\alpha_1+1$ and $n=\alpha_2+1$. So, we can say that Theorem 3.1 holds for all values of m and n.

Example. 3.1

Let $\mathcal{R}=\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}$ be a set of experts with the given weight vector $\omega_i=(0.38,0.45,0.17)^T$. The group of experts describes the beauty of a house under-considered attributes $\text{Å}=\{e_1=\textit{lawn},e_2=\textit{security system}\}$ with their corresponding sub-attributes Lawn $=e_1=\{e_{11}=\textit{with grass},e_{12}=\textit{without grass}\}$, security system $=e_2=\{e_{21}=\textit{guards},e_{22}=\textit{cameras}\}$. Let $\text{Å}=e_1\times e_2$ be a set of sub-attributes $\text{Å}=e_1\times e_2=\{e_{11},e_{12}\}\times\{e_{21},e_{22}\}=\{(e_{11},e_{21}),(e_{11},e_{22}),(e_{12},e_{21}),(e_{12},e_{22})\}$

$\text{Å}=\{\check{d}_1,\check{d}_2,\check{d}_3,\check{d}_4\}$ be a set of multi sub-attributes with weights $\nu_j=(0.2,0.2,0.2,0.4)^T$. The rating values for each alternative in the form of IVPFHSN $(\mathcal{F},\text{Å})=\left(\left[\kappa_{\check{d}_{ij}}^{l},\kappa_{\check{d}_{ij}}^{u}\right],\left[\delta_{\check{d}_{ij}}^{l},\delta_{\check{d}_{ij}}^{u}\right]\right)_{3\times4}$ given as:

$$(\mathcal{F},\text{Å})=\begin{bmatrix}([0.3,0.8],[0.4,0.5]) & ([0.4,0.6],[0.3,0.7]) & ([0.5,0.8],[0.5,0.6]) & ([0.4,0.9],[0.3,0.7])\\ ([0.1,0.5],[0.2,0.3]) & ([0.3,0.8],[0.5,0.7]) & ([0.2,0.4],[0.2,0.3]) & ([0.3,0.8],[0.6,0.7])\\ ([0.2,0.9],[0.2,0.3]) & ([0.5,0.7],[0.2,0.6]) & ([0.2,0.4],[0.2,0.8]) & ([0.3,0.8],[0.5,0.8])\end{bmatrix}$$

IVPFHSWA $(\mathcal{F}_{\check{d}_{11}},\mathcal{F}_{\check{d}_{12}},\ldots,\mathcal{F}_{\check{d}_{34}})$

$$=\left(\sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\left[\kappa_{\check{d}_{ij}}^{l},\kappa_{\check{d}_{ij}}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(\left[\delta_{\check{d}_{ij}}^{l},\delta_{\check{d}_{ij}}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$= \begin{pmatrix} \sqrt{1 - \left(\begin{Bmatrix} [0.36, 0.91]^{0.38} [0.75, 0.84]^{0.45} \\ [0.19, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.38} \\ [0.36, 0.91]^{0.45} [0.51, 0.75]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.36, 0.75]^{0.38} [0.84, 0.96]^{0.45} \\ [0.84, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.19, 0.84]^{0.38} [0.36, 0.91]^{0.45} \\ [0.36, 0.91]^{0.17} \end{Bmatrix}^{0.4} \right)}, \\ \left(\begin{Bmatrix} [0.4, 0.5]^{0.38} [0.2, 0.3]^{0.45} \\ [0.2, 0.3]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.38} [0.5, 0.7]^{0.45} \\ [0.2, 0.6]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.5, 0.6]^{0.38} [0.2, 0.3]^{0.45} \\ [0.2, 0.8]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.38} [0.6, 0.7]^{0.45} \\ [0.5, 0.8]^{0.17} \end{Bmatrix}^{0.4} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left(\begin{Bmatrix} [0.36, 0.91]^{0.38} [0.75, 0.84]^{0.45} \\ [0.19, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.38} \\ [0.36, 0.91]^{0.45} [0.51, 0.75]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.36, 0.75]^{0.38} [0.84, 0.96]^{0.45} \\ [0.84, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.19, 0.84]^{0.38} [0.36, 0.91]^{0.45} \\ [0.36, 0.91]^{0.17} \end{Bmatrix}^{0.4} \right)}, \\ \left(\begin{Bmatrix} [0.4, 0.5]^{0.38} [0.2, 0.3]^{0.45} \\ [0.2, 0.3]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.38} [0.5, 0.7]^{0.45} \\ [0.2, 0.6]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.5, 0.6]^{0.38} [0.2, 0.3]^{0.45} \\ [0.2, 0.8]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.38} [0.6, 0.7]^{0.45} \\ [0.5, 0.8]^{0.17} \end{Bmatrix}^{0.4} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left(\begin{Bmatrix} [0.96, 0.68] [0.92, 0.88] \\ [0.99, 0.75] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.93, 0.84] \\ [0.95, 0.63] [0.95, 0.89] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.89, 0.68] [0.98, 0.92] \\ [0.99, 0.9] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.93, 0.53] [0.95, 0.63] \\ [0.98, 0.84] \end{Bmatrix}^{0.4} \right)}, \\ \left(\begin{Bmatrix} [0.77, 0.70] [0.58, 0.48] \\ [0.81, 0.76] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.87, 0.63] [0.85, 0.73] \\ [0.92, 0.76] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.82, 0.76] [0.58, 0.48] \\ [0.96, 0.76] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.87, 0.63] [0.85, 0.79] \\ [0.96, 0.76] \end{Bmatrix}^{0.4} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left([0.87, 0.45]^{0.2} [0.84, 0.4]^{0.2} [0.86, 0.61]^{0.2} [0.86, 0.28]^{0.4} \right)}, \\ \left([0.36, 0.25]^{0.2} [0.68, 0.34]^{0.2} [0.46, 0.28]^{0.2} [0.71, 0.38]^{0.4} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - ([0.8524, 0.9725] [0.8325, 0.9657] [0.9059, 0.9703] [0.6009, 0.9414])}, \\ ([0.7579, 0.8152] [0.8059, 0.9258] [0.7752, 0.8561] [0.6791, 0.8719]) \end{pmatrix}$$

$$= \left(\sqrt{1 - [0.3863, 0.8579]}, [0.3215, 0.5634] \right)$$

$$= \left(\sqrt{[0.6137, 0.1421]}, [0.3215, 0.5634] \right)$$

$$= [0.3769, 0.7833], [0.3215, 0.5634].$$

3.1 Properties of IVPFHSWA Operator

3.1.1 Idempotency

If $\mathcal{F}_{\check{d}_{ij}} = \mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right) \forall i, j$. Then
IVPFHSWA $\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \mathcal{F}_{\check{d}_k}$

Proof. As we know that all $\mathcal{F}_{\check{d}_{ij}} = \mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$, then, we have
IVPFHSWA $\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right)$

$$= \left(\sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$= \left(\sqrt{1 - \left(\left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}}, \left(\left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}\right)$$

As $\sum_{j=1}^{m} \nu_j = 1$ and $\sum_{i=1}^{n} \omega_i = 1$, then we have

$$= \left(\sqrt{1 - \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)}, \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$$

$$= \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$$

$$= \mathcal{F}_{\check{d}_k}.$$

3.1.2 Boundedness

Let $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be a collection of IVPFHSNs where $\mathcal{F}^-_{\check{d}_{ij}} = \left(\underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \underset{j}{max}\ \underset{i}{max}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$ and $\mathcal{F}^+_{\check{d}_{ij}} = \left(\underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \underset{j}{min}\ \underset{i}{min}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$, then
$\mathcal{F}^-_{\check{d}_{ij}} \leq$ IVPFHSWA $\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) \leq \mathcal{F}^+_{\check{d}_{ij}}$

Proof. As we know that $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be an IVPFHSN, then

$$\underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\} \leq \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2 \leq \underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Rightarrow 1 - \underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\} \leq 1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2 \leq 1 - \underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n}\omega_i} \leq \prod_{i=1}^{n}\left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n}\omega_i}$$

$$\Leftrightarrow \left(1-\max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^n v_j} \leq \prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j} \leq \left(1-\min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^n v_j}$$

$$\Leftrightarrow 1-\max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\} \leq \prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j} \leq 1-\min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\} \leq 1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j} \leq \max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\} \leq \sqrt{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j}} \leq \max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\} \quad (a)$$

Similarly,

$$\min_j\min_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\} \leq \prod_{j=1}^m\left(\prod_{i=1}^n\left(\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{v_j} \leq \max_j\max_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\} \quad (b)$$

Let IVPFHSWA$(\mathcal{F}_{\check{d}_{11}},\mathcal{F}_{\check{d}_{12}},\ldots,\mathcal{F}_{\check{d}_{nm}}) = \left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right],\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right) = \mathcal{F}_{\check{d}_{ij}}$. So, (a) and (b) can be transferred into the form:

$\min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\} \leq \mathcal{F}_{\check{d}_k} \leq \max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\}$ and $\min_j\min_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\} \leq \mathcal{F}_{\check{d}_k} \leq \max_j\max_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\}$, respectively.

Using the score function, we have

$$S\left(\mathcal{F}_{\check{d}_k}\right) = \frac{\left(\kappa^l_{\check{d}_k}\right)^2+\left(\kappa^u_{\check{d}_k}\right)^2-\left(\delta^l_{\check{d}_k}\right)^2-\left(\delta^u_{\check{d}_k}\right)^2}{2} \leq \max_j\max_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\} - \min_j\min_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\} = S\left(\mathcal{F}^-_{\check{d}_k}\right)$$

$$S\left(\mathcal{F}_{\check{d}_k}\right) = \frac{\left(\kappa^l_{\check{d}_k}\right)^2+\left(\kappa^u_{\check{d}_k}\right)^2-\left(\delta^l_{\check{d}_k}\right)^2-\left(\delta^u_{\check{d}_k}\right)^2}{2} \geq \min_j\min_i\left\{\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right\} - \max_j\max_i\left\{\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right\} = S\left(\mathcal{F}^+_{\check{d}_k}\right)$$

By order relation between two IVPFHSNs, we have

$\mathcal{F}^-_{\check{d}_k} \leq$ IVPFHSWA$(\mathcal{F}_{\check{d}_{11}},\mathcal{F}_{\check{d}_{12}},\ldots,\mathcal{F}_{\check{d}_{nm}}) \leq \mathcal{F}^+_{\check{d}_k}$.

3.1.3 Shift Invariance

Let $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k},\kappa^u_{\check{d}_k}\right],\left[\delta^l_{\check{d}_k},\delta^u_{\check{d}_k}\right]\right)$ be an IVPFHSN. Then

$$IVPFHSWA(\mathcal{F}_{\check{d}_{11}}\oplus\mathcal{F}_{\check{d}_k},\mathcal{F}_{\check{d}_{12}}\oplus\mathcal{F}_{\check{d}_k},\ldots,\mathcal{F}_{\check{d}_{nm}}\oplus\mathcal{F}_{\check{d}_k}) = IVPFHSWA(\mathcal{F}_{\check{d}_{11}},\mathcal{F}_{\check{d}_{12}},\ldots,\mathcal{F}_{\check{d}_{nm}})\oplus\mathcal{F}_{\check{d}_k}$$

Proof. As $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k},\kappa^u_{\check{d}_k}\right],\left[\delta^l_{\check{d}_k},\delta^u_{\check{d}_k}\right]\right)$ and $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right],\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right)$ be two IVPFHSNs. Then, using Definition 3.2 (1)

$$\mathcal{F}_{\check{d}_k}\oplus\mathcal{F}_{\check{d}_{ij}} = \left(\sqrt{\left[\kappa^l_{\check{d}_k},\kappa^u_{\check{d}_k}\right]+\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2-\left[\kappa^l_{\check{d}_k},\kappa^u_{\check{d}_k}\right]\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]^2},\left[\delta^l_{\check{d}_k},\delta^u_{\check{d}_k}\right]\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right),$$

So,

$$IVPFHSWA\left(\mathcal{F}_{\check{d}_{11}}\oplus\mathcal{F}_{\check{d}_k},\mathcal{F}_{\check{d}_{12}}\oplus\mathcal{F}_{\check{d}_k},\ldots,\mathcal{F}_{\check{d}_{nm}}\oplus\mathcal{F}_{\check{d}_k}\right) == \oplus_{j=1}^m v_j\left(\oplus_{i=1}^n\omega_i\left(\mathcal{F}_{\check{d}_{ij}}\oplus\mathcal{F}_{\check{d}_k}\right)\right)$$

$$= \left(\sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \left(1 - \left[\kappa_{\check{d}_k}^{l}, \kappa_{\check{d}_k}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)^{\omega_i} \left(\left[\delta_{\check{d}_k}^{l}, \delta_{\check{d}_k}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\sqrt{1 - \left(1 - \left[\kappa_{\check{d}_k}^{l}, \kappa_{\check{d}_k}^{u} \right]^2 \right) \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \left[\delta_{\check{d}_k}^{l}, \delta_{\check{d}_k}^{u} \right] \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\left(\sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right) \oplus \left(\left[\kappa_{\check{d}_k}^{l}, \kappa_{\check{d}_k}^{u} \right], \left[\delta_{\check{d}_k}^{l}, \delta_{\check{d}_k}^{u} \right] \right) \right)$$

$$= IVPFHSWA(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}) \oplus \mathcal{F}_{\check{d}_k}.$$

3.1.4 Homogeneity

Prove that IVPFHSWA $(\beta\mathcal{F}_{\check{d}_{11}}, \beta\mathcal{F}_{\check{d}_{12}}, \ldots, \beta\mathcal{F}_{\check{d}_{nm}}) = \beta$ IVPFHSWA $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}})$ for any positive real number β.

Proof. Let $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right], \left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)$ be an IVPFHSN and $\beta > 0$. Then using Definition 3.2, we have

$$\beta\mathcal{F}_{\check{d}_{ij}} = \left(\sqrt{1 - \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\beta}}, \left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right]^{\beta} \right)$$

So,

$$\left(\beta\mathcal{F}_{\check{d}_{11}}, \beta\mathcal{F}_{\check{d}_{12}}, \ldots, \beta\mathcal{F}_{\check{d}_{nm}} \right)$$

$$= \left(\sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\beta\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)^{\beta\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\sqrt{1 - \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} \right)^{\beta}}, \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)^{\beta} \right)$$

$= \beta$ IVPFHSWA $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}})$.

Definition 3.4. Let $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa_{\check{d}_k}^{l}, \kappa_{\check{d}_k}^{u} \right], \left[\delta_{\check{d}_k}^{l}, \delta_{\check{d}_k}^{u} \right] \right)$ be a collection of IVPFHSNs, and ω_i and ν_j are the weight vector for experts and multi sub-parameters, respectively, with given conditions $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1; \nu_j > 0, \sum_{j=1}^{m} \nu_j = 1$. Then, the IVPFHSWG operator is defined as IVPFHSWG: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}} \right) = \otimes_{j=1}^{m} \nu_j \left(\otimes_{i=1}^{n} \omega_i \mathcal{F}_{\check{d}_{ij}} \right)$$

Theorem 3.2. Let $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa_{\check{d}_{ij}}^{l}, \kappa_{\check{d}_{ij}}^{u} \right], \left[\delta_{\check{d}_{ij}}^{l}, \delta_{\check{d}_{ij}}^{u} \right] \right)$ be a collection of IVPFHSNs, where ($i = 1, 2, 3 \ldots\ldots, n$ and $j = 1, 2, 3, \ldots, m$), and the aggregated value is also an IVPFHSN, such as

$$\text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}} \right)$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

ω_i and ν_j represents the expert's and multi sub-attributes weights, respectively, such as $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$, $\nu_j > 0$, $\sum_{j=1}^{m} \nu_j = 1$.

Proof. Using mathematical induction, we can prove the IVPFHSWG operator as follows:

For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m} \mathcal{F}_{\check{d}_{1j}}^{\nu_j}$$

$$\text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right)$$

$$= \left(\prod_{j=1}^{m} \left(\left[\kappa^{l}_{\check{d}_{1j}}, \kappa^{u}_{\check{d}_{1j}} \right] \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(1 - \left[\delta^{l}_{\check{d}_{1j}}, \delta^{u}_{\check{d}_{1j}} \right]^2 \right)^{\nu_j}} \right)$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{1} \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right).$$

For $m = 1$, we get $\nu_1 = 1$. Then, we have

$$\text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{21}}, \ldots, \mathcal{F}_{\check{d}_{n1}}\right) = \otimes_{i=1}^{n} \left(\mathcal{F}_{\check{d}_{n1}}\right)^{\omega_i}$$

$$= \left(\prod_{i=1}^{n} \left(\left[\kappa^{l}_{\check{d}_{i1}}, \kappa^{u}_{\check{d}_{i1}} \right] \right)^{\omega_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \left[\delta^{l}_{\check{d}_{i1}}, \delta^{u}_{\check{d}_{i1}} \right]^2 \right)^{\omega_i}} \right)$$

$$= \left(\prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

So, for $n = 1$ and $m = 1$ the IVPFHSWG operators holds.

Now, for $m = \alpha_1 + 1, n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$, such as

$$\otimes_{j=1}^{\alpha_1+1} \left(\otimes_{i=1}^{\alpha_2} \left(\mathcal{F}_{\check{d}_{ij}} \right)^{\omega_i} \right)^{\nu_j}$$

$$= \left(\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}} \right] \right)^{\omega_i}, \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right) \right)$$

$$\otimes_{j=1}^{\alpha_1} \left(\otimes_{i=1}^{\alpha_2+1} \left(\mathcal{F}_{\check{d}_{ij}} \right)^{\omega_i} \right)^{\nu_j}$$

$$= \left(\prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

For $m = \alpha_1 + 1$ and $n = \alpha_2 + 1$, we have

$$\otimes_{j=1}^{\alpha_1+1} \left(\otimes_{i=1}^{\alpha_2+1} \left(\mathcal{F}_{\check{d}_{ij}} \right)^{\omega_i} \right)^{\nu_j} = \otimes_{j=1}^{\alpha_1+1} \left(\otimes_{i=1}^{\alpha_2} \left(\mathcal{F}_{\check{d}_{ij}} \right)^{\omega_i} \otimes \left(\mathcal{F}_{\check{d}_{(\alpha_2+1)j}} \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}$$

$$= \otimes_{j=1}^{\alpha_1+1} \otimes_{i=1}^{\alpha_2} \left(\left(\mathcal{F}_{\check{d}_{ij}} \right)^{\omega_i} \right)^{\nu_j} \otimes_{j=1}^{\alpha_1+1} \left(\left(\mathcal{F}_{\check{d}_{(\alpha_2+1)j}} \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}$$

$$=\left(\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\otimes\prod_{j=1}^{\alpha_1+1}\left(\left(\left[\kappa^l_{\check{d}_{(\alpha_2+1)j}},\kappa^u_{\check{d}_{(\alpha_2+1)j}}\right]\right)^{\omega_{(\alpha_2+1)}}\right)^{\nu_j},\right.$$

$$\left.\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\otimes\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\left(1-\left[\delta^l_{\check{d}_{(\alpha_2+1)j}},\delta^u_{\check{d}_{(\alpha_2+1)j}}\right]^2\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}}\right)$$

$$=\left(\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j},\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

So, it is proved the for $m=\alpha_1+1$ and $n=\alpha_2+1$ holds. So, the IVPFHSWG operator holds for all values of m and n.

Example 3.2. Let $\mathcal{R}=\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}$ be a set of experts with the given weight vector $\omega_i=(0.38,0.45,0.17)^T$. The group of experts describes the beauty of a house under-considered attributes $\text{Å}=\{e_1=lawn, e_2=security\ system\}$ with their corresponding sub-attributes Lawn $=e_1=\{e_{11}=with\ grass, e_{12}=without\ grass\}$, security system $=e_2=\{e_{21}=guards, e_{22}=cameras\}$. Let $\text{Å}=e_1\times e_2$ be a set of sub-attributes

$\text{Å}=e_1\times e_2=\{e_{11},e_{12}\}\times\{e_{21},e_{22}\}=\left\{(e_{11},e_{21}),(e_{11},e_{22}),(e_{12},e_{21}),(e_{12},e_{22})\right\}$

$\text{Å}=\left\{\check{d}_1,\check{d}_2,\check{d}_3,\check{d}_4\right\}$ be a set of multi sub-attributes with weights $\nu_j=(0.2,0.2,0.2,0.4)^T$. The rating values for each alternative in the form of IVPFHSN $(\mathcal{F},\text{Å})=\left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right],\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]\right)_{3\times4}$ given as:

$$(\mathcal{F},\text{Å})=\begin{bmatrix}([0.3,0.8],[0.4,0.5]) & ([0.4,0.6],[0.3,0.7]) & ([0.5,0.8],[0.5,0.6]) & ([0.4,0.9],[0.3,0.7])\\ ([0.1,0.5],[0.2,0.3]) & ([0.3,0.8],[0.5,0.7]) & ([0.2,0.4],[0.2,0.3]) & ([0.3,0.8],[0.6,0.7])\\ ([0.2,0.9],[0.2,0.3]) & ([0.5,0.7],[0.2,0.6]) & ([0.2,0.4],[0.2,0.8]) & ([0.3,0.8],[0.5,0.8])\end{bmatrix}$$

By using the above theorem, we have

IVPFHSWG $\left(\mathcal{F}_{\check{d}_{11}},\mathcal{F}_{\check{d}_{12}},\ldots,\mathcal{F}_{\check{d}_{34}}\right)$

$$=\left(\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(\left[\kappa^l_{\check{d}_{ij}},\kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j},\sqrt{1-\prod_{j=1}^{3}\left(\prod_{i=1}^{4}\left(1-\left[\delta^l_{\check{d}_{ij}},\delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left\{\begin{array}{c}[0.3,0.8]^{0.38}[0.1,0.5]^{0.45}\\ [0.2,0.9]^{0.17}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.4,0.6]^{0.38}[0.3,0.8]^{0.45}\\ [0.5,0.7]^{0.17}\end{array}\right\}^{0.2}\\ \left\{\begin{array}{c}[0.5,0.8]^{0.38}[0.2,0.4]^{0.45}\\ [0.2,0.4]^{0.17}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.4,0.9]^{0.38}[0.3,0.8]^{0.45}\\ [0.3,0.8]^{0.17}\end{array}\right\}^{0.4}\end{array}\right),\\ \sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.75,0.84]^{0.38}[0.91,0.96]^{0.45}\\ [0.91,0.96]^{0.17}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.51,0.91]^{0.38}\\ [0.51,0.75]^{0.45}[0.64,0.96]^{0.17}\end{array}\right\}^{0.2}\\ \left\{\begin{array}{c}[0.64,0.75]^{0.38}[0.91,0.96]^{0.45}\\ [0.36,0.96]^{0.17}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.51,0.91]^{0.38}[0.51,0.64]^{0.45}\\ [0.36,0.75]^{0.17}\end{array}\right\}^{0.4}\end{array}\right)}\end{array}\right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.92, 0.63]^{0.38} [0.73, 0.35]^{0.45} \\ [0.98, 0.76]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.82, 0.70]^{0.38} [0.90, 0.38]^{0.45} \\ [0.94, 0.89]^{0.17} \end{Bmatrix}^{0.2} \\ \begin{Bmatrix} [0.92, 0.77]^{0.38} [0.66, 0.48]^{0.45} \\ [0.85, 0.76]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.96, 0.70]^{0.38} [0.90, 0.58]^{0.45} \\ [0.96, 0.81]^{0.17} \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.75, 0.84]^{0.38} [0.91, 0.96]^{0.45} \\ [0.91, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.51, 0.91]^{0.38} \\ [0.51, 0.75]^{0.45} [0.64, 0.96]^{0.17} \end{Bmatrix}^{0.2} \\ \begin{Bmatrix} [0.64, 0.75]^{0.38} [0.91, 0.96]^{0.45} \\ [0.36, 0.96]^{0.17} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.51, 0.91]^{0.38} [0.51, 0.64]^{0.45} \\ [0.36, 0.75]^{0.17} \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= \left(\frac{\left([0.66, 0.17]^{0.2} [0.69, 0.24]^{0.2} [0.52, 0.28]^{0.2} [0.83, 0.33]^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.93, 0.89] [0.98, 0.96] \\ [0.99, 0.98] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.96, 0.77] \\ [0.88, 0.74] [0.99, 0.93] \end{Bmatrix}^{0.2} \\ \begin{Bmatrix} [0.89, 0.84] [0.98, 0.96] \\ [0.99, 0.84] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.96, 0.77] [0.81, 0.74] \\ [0.95, 0.84] \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= \left(\frac{(0.7016, 0.9202] [0.7516, 0.9284] [0.7752, 0.8774] [0.6418, 0.9282])}{\sqrt{1 - ([0.9023, 0.8373]^{0.2} [0.8363, 0.5299]^{0.2} [0.8634, 0.6773]^{0.2} [0.7387, 0.4786]^{0.4})}} \right)$$

$$= ([0.2623, 0.6957], \sqrt{1 - ([0.9651, 0.9796] [0.8807, 0.9649] [0.9250, 0.9710] [0.7447, 0.8859])})$$

$$= [0.2623, 0.6957], \sqrt{1 - [0.5855, 0.8131]}$$

$$= [0.2623, 0.6957], \sqrt{[0.4146, 0.1869]}$$

$$= [0.2623, 0.6957], [0.4323, 0.6438].$$

3.2 Properties of IVPFSWG

3.2.1 Idempotency

If $\mathcal{F}_{\check{d}_{ij}} = \mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}} \right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}} \right] \right) \forall i, j$. Then IVPFHSWG $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}) = \mathcal{F}_{\check{d}_k}$.

Proof. As we know that all $\mathcal{F}_{\check{d}_{ij}} = \mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}} \right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}} \right] \right)$, then, we have

IVPFHSWG $\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right)$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{v_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{v_j}} \right)$$

$$= \left(\left(\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}} \right] \right)^{\sum_{i=1}^{n} \omega_i} \right)^{\sum_{j=1}^{m} v_j}, \sqrt{1 - \left(\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}} \right]^2 \right)^{\sum_{i=1}^{n} \omega_i} \right)^{\sum_{j=1}^{m} v_j}} \right)$$

As $\sum_{j=1}^{m} \nu_j = 1$ and $\sum_{i=1}^{n} \omega_i = 1$, then we have

$$= \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \sqrt{1 - \left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)}\right)$$

$$= \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$$

$$= \mathcal{F}_{\check{d}_k}.$$

3.2.2 Boundedness

Let $\mathcal{F}_{\check{d}_{ij}}$ be a collection of IVPFHSNs, where $\mathcal{F}^-_{\check{d}_{ij}} = \left(\begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$ and $\mathcal{F}^+_{\check{d}_{ij}} = \left(\begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$, then

$\mathcal{F}^-_{\check{d}_{ij}} \leq \text{IVPFHSWG}\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) \leq \mathcal{F}^+_{\check{d}_{ij}}$

Proof. As we know that $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be an IVPFHSN, then

$$\begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\} \leq \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2 \leq \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Rightarrow 1 - \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\} \leq 1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2 \leq 1 - \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i} \leq \prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i}$$

$$\Leftrightarrow \left(1 - \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \left(1 - \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j}$$

$$\Leftrightarrow 1 - \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq 1 - \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\} \leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\} \leq \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \leq \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\} \quad (C)$$

Similarly,

$$\begin{matrix} min\, min \\ j \quad i \end{matrix}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j} \leq \begin{matrix} max\, max \\ j \quad i \end{matrix}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\} \quad (D)$$

If IVPFHSWG $\left(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}\right) = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right) = \mathcal{F}_{\check{d}_k}$, then inequalities (C) and (D) can be transferred into the form.

$\underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\} \leq \mathcal{M}_\sigma \leq \underset{j}{max}\,\underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}$ and $\underset{j}{min}\,\underset{i}{min}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\} \leq \mathcal{F}_{\check{d}_k} \leq \underset{j}{max}\,\underset{i}{max}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}$, respectively.

Using the score function,

$$S\left(\mathcal{F}_{\check{d}_k}\right) = \frac{\left(\kappa^l_{\check{d}_k}\right)^2 + \left(\kappa^u_{\check{d}_k}\right)^2 - \left(\delta^l_{\check{d}_k}\right)^2 - \left(\delta^u_{\check{d}_k}\right)^2}{2} \leq \underset{j}{max}\,\underset{i}{max}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\} - \underset{j}{min}\,\underset{i}{min}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\} = S\left(\mathcal{F}^-_{\check{d}_{ij}}\right)$$

$$S\left(\mathcal{F}_{\check{d}_k}\right) = \frac{\left(\kappa^l_{\check{d}_k}\right)^2 + \left(\kappa^u_{\check{d}_k}\right)^2 - \left(\delta^l_{\check{d}_k}\right)^2 - \left(\delta^u_{\check{d}_k}\right)^2}{2} \geq \underset{j}{min}\,\underset{i}{min}\left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\} - \underset{j}{max}\,\underset{i}{max}\left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\} = S\left(\mathcal{F}^+_{\check{d}_{ij}}\right)$$

By order relation between two IVPFHSNs, we have

$\mathcal{F}^-_{\check{d}_k} \leq$ IVPFHSWG $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}) \leq \mathcal{F}^+_{\check{d}_k}$.

3.2.3 Shift Invariance

Let $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right], \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]\right)$ be an IVPFHSN. Then

$IVPFHSWG(\mathcal{F}_{\check{d}_{11}} \otimes \mathcal{F}_{\check{d}_k}, \mathcal{F}_{\check{d}_{12}} \otimes \mathcal{F}_{\check{d}_k}, \ldots, \mathcal{F}_{\check{d}_{nm}} \otimes \mathcal{F}_{\check{d}_k}) = IVPFHSWG(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}) \otimes \mathcal{F}_{\check{d}_k}$

Proof. As $\mathcal{F}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right], \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]\right)$ and $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be two IVPFHSNs. Then, using Definition 3.2 (2)

$$\mathcal{F}_{\check{d}_k} \otimes \mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_k}\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_k}\kappa^u_{\check{d}_{ij}}\right], \left[\sqrt{\delta^l_{\check{d}_k}{}^2 + \delta^l_{\check{d}_{ij}}{}^2 - \delta^l_{\check{d}_k}{}^2\delta^l_{\check{d}_{ij}}{}^2}, \sqrt{\delta^u_{\check{d}_k}{}^2 + \delta^u_{\check{d}_{ij}}{}^2 - \delta^u_{\check{d}_k}{}^2\delta^u_{\check{d}_{ij}}{}^2}\right]\right)$$

So,

$$IVPFHSWG\left(\mathcal{F}_{\check{d}_{11}} \otimes \mathcal{F}_{\check{d}_k}, \mathcal{F}_{\check{d}_{12}} \otimes \mathcal{F}_{\check{d}_k}, \ldots, \mathcal{F}_{\check{d}_{nm}} \otimes \mathcal{F}_{\check{d}_k}\right)$$

$$= \otimes^m_{j=1}\nu_j\left(\otimes^n_{i=1}\omega_i\left(\mathcal{F}_{\check{d}_{ij}} \otimes \mathcal{F}_{\check{d}_k}\right)\right)$$

$$= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\left(1 - \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

$$= \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right]\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \left(1 - \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]^2\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

$$= \left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right) \otimes \left(\left[\kappa^l_{\check{d}_k}, \kappa^u_{\check{d}_k}\right], \left[\delta^l_{\check{d}_k}, \delta^u_{\check{d}_k}\right]\right)\right)$$

$IVPFHSWG(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}}) \otimes \mathcal{F}_{\check{d}_k}$.

3.2.4 Homogeneity

Prove that IVPFHSWG $(\beta\mathcal{F}_{\check{d}_{11}}, \beta\mathcal{F}_{\check{d}_{12}}, \ldots, \beta\mathcal{F}_{\check{d}_{nm}}) = \beta$ IVPFHSWG $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}})$ for any positive real number β.

Proof. Let $\mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}}\right], \left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}}\right]\right)$ be an IVPFHSN and $\beta > 0$. Then using Definition 3.2, we have

$$\mathcal{F}_{\check{d}_k}{}^{\beta} = \left(\left[\kappa^{l\ \beta}_{\check{d}_k}, \kappa^{u\ \beta}_{\check{d}_k}\right], \sqrt{1-\left(1-\left[\delta^{l}_{\check{d}_k}, \delta^{u}_{\check{d}_k}\right]^2\right)^{\beta}}\right)$$

So,

IVPFHSWG $\left(\beta\mathcal{F}_{\check{d}_{11}}, \beta\mathcal{F}_{\check{d}_{12}}, \ldots, \beta\mathcal{F}_{\check{d}_{nm}}\right)$

$$= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^{l\ \beta}_{\check{d}_{ij}}, \kappa^{u\ \beta}_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(\left(1-\left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\beta}\right)\right)^{\nu_j}}\right)$$

$$= \left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}}\right]\right)^{\beta\omega_i}\right)^{\nu_j}\right), \sqrt{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}}\right)$$

$$= \left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^{l}_{\check{d}_{ij}}, \kappa^{u}_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}, \sqrt{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\check{d}_{ij}}, \delta^{u}_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}}\right)$$

$= \beta$ IVPFHSWG $(\mathcal{F}_{\check{d}_{11}}, \mathcal{F}_{\check{d}_{12}}, \ldots, \mathcal{F}_{\check{d}_{nm}})$.

4 Multi-Criteria Group Decision-Making Approach Based on Proposed Operators

A decision-making method has been present to resolve the MCGDM obstacles to authenticate the implication of the planned AOs. Also, a statistical illustration has been offered to confirm the pragmatism of the developed methodology.

4.1 Proposed MCGDM Approach

Consider $\mathfrak{I} = \left\{\mathfrak{I}^1, \mathfrak{I}^2, \mathfrak{I}^3, \ldots, \mathfrak{I}^s\right\}$ be the set of s alternatives, $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_r\}$ be the set of r decision-makers. The weights of experts are given as $\omega_i = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ such that $\omega_i > 0$, $\sum_{i=1}^{n}\omega_i = 1$. Suppose Let $\mathfrak{L} = \{e_1, e_2, e_3, \ldots, e_m\}$ be the set of attributes with their corresponding multi sub-attributes such as $\mathfrak{L}' = \left\{\left(e_{1\rho} \times e_{2\rho} \times \ldots \times e_{m\rho}\right) \text{ for all } \rho \in \{1, 2, \ldots, t\}\right\}$ with weights $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)^T$ such that $\nu_i > 0$, $\sum_{i=1}^{n}\nu_i = 1$. and can be stated as $\mathfrak{L}' = \left\{\check{d}_{\partial} : \partial \in \{1, 2, \ldots, m\}\right\}$. The group of experts $\{\kappa^i : i = 1, 2, \ldots, n\}$ assess the alternatives $\{\mathfrak{H}^{(z)} : z = 1, 2, \ldots, s\}$ under the chosen sub-attributes $\{\check{d}_{\partial} : \partial = 1, 2, \ldots, k\}$ in the form of IVPFHSNs such as $\left(\mathfrak{I}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\left[\kappa^{l}_{\check{d}_{ik}}, \kappa^{u}_{\check{d}_{ik}}\right], \left[\delta^{l}_{\check{d}_{ik}}, \delta^{u}_{\check{d}_{ik}}\right]\right)_{n*m}$. Where $0 \le \kappa^{l}_{\check{d}_{ik}}, \kappa^{u}_{\check{d}_{ik}}, \delta^{l}_{\check{d}_{ik}}, \delta^{u}_{\check{d}_{ik}} \le 1$ and $0 \le \left(\kappa^{u}_{\check{d}_{ik}}\right)^2 + \left(\delta^{u}_{\check{d}_{ik}}\right)^2 \le 1$ for all i, k. The decision-makers give their judgment in the form of IVPFHSNs Θ_k for each alternative. The stepwise algorithm is based on established operators given such as follows:

Step-1: Obtain a decision matrix in IVPFHSNs for each alternative according to the expert's opinion.

$$\left(\mathfrak{I}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\left[\kappa^{l}_{\check{d}_{ik}}, \kappa^{u}_{\check{d}_{ik}}\right], \left[\delta^{l}_{\check{d}_{ik}}, \delta^{u}_{\check{d}_{ik}}\right]\right)_{n*m}$$

$$= \begin{bmatrix} \left(\left[\kappa^l_{\check{d}_{11}}, \kappa^u_{\check{d}_{11}}\right], \left[\delta^l_{\check{d}_{11}}, \delta^u_{\check{d}_{11}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{12}}, \kappa^u_{\check{d}_{12}}\right], \left[\delta^l_{\check{d}_{12}}, \delta^u_{\check{d}_{12}}\right]\right) & \cdots & \left(\left[\kappa^l_{\check{d}_{1m}}, \kappa^u_{\check{d}_{1m}}\right], \left[\delta^l_{\check{d}_{1m}}, \delta^u_{\check{d}_{1m}}\right]\right) \\ \left(\left[\kappa^l_{\check{d}_{21}}, \kappa^u_{\check{d}_{21}}\right], \left[\delta^l_{\check{d}_{21}}, \delta^u_{\check{d}_{21}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{22}}, \kappa^u_{\check{d}_{22}}\right], \left[\delta^l_{\check{d}_{22}}, \delta^u_{\check{d}_{22}}\right]\right) & & \left(\left[\kappa^l_{\check{d}_{2n}}, \kappa^u_{\check{d}_{2n}}\right], \left[\delta^l_{\check{d}_{2m}}, \delta^u_{\check{d}_{2m}}\right]\right) \\ \vdots & & \ddots & \vdots \\ \left(\left[\kappa^l_{\check{d}_{n1}}, \kappa^u_{\check{d}_{n1}}\right], \left[\delta^l_{\check{d}_{n1}}, \delta^u_{\check{d}_{n1}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{n2}}, \kappa^u_{\check{d}_{n2}}\right], \left[\delta^l_{\check{d}_{n2}}, \delta^u_{\check{d}_{n2}}\right]\right) & \cdots & \left(\left[\kappa^l_{\check{d}_{nm}}, \kappa^u_{\check{d}_{nm}}\right], \left[\delta^l_{\check{d}_{nm}}, \delta^u_{\check{d}_{nm}}\right]\right) \end{bmatrix}$$

Step-2: Convert the cost type attributes to benefit type using the normalization rule and establish the normalized decision matrices.

$$\mathcal{F}_{\check{d}_{ik}} = \begin{cases} \mathcal{F}^c_{\check{d}_{ij}} = \left(\left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right], \left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right]\right)_{n\times m} \textit{cost type parameter} \\ \mathcal{F}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right], \left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right]\right)_{n\times m} \textit{benefit type parameter} \end{cases}$$

Step-3: Calculate the aggregated values for each alternative using developed IVPFHSWA and IVPFHSWG.

Step-4: Calculate the score values for each alternative.

Step-5: Examine the ranking of the alternatives.

4.2 Numerical Example

It is an intelligent transformation of fossil waste energy, such as natural gas first converted into hydrogen. In inference, despite the overdevelopment of fossil fuels and the potential for global warming, the most important renewable energy sources will originate from the description of financial or environmental reasons. The recently formed hydrogen fuel is different in weight and volume from the commonly used hydrogen fuel in power performance. This hydrogen, irrelevant to its energy capacity, is the most prominent feature. The energy content per kilogram of hydrogen is 120 MJ. The advantage of methanol is an extraordinary six times [38]. Hydrogen has a bit of volumetric energy compactness associated with its particular gravimetric density. The compactness of hydrogen is determined by its accumulation state. A stable thickness of up to 700 bar is not a large enough property for hydrocarbons like gasoline and diesel. Only liquid hydrogen can affect a reasonable amount, still less than a quarter of the amount of gasoline. Therefore, hydrogen containers for motor tenders will conquer more than previously used fluid hydrocarbon containers [39]. Cryogenic storage containers are also considered cryogenic storage containers. The dewar is a double-walled super-insulated container. Its vehicles fluid oxygen, nitrogen, hydrogen, helium, and argon, temperatures <110 K/163°C.

The most significant features (parameters) to deliberate when electing a materiality dashboard DM. The assortment method initiates with a preliminary screening of the material used for the dashboard and is captivated by the validation configuration in-built into the application. Throughout the airing progression, potentially proper materials are acknowledged. Defining the ingredients that can be used by the preliminary MS of the dashboard fashioning is serious. Then select from four material assessment abilities: $\mathfrak{I}^1 =$ Ti–6Al–4V, $\mathfrak{I}^2 =$ SS301–FH, $\mathfrak{I}^3 =$ 70Cu–30Zn, and $\mathfrak{I}^4 =$ Inconel 718. The aspect of material assortment is specified as follows: $\mathfrak{L} = \{d_1 =$ Specific gravity $=$ attaining data around the meditation of resolutions of numerous materials, $d_2 =$ Toughness index, $d_3 =$ Yield stress, $d_4 =$ Easily accessible$\}$. The corresponding subattributes of the considered parameters, Specific gravity = attaining data around the meditation of resolutions of numerous materials = $d_1 = \{d_{11} =$ assess corporal variations, $d_{12} =$ govern the degree of regularity among tasters}, Toughness index $= d_2 = \{d_{21} =$ CharpyV − Notch Impact Energy, $d_{22} =$ Plane Strain Fracture Toughness}, Yield stress $= d_3 = \{d_{31} =$ Yield stress}, Easily accessible $= d_4 = \{d_{41} =$ Easily accessible}. Let $\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4$ be a set of sub-attributes

$$\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times \{d_{31}\} \times \{d_{41}\}$$

$= \begin{Bmatrix} (d_{11}, d_{21}, d_{31}d_{41}), (d_{11}, d_{22}, d_{31}, d_{41}), \\ (d_{12}, d_{21}, d_{31}, d_{41}), (d_{12}, d_{22}, d_{31}, d_{41}) \end{Bmatrix}$, $\mathfrak{L}' = \left\{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\right\}$ be a set of all sub-attributes with weights $(0.3, 0.1, 0.2, 0.4)^T$. Let $\{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4\}$ be a set of four experts with weights $(0.1, 0.2, 0.4, 0.3)^T$. To judge the optimal alternative, experts deliver their preferences in IVPFHSNs.

4.2.1 By IVPFHSWA Operator

Step-1: Decision-maker's opinion on IVPFHSNs is given in Tables 1–4.

Table 1: Decision matrix for $\mathfrak{I}^1$ in the form of IVPFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
$\mathcal{U}_1$	$([0.4, 0.5], [0.2, 0.5])$	$([0.7, 0.8], [0.5, 0.6])$	$([0.4, 0.6], [0.2, 0.5])$	$([0.2, 0.4], [0.2, 0.6])$
$\mathcal{U}_2$	$([0.2, 0.7], [0.2, 0.6])$	$([0.1, 0.6], [0.4, 0.5])$	$([0.2, 0.3], [0.4, 0.8])$	$([0.2, 0.5], [0.4, 0.7])$
$\mathcal{U}_3$	$([0.3, 0.5], [0.1, 0.4])$	$([0.4, 0.6], [0.2, 0.7])$	$([0.4, 0.7], [0.3, 0.7])$	$([0.5, 0.7], [0.2, 0.4])$
$\mathcal{U}_4$	$([0.4, 0.6], [0.3, 0.7])$	$([0.4, 0.5], [0.3, 0.7])$	$([0.3, 0.6], [0.3, 0.5])$	$([0.3, 0.6], [0.3, 0.5])$

Table 2: Decision matrix for $\mathfrak{I}^2$ in the form of IVPFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
$\mathcal{U}_1$	$([0.3, 0.6], [0.5, 0.6])$	$([0.2, 0.7], [0.5, 0.7])$	$([0.2, 0.7], [0.4, 0.5])$	$([0.6, 0.7], [0.5, 0.8])$
$\mathcal{U}_2$	$([0.3, 0.5], [0.5, 0.8])$	$([0.1, 0.4], [0.4, 0.5])$	$([0.1, 0.5], [0.3, 0.7])$	$([0.4, 0.5], [0.3, 0.6])$
$\mathcal{U}_3$	$([0.2, 0.6], [0.1, 0.4])$	$([0.1, 0.2], [0.2, 0.9])$	$([0.4, 0.7], [0.3, 0.8])$	$([0.5, 0.8], [0.2, 0.6])$
$\mathcal{U}_4$	$([0.2, 0.3], [0.3, 0.8])$	$([0.3, 0.5], [0.2, 0.8])$	$([0.3, 0.7], [0.2, 0.6])$	$([0.1, 0.7], [0.3, 0.6])$

Table 3: Decision matrix for $\mathfrak{I}^3$ in the form of IVPFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
$\mathcal{U}_1$	$([0.3, 0.4], [0.2, 0.7])$	$([0.3, 0.4], [0.4, 0.6])$	$([0.5, 0.6], [0.4, 0.5])$	$([0.3, 0.4], [0.3, 0.6])$
$\mathcal{U}_2$	$([0.4, 0.6], [0.3, 0.7])$	$([0.3, 0.5], [0.2, 0.3])$	$([0.3, 0.5], [0.5, 0.8])$	$([0.2, 0.6], [0.2, 0.4])$
$\mathcal{U}_3$	$([0.2, 0.4], [0.3, 0.4])$	$([0.3, 0.5], [0.3, 0.7])$	$([0.3, 0.7], [0.3, 0.8])$	$([0.1, 0.3], [0.5, 0.6])$
$\mathcal{U}_4$	$([0.3, 0.7], [0.3, 0.7])$	$([0.3, 0.5], [0.2, 0.4])$	$([0.2, 0.5], [0.3, 0.6])$	$([0.3, 0.4], [0.3, 0.7])$

Table 4: Decision matrix for $\mathfrak{I}^4$ in the form of IVPFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
$\mathcal{U}_1$	([0.3, 0.5], [0.2, 0.6])	([0.2, 0.6], [0.4, 0.7])	([0.2, 0.5], [0.3, 0.6])	([0.5, 0.7], [0.6, 0.8])
$\mathcal{U}_2$	([0.2, 0.7], [0.3, 0.8])	([0.1, 0.5], [0.4, 0.7])	([0.5, 0.7], [0.4, 0.5])	([0.2, 0.5], [0.3, 0.4])
$\mathcal{U}_3$	([0.2, 0.5], [0.1, 0.6])	([0.2, 0.5], [0.1, 0.5])	([0.2, 0.4], [0.2, 0.7])	([0.3, 0.5], [0.1, 0.5])
$\mathcal{U}_4$	([0.2, 0.4], [0.5, 0.8])	([0.2, 0.5], [0.5, 0.8])	([0.2, 0.7], [0.3, 0.6])	([0.2, 0.5], [0.4, 0.5])

Step-2: There is no need to normalize because all parameters are the same type.

Step-3: Compute the aggregated values employing the developed IVPFHSWA operator for each alternative.

$$\Theta_1 = \left(\sqrt{1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j}}, \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{v_j}\right)$$

$$\Theta_1 = \left(\begin{array}{l} \sqrt{1 - \left(\begin{array}{l} \begin{Bmatrix} [0.98, 0.97]\,[0.99, 0.87] \\ [0.96, 0.89]\,[0.95, 0.87] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.93, 0.90]\,[0.99, 0.91] \\ [0.93, 0.84]\,[0.95, 0.92] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.98, 0.96]\,[0.99, 0.98] \\ [0.93, 0.76]\,[0.97, 0.95] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.99, 0.98]\,[0.99, 0.94] \\ [0.89, 0.76]\,[0.97, 0.87] \end{Bmatrix}^{0.4} \end{array}\right)}, \\ \left(\begin{array}{l} \begin{Bmatrix} [0.93, 0.85]\,[0.90, 0.72] \\ [0.69, 0.39]\,[0.89, 0.69] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.95, 0.93]\,[0.87, 0.83] \\ [0.87, 0.52]\,[0.89, 0.69] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.93, 0.85]\,[0.96, 0.83] \\ [0.87, 0.62]\,[0.81, 0.69] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.95, 0.85]\,[0.93, 0.83] \\ [0.69, 0.52]\,[0.81, 0.69] \end{Bmatrix}^{0.4} \end{array}\right) \end{array}\right)$$

$$\Theta_1 = \left(\begin{array}{l} \sqrt{1 - \left([0.88, 0.65]^{0.3}\,[0.81, 0.63]^{0.1}\,[0.87, 0.68]^{0.2}\,[0.85, 0.61]^{0.4}\right)}, \\ \left([0.51, 0.16]^{0.3}\,[0.64, 0.28]^{0.1}\,[0.63, 0.30]^{0.2}\,[0.49, 0.25]^{0.4}\right) \end{array}\right)$$

$$\Theta_1 = \left(\begin{array}{l} \sqrt{1 - ([0.8788, 0.9624]\,[0.9548, 0.9791]\,[0.9258, 0.9725]\,[0.8206, 0.9371])}, \\ ([0.5771, 0.8171]\,[0.8805, 0.9563]\,[0.7860, 0.9117]\,[0.5743, 0.7517]) \end{array}\right)$$

$$\Theta_1 = \left(\sqrt{1 - [0.6374, 0.8587]}, [0.2294, 0.5355]\right)$$

$$\Theta_1 = \left(\sqrt{[0.3626, 0.1413]}, [0.2294, 0.5355]\right)$$

$$\Theta_1 = [0.3759, 0.6022], [0.2294, 0.5355].$$

$$\Theta_2 = \left(\sqrt{1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1 - \left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j}}, \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{v_j}\right)$$

$$\Theta_2 = \left(\begin{array}{c} \sqrt{1-\left(\begin{array}{cc} \left\{\begin{array}{c}[0.64,0.91]^{0.1}\,[0.71,0.91]^{0.2}\\ [0.64,0.96]^{0.4}\,[0.91,0.96]^{0.3}\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.51,0.96]^{0.1}\,[0.84,0.99]^{0.2}\\ [0.96,0.99]^{0.4}\,[0.75,0.91]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.51,0.96]^{0.1}\,[0.75,0.99]^{0.2}\\ [0.51,0.84]^{0.4}\,[0.51,0.91]^{0.3}\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.51,0.64]^{0.1}\,[0.75,0.84]^{0.2}\\ [0.36,0.75]^{0.4}\,[0.51,0.99]^{0.3}\end{array}\right\}^{0.4}\end{array}\right)}, \\ \left(\begin{array}{cc} \left\{\begin{array}{c}[0.5,0.6]^{0.1}\,[0.5,0.8]^{0.2}\\ [0.1,0.4]^{0.4}\,[0.3,0.8]^{0.3}\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.5,0.7]^{0.1}\,[0.4,0.5]^{0.2}\\ [0.2,0.9]^{0.4}\,[0.2,0.8]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.4,0.5]^{0.1}\,[0.3,0.7]^{0.2}\\ [0.3,0.8]^{0.4}\,[0.2,0.6]^{0.3}\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.5,0.8]^{0.1}\,[0.3,0.6]^{0.2}\\ [0.2,0.6]^{0.4}\,[0.3,0.6]^{0.3}\end{array}\right\}^{0.4}\end{array}\right) \end{array}\right)$$

$$\Theta_2 = \left(\begin{array}{c} \sqrt{1-\left(\begin{array}{cc} \left\{\begin{array}{c}[0.99,0.96]\,[0.98,0.93]\\ [0.98,0.84]\,[0.95,0.87]\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.93,0.90]\,[0.99,0.91]\\ [0.93,0.84]\,[0.95,0.92]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.98,0.96]\,[0.99,0.98]\\ [0.93,0.76]\,[0.97,0.87]\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.99,0.98]\,[0.99,0.94]\\ [0.89,0.76]\,[0.97,0.87]\end{array}\right\}^{0.4}\end{array}\right)}, \\ \left(\begin{array}{cc} \left\{\begin{array}{c}[0.93,0.85]\,[0.90,0.72]\\ [0.69,0.39]\,[0.89,0.69]\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.95,0.93]\,[0.87,0.83]\\ [0.87,0.52]\,[0.89,0.69]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.93,0.85]\,[0.96,0.83]\\ [0.87,0.62]\,[0.81,0.69]\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.95,0.85]\,[0.93,0.83]\\ [0.69,0.52]\,[0.81,0.69]\end{array}\right\}^{0.4}\end{array}\right) \end{array}\right)$$

$$\Theta_2 = \left(\begin{array}{c}\sqrt{1-\left([0.90,0.65]^{0.3}\,[0.81,0.63]^{0.1}\,[0.87,0.68]^{0.2}\,[0.85,0.61]^{0.4}\right)},\\ \left([0.51,0.16]^{0.3}\,[0.64,0.28]^{0.1}\,[0.63,0.30]^{0.2}\,[0.49,0.25]^{0.4}\right)\end{array}\right)$$

$$\Theta_2 = \left(\begin{array}{c}\sqrt{1-([0.8788,0.9689]\,[0.9548,0.9791]\,[0.9258,0.9725]\,[0.8206,0.9371])},\\ ([0.5771,0.8171]\,[0.8805,0.9563]\,[0.7860,0.9117]\,[0.5743,0.7517])\end{array}\right)$$

$$\Theta_2 = \left(\sqrt{1-[0.6374,0.8645]},(0.2294,0.5355)\right)$$

$$\Theta_2 = \left(\sqrt{[0.3626,0.1355]},[0.2294,0.5355]\right)$$

$$\Theta_2 = [0.3681,0.6022],[0.2294,0.5355].$$

$$\Theta_3 = \left(\sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\kappa^{l}_{\check{d}_{ij}},\kappa^{u}_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\delta^{l}_{\check{d}_{ij}},\delta^{u}_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$\Theta_3 = \left(\begin{array}{c} \sqrt{1-\left(\begin{array}{cc} \left\{\begin{array}{c}[0.84,0.91]^{0.1}\,[0.64,0.84]^{0.2}\\ [0.84,0.96]^{0.4}\,[0.51,0.91]^{0.3}\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.84,0.91]^{0.1}\,[0.75,0.91]^{0.2}\\ [0.75,0.91]^{0.4}\,[0.75,0.91]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.64,0.75]^{0.1}\,[0.75,0.91]^{0.2}\\ [0.51,0.91]^{0.4}\,[0.75,0.96]^{0.3}\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.84,0.91]^{0.1}\,[0.64,0.96]^{0.2}\\ [0.91,0.99]^{0.4}\,[0.84,0.91]^{0.3}\end{array}\right\}^{0.4}\end{array}\right)}, \\ \left(\begin{array}{cc} \left\{\begin{array}{c}[0.2,0.7]^{0.1}\,[0.3,0.7]^{0.2}\\ [0.3,0.4]^{0.4}\,[0.3,0.7]^{0.3}\end{array}\right\}^{0.3} & \left\{\begin{array}{c}[0.4,0.6]^{0.1}\,[0.2,0.3]^{0.2}\\ [0.3,0.7]^{0.4}\,[0.2,0.4]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.4,0.5]^{0.1}\,[0.5,0.8]^{0.2}\\ [0.3,0.8]^{0.4}\,[0.3,0.6]^{0.3}\end{array}\right\}^{0.2} & \left\{\begin{array}{c}[0.3,0.6]^{0.1}\,[0.2,0.4]^{0.2}\\ [0.5,0.6]^{0.4}\,[0.3,0.7]^{0.3}\end{array}\right\}^{0.4}\end{array}\right) \end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.99,0.98]\,[0.96,0.91]\\ [0.98,0.93]\,[0.97,0.82]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.99,0.98]\,[0.98,0.94]\\ [0.96,0.89]\,[0.97,0.92]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.97,0.96]\,[0.98,0.94]\\ [0.96,0.76]\,[0.99,0.92]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.99,0.98]\,[0.99,0.91]\\ [0.99,0.96]\,[0.97,0.95]\end{array}\right\}^{0.4}\end{array}\right)},\\ \left(\begin{array}{c}\left\{\begin{array}{c}[0.96,0.85]\,[0.93,0.79]\\ [0.69,0.62]\,[0.89,0.69]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.95,0.91]\ 0.79,0.72]\\ [0.86,0.62]\,[0.76,0.62]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.93,0.91]\,[0.96,0.87]\\ [0.91,0.62]\,[0.86,0.69]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.95,0.88]\,[0.83,0.72]\\ [0.81,0.76]\,[0.89,0.69]\end{array}\right\}^{0.4}\end{array}\right)\end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\sqrt{1-\left([0.90,0.68]^{0.3}\,[0.90,0.75]^{0.1}\,[0.90,0.63]^{0.2}\,[0.94,0.81]^{0.4}\right)},\\ \left([0.55,0.29]^{0.3}\,[0.49,0.25]^{0.1}\,[0.69,0.34]^{0.2}\,[0.57,0.33]^{0.4}\right)\end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\sqrt{1-([0.8907,0.9689]\,[0.9716,0.9895]\,[0.9117,0.9791]\,[0.9192,0.9755])},\\ ([0.6898,0.8358]\,[0.8705,0.9311]\,[0.8059,0.9285]\,[0.6418,0.7986])\end{array}\right)$$

$$\Theta_3 = \left(\sqrt{1-[0.7252,0.9157]},(0.3183,0.4637)\right)$$

$$\Theta_3 = \left(\sqrt{[0.7252,0.9157]},(0.3183,0.4637)\right)$$

$$\Theta_3 = [0.2903,0.5242],[0.3183,0.4637].$$

$$\Theta_4 = \left(\sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\kappa^{l}_{\check{d}_{ij}},\kappa^{u}_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j}},\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\delta^{l}_{\check{d}_{ij}},\delta^{u}_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{v_j}\right)$$

$$\Theta_4 = \left(\begin{array}{c}\sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.75,0.91]^{0.1}\,[0.51,0.84]^{0.2}\\ [0.75,0.96]^{0.4}\,[0.51,0.75]^{0.3}\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.51,0.96]^{0.1}\,[0.51,0.84]^{0.2}\\ [0.51,0.75]^{0.4}\,[0.75,0.96]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.75,0.96]^{0.1}\,[0.75,0.99]^{0.2}\\ [0.84,0.96]^{0.4}\,[0.75,0.91]^{0.3}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.84,0.96]^{0.1}\,[0.36,0.75]^{0.2}\\ [0.51,0.96]^{0.4}\,[0.75,0.96]^{0.3}\end{array}\right\}^{0.4}\end{array}\right)},\\ \left(\begin{array}{c}\left\{\begin{array}{c}[0.2,0.6]^{0.1}\,[0.2,0.6]^{0.2}\\ [0.3,0.6]^{0.4}\,[0.6,0.8]^{0.3}\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.3,0.8]^{0.1}\,[0.1,0.5]^{0.2}\\ [0.4,0.5]^{0.4}\,[0.3,0.4]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.1,0.6]^{0.1}\,[0.2,0.5]^{0.2}\\ [0.2,0.7]^{0.4}\,[0.1,0.5]^{0.3}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.5,0.8]^{0.1}\,[0.2,0.5]^{0.2}\\ [0.3,0.6]^{0.4}\,[0.4,0.5]^{0.3}\end{array}\right\}^{0.4}\end{array}\right)\end{array}\right)$$

$$\Theta_4 = \left(\begin{array}{c}\sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.99,0.97]\,[0.96,0.87]\\ [0.98,0.89]\,[0.92,0.82]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.99,0.93]\,[0.96,0.87]\\ [0.89,0.76]\,[0.99,0.92]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.99,0.97]\,[0.99,0.94]\\ [0.98,0.93]\,[0.97,0.92]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.99,0.98]\,[0.94,0.81]\\ [0.98,0.76]\,[0.99,0.92]\end{array}\right\}^{0.4}\end{array}\right)},\\ \left(\begin{array}{c}\left\{\begin{array}{c}[0.95,0.85]\,[0.90,0.72]\\ [0.81,0.62]\,[0.93,0.86]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.98,0.89]\,[0.87,0.63]\\ [0.76,0.69]\,[0.76,0.69]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.95,0.79]\,[0.87,0.72]\\ [0.87,0.52]\,[0.81,0.50]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.98,0.93]\,[0.87,0.72]\\ [0.81,0.62]\,[0.81,0.76]\end{array}\right\}^{0.4}\end{array}\right)\end{array}\right)$$

$$\Theta_4 = \begin{pmatrix} \sqrt{1 - \left([0.86, 0.61]^{0.3}\,[0.84, 0.56]^{0.1}\,[0.93, 0.78]^{0.2}\,[0.90, 0.55]^{0.4}\right)}, \\ \left([0.64, 0.33]^{0.3}\,[0.49, 0.27]^{0.1}\,[0.58, 0.15]^{0.2}\,[0.56, 0.31]^{0.4}\right) \end{pmatrix}$$

$$\Theta_4 = \begin{pmatrix} \sqrt{1 - ([0.8622, 0.9558]\,[0.9437, 0.9827]\,[0.9515, 0.9856]\,[0.7873, 0.9587])}, \\ ([0.7170, 0.8747]\,[0.8773, 0.9311]\,[0.6842, 0.8968]\,[0.6259, 0.7930]) \end{pmatrix}$$

$$\Theta_4 = \left(\sqrt{1 - [0.6095, 0.8875]}, [0.2694, 0.5792]\right)$$

$$\Theta_4 = \left(\sqrt{[0.3905, 0.1125]}, [0.2694, 0.5792]\right)$$

$$\Theta_4 = [0.3354, 0.6249], [0.2694, 0.5792].$$

Step-4: Using score function $S = \frac{\left(\kappa^l_{\check{d}_{ij}}\right)^2 + \left(\kappa^u_{\check{d}_{ij}}\right)^2 - \left(\delta^l_{\check{d}_{ij}}\right)^2 - \left(\delta^u_{\check{d}_{ij}}\right)^2}{2}$ for the IVPFSSS to calculate the score values for all alternatives. $S(\Theta_1) = 0.0599$, $S(\Theta_2) = 0.0578$, $S(\Theta_3) = 0.0266$, and $S(\Theta_4) = -0.0382$.

Step-5: From the above calculation, we get $S(\Theta_1) > S(\Theta_2) > S(\Theta_3) > S(\Theta_4)$, which shows that $\Im^1$ is the best alternative. So, $\Im^1 > \Im^2 > \Im^3 > \Im^4$.

4.2.2 By IVPFHSWG Operator

Step-1 and Step-2 are similar to Section 4.2.1.

Step-3: Compute the aggregated values employing the developed IVPFHSWG operator for each alternative.

$$\Theta_1 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$\Theta_1 = \left(\begin{array}{c} \left(\begin{array}{c} \begin{Bmatrix} [0.4, 0.5]^{0.1}\,[0.2, 0.7]^{0.2} \\ [0.3, 0.5]^{0.4}\,[0.4, 0.6]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.7, 0.8]^{0.1}\,[0.1, 0.6]^{0.2} \\ [0.4, 0.6]^{0.4}\,[0.4, 0.5]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.4, 0.6]^{0.1}\,[0.2, 0.3]^{0.2} \\ [0.4, 0.7]^{0.4}\,[0.3, 0.6]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.2, 0.4]^{0.1}\,[0.2, 0.5]^{0.2} \\ [0.5, 0.7]^{0.4}\,[0.3, 0.6]^{0.3} \end{Bmatrix}^{0.4} \end{array} \right), \\ \sqrt{1 - \left(\begin{array}{c} \begin{Bmatrix} [0.75, 0.84]^{0.1}\,[0.64, 0.96]^{0.2} \\ [0.84, 0.99]^{0.4}\,[0.51, 0.91]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.64, 0.75]^{0.1}\,[0.75, 0.84]^{0.2} \\ [0.51, 0.96]^{0.4}\,[0.51, 0.91]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.96]^{0.1}\,[0.36, 0.84]^{0.2} \\ [0.51, 0.91]^{0.4}\,[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.96]^{0.1}\,[0.51, 0.84]^{0.2} \\ [0.84, 0.91]^{0.4}\,[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{array} \right)} \end{array} \right)$$

$$\Theta_1 = \left(\begin{array}{c} \left(\begin{array}{c} \begin{Bmatrix} [0.93, 0.91]\,[0.93, 0.72] \\ [0.76, 0.62]\,[0.86, 0.76] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.98, 0.96]\,[0.90, 0.63] \\ [0.81, 0.69]\,[0.81, 0.76] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.95, 0.91]\,[0.79, 0.72] \\ [0.87, 0.69]\,[0.86, 0.69] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.91, 0.85]\,[0.87, 0.72] \\ [0.87, 0.76]\,[0.86, 0.69] \end{Bmatrix}^{0.4} \end{array} \right), \\ \sqrt{1 - \left(\begin{array}{c} \begin{Bmatrix} [0.98, 0.97]\,[0.99, 0.91] \\ [0.98, 0.76]\,[0.97, 0.82] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.97, 0.96]\,[0.97, 0.94] \\ [0.81, 0.69]\,[0.81, 0.76] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.99, 0.97]\,[0.96, 0.81] \\ [0.96, 0.76]\,[0.97, 0.92] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.99, 0.96]\,[0.96, 0.87] \\ [0.96, 0.93]\,[0.97, 0.89] \end{Bmatrix}^{0.4} \end{array} \right)} \end{array} \right)$$

$$\Theta_1 = \left(\frac{\begin{pmatrix} [0.56, 0.31]^{0.3} [0.58, 0.32]^{0.1} \\ [0.56, 0.31]^{0.2} [0.59, 0.32]^{0.4} \end{pmatrix},}{\sqrt{1 - \begin{pmatrix} [0.92, 0.55]^{0.3} [0.62, 0.47]^{0.1} \\ [0.88, 0.55]^{0.2} [0.88, 0.69]^{0.4} \end{pmatrix}}} \right)$$

$$\Theta_1 = \left(\frac{([0.7037, 0.8403] [0.8923, 0.9469] [0.7911, 0.9747] [0.6339, 0.8097]),}{\sqrt{1 - ([0.8358, 0.9753] [0.9273, 0.9533] [0.8873, 0.9747] [0.8620, 0.9501])}} \right)$$

$$\Theta_1 = [0.3149, 0.6279], \sqrt{1 - [0.5928, 0.8610]}$$

$$\Theta_1 = [0.3149, 0.6279], \sqrt{[0.4072, 0.139]}$$

$$\Theta_1 = [0.3149, 0.6279], [0.3728, 0.6381].$$

$$\Theta_2 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}} \right] \right)^{\omega_i} \right)^{v_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}} \right]^2 \right)^{\omega_i} \right)^{v_j}} \right)$$

$$\Theta_2 = \left(\frac{\begin{pmatrix} \begin{Bmatrix} [0.2, 0.5]^{0.1} [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.2, 0.6]^{0.1} [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} [0.4, 0.7]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} [0.3, 0.5]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix},}{\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.75, 0.84]^{0.1} [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} [0.84, 0.96]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.51, 0.96]^{0.1} [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} [0.75, 0.96]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} [0.64, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}}} \right)$$

$$\Theta_2 = \left(\frac{\begin{pmatrix} \begin{Bmatrix} [0.93, 0.85] [0.96, 0.93] \\ [0.76, 0.52] [0.86, 0.62] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.95, 0.85] [0.90, 0.63] \\ [0.91, 0.69] [0.89, 0.76] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.91, 0.79] [0.90, 0.83] \\ [0.87, 0.62] [0.76, 0.62] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.96, 0.89] [0.87, 0.83] \\ [0.76, 0.62] [0.81, 0.69] \end{Bmatrix}^{0.4} \end{pmatrix},}{\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.98, 0.97] [0.94, 0.91] \\ [0.93, 0.84] [0.79, 0.95] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.99, 0.93] [0.96, 0.94] \\ [0.98, 0.96] [0.99, 0.92] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.99, 0.97] [0.99, 0.87] \\ [0.93, 0.51] [0.97, 0.92] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.98, 0.96] [0.98, 0.87] \\ [0.96, 0.84] [0.97, 0.87] \end{Bmatrix}^{0.4} \end{pmatrix}}} \right)$$

$$\Theta_2 = \left(\frac{\begin{pmatrix} [0.58, 0.25]^{0.3} [0.69, 0.28]^{0.1} \\ [0.54, 0.25]^{0.2} [0.51, 0.32]^{0.4} \end{pmatrix},}{\sqrt{1 - \begin{pmatrix} [0.68, 0.70]^{0.3} [0.92, 0.77]^{0.1} \\ [0.88, 0.39]^{0.2} [0.89, 0.61]^{0.4} \end{pmatrix}}} \right)$$

$$\Theta_2 = \left(\frac{([0.6597, 0.8492] [0.8804, 0.9635] [0.7578, 0.8840] [0.6339, 0.7639]),}{\sqrt{1 - ([0.8985, 0.8907] [0.9742, 0.9917] [0.8283, 0.9747] [0.8206, 0.9544])}} \right)$$

$$\Theta_2 = [0.2409, 0.5525], \sqrt{1 - [0.5949, 0.8217]}$$

$$\Theta_2 = [0.2409, 0.5525], \sqrt{[0.4051, 0.1783]}$$

$$\Theta_2 = [0.2409, 0.5525], [0.4222, 0.6365].$$

$$\Theta_3 = \left(\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right)^{\omega_i}\right)^{v_j}, \sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{v_j}}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\left(\begin{array}{c}\left\{\begin{array}{c}[0.3, 0.4]^{0.1}\,[0.4, 0.6]^{0.2}\\ [0.2, 0.4]^{0.4}\,[0.3, 0.7]^{0.3}\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.3, 0.4]^{0.1}\,[0.3, 0.5]^{0.2}\\ [0.3, 0.5]^{0.4}\,[0.3, 0.5]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.5, 0.6]^{0.1}\,[0.3, 0.5]^{0.2}\\ [0.3, 0.7]^{0.4}\,[0.2, 0.5]^{0.3}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.3, 0.4]^{0.1}\,[0.2, 0.6]^{0.2}\\ [0.1, 0.3]^{0.4}\,[0.3, 0.4]^{0.3}\end{array}\right\}^{0.4}\end{array}\right),\\ \hline \sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.51, 0.96]^{0.1}\,[0.51, 0.91]^{0.2}\\ [0.84, 0.91]^{0.4}\,[0.51, 0.91]^{0.3}\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.64, 0.91]^{0.1}\,[0.91, 0.96]^{0.2}\\ [0.51, 0.91]^{0.4}\,[0.84, 0.96]^{0.3}\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.75, 0.84]^{0.1}\,[0.36, 0.75]^{0.2}\\ [0.36, 0.91]^{0.4}\,[0.64, 0.91]^{0.3}\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.64, 0.91]^{0.1}\,[0.84, 0.96]^{0.2}\\ [0.64, 0.75]^{0.4}\,[0.51, 0.91]^{0.3}\end{array}\right\}^{0.4}\end{array}\right)}\end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\left(\begin{array}{c}\left\{\begin{array}{c}[0.91, 0.89]\,[0.90, 0.83]\\ [0.69, 0.52]\,[0.89, 0.69]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.91, 0.89]\,[0.87, 0.79]\\ [0.76, 0.62]\,[0.81, 0.69]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.95, 0.93]\,[0.87, 0.79]\\ [0.87, 0.62]\,[0.81, 0.62]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.91, 0.89]\,[0.90, 0.72]\\ [0.62, 0.39]\,[0.76, 0.69]\end{array}\right\}^{0.4}\end{array}\right),\\ \hline \sqrt{1-\left(\begin{array}{c}\left\{\begin{array}{c}[0.99, 0.93]\,[0.98, 0.87]\\ [0.96, 0.93]\,[0.97, 0.82]\end{array}\right\}^{0.3}\left\{\begin{array}{c}[0.99, 0.96]\,[0.99, 0.98]\\ [0.96, 0.76]\,[0.99, 0.95]\end{array}\right\}^{0.1}\\ \left\{\begin{array}{c}[0.98, 0.97]\,[0.94, 0.81]\\ [0.96, 0.66]\,[0.97, 0.87]\end{array}\right\}^{0.2}\left\{\begin{array}{c}[0.99, 0.96]\,[0.99, 0.96]\\ [0.89, 0.84]\,[0.97, 0.82]\end{array}\right\}^{0.4}\end{array}\right)}\end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}\left(\begin{array}{c}[0.50, 0.26]^{0.3}\,[0.49, 0.30]^{0.1}\\ [0.58, 0.28]^{0.2}\,[0.38, 0.17]^{0.4}\end{array}\right),\\ \hline \sqrt{1-\left(\begin{array}{c}[0.90, 0.62]^{0.3}\,[0.93, 0.68]^{0.1}\\ [0.86, 0.45]^{0.2}\,[0.85, 0.63]^{0.4}\end{array}\right)}\end{array}\right)$$

$$\Theta_3 = \left(\begin{array}{c}([0.6676, 0.8122]\,[0.8866, 0.9311]\,[0.7752, 0.8968]\,[0.4922, 0.6791]),\\ \hline \sqrt{1-([0.8663, 0.9689]\,[0.9622, 0.9928]\,[0.8524, 0.9703]\,[0.8312, 0.9371])}\end{array}\right)$$

$$\Theta_3 = [0.2258, 0.4605], [0.3541, 0.6398].$$

$$\Theta_4 = \left(\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\left[\kappa^l_{ij}, \kappa^u_{ij}\right]\right)^{\omega_i}\right)^{v_j}, \sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\delta^l_{ij}, \delta^u_{ij}\right]^2\right)^{\omega_i}\right)^{v_j}}\right)$$

$$\Theta_4 = \left(\frac{\left(\begin{Bmatrix} [0.3, 0.5]^{0.1} & [0.2, 0.7]^{0.2} \\ [0.2, 0.5]^{0.4} & [0.2, 0.4]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.2, 0.6]^{0.1} & [0.1, 0.5]^{0.2} \\ [0.2, 0.5]^{0.4} & [0.2, 0.5]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.2, 0.5]^{0.1} & [0.5, 0.7]^{0.2} \\ [0.2, 0.4]^{0.4} & [0.2, 0.7]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.5, 0.7]^{0.1} & [0.2, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} & [0.2, 0.5]^{0.3} \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.64, 0.96]^{0.1} & [0.36, 0.91]^{0.2} \\ [0.64, 0.99]^{0.4} & [0.36, 0.75]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.51, 0.84]^{0.1} & [0.51, 0.84]^{0.2} \\ [0.75, 0.99]^{0.4} & [0.36, 0.75]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.84, 0.91]^{0.1} & [0.75, 0.84]^{0.2} \\ [0.51, 0.96]^{0.4} & [0.64, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.36, 0.64]^{0.1} & [0.84, 0.91]^{0.2} \\ [0.75, 0.99]^{0.4} & [0.75, 0.84]^{0.3} \end{Bmatrix}^{0.4} \right)}} \right)$$

$$\Theta_4 = \left(\frac{\left(\begin{Bmatrix} [0.93, 0.88] & [0.93, 0.72] \\ [0.76, 0.52] & [0.76, 0.62] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.95, 0.85] & [0.87, 0.63] \\ [0.76, 0.52] & [0.81, 0.62] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.93, 0.85] & [0.93, 0.87] \\ [0.69, 0.52] & [0.89, 0.62] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.96, 0.93] & [0.87, 0.72] \\ [0.76, 0.62] & [0.81, 0.62] \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.99, 0.93] & [0.98, 0.87] \\ [0.96, 0.93] & [0.97, 0.82] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.99, 0.96] & [0.99, 0.98] \\ [0.96, 0.76] & [0.99, 0.95] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.98, 0.97] & [0.94, 0.81] \\ [0.96, 0.66] & [0.97, 0.87] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.99, 0.96] & [0.99, 0.96] \\ [0.89, 0.84] & [0.97, 0.82] \end{Bmatrix}^{0.4} \right)}} \right)$$

$$\Theta_4 = \left(\frac{\begin{pmatrix} [0.49, 0.20]^{0.3} & [0.51, 0.17]^{0.1} \\ [0.53, 0.24]^{0.2} & [0.51, 0.26]^{0.4} \end{pmatrix},}{\sqrt{1 - \begin{pmatrix} [0.74, 0.62]^{0.3} & [0.93, 0.68]^{0.1} \\ [0.86, 0.45]^{0.2} & [0.85, 0.63]^{0.4} \end{pmatrix}}} \right)$$

$$\Theta_4 = \left(\frac{([0.6170, 0.8073]\,[0.8376, 0.9348]\,[0.7517, 0.8807]\,[0.5834, 0.7639])\,,}{\sqrt{1 - ([0.8663, 0.9136]\,[0.9625, 0.9928]\,[0.8523, 0.9702]\,[0.8312, 0.9371])}} \right)$$

$$\Theta_4 = [0.2266, 0.5077]\,, \sqrt{1 - [0.5907, 0.8546]}$$

$$\Theta_4 = [0.2266, 0.5077]\,, \sqrt{[0.4093, 0.1454]}$$

$$\Theta_4 = [0.2266, 0.5077], [0.3813, 0.6398].$$

Step-4: Use the score function $S = \frac{(\kappa^l)^2 + (\kappa^u)^2 - (\delta^l)^2 - (\delta^u)^2}{2}$ interval-valued for the Pythagorean fuzzy soft set to calculate the score values for all alternatives such as $S(\Theta_1) = 0.0752$, $S(\Theta_2) = 0.0654$, $S(\Theta_3) = 0.0241$, and $S(\Theta_4) = 0.0114$.

Step-5: From the above calculation, we get the ranking of alternatives $S(\Theta_1) > S(\Theta_2) > S(\Theta_3) > S(\Theta_4)$. Which shows that $\mathfrak{I}^1$ is the best alternative. So, $\mathfrak{I}^1 > \mathfrak{I}^2 > \mathfrak{I}^3 > \mathfrak{I}^4$.

Subsequently, the material assessment wonders at the theoretical level through the depiction phase of the strategy; there is more possibility to the extent of the correctness of the specific materials. Face-centered cube materials are typically used at minor temperatures −163°C and $\mathfrak{I}^1$ = Ti–6Al–4V ratings first. This is steadfast in employing initial investigations and real-world maneuvers. Austenitic steels are still classically used in melted nitrogen or hydrogen storing vessels [40].

5 Comparative Studies

A comparison among the projected model and prevalent approaches is planned to validate the efficacy of the offered technique in the subsequent section.

5.1 Supremacy of the Planned Technique

The intended method is proficient and realistic; in the IVPFHSS setting, we construct an inventive MCGDM model on the IVPFHSWA and PFHSEWG operators. Our planned model is more talented than prevalent techniques and can produce the most subtle implications in MCGDM difficulties. The cooperative model is multipurpose and conversant, adjusting to evolving instability, commitment, and output. Different models have particular ranking processes, so there is a straight modification among the rankings of the anticipated methods conferring to their expectations. This systematic study and assessment determined that the outcomes attained from prevailing procedures are irregularly equated to hybrid structures. Also, due to some favorable situations, many mixed IVFS, IVIFS, IVPFS, IVIFSS, and IVPFSS grow into special in IVPFHSS. It is easy to syndicate insufficient and ambiguous data in DM procedures. Imprecise and anxious facts are mixed in the DM procedure. Hence, our scheduled method will be more proficient, crucial, superior, and better than numerous mixed FS structures. Table 5 below presents the projected technique and the characteristic analysis of some existing models.

Table 5: Feature analysis of different models with a proposed model

	Fuzzy information	Aggregated attributes information	Aggregated sub-attributes information of any attribute	Aggregated information in intervals form
IVFS [2]	✓	×	×	✓
IVIFWA [41]	✓	×	×	✓
IVIFWG [42]	✓	×	×	✓
IVPFWA [14]	✓	×	×	✓
IVPFWG [15]	✓	×	×	✓
IFSWA [20]	✓	✓	×	×
IFSWG [20]	✓	✓	×	×
IVIFSWA [22]	✓	✓	×	✓
IVIFSWG [22]	✓	✓	×	✓
PFSWA [24]	✓	✓	×	×
PFSWG [24]	✓	✓	×	×
PFSIWA [25]	✓	✓	×	×
PFSIWG [25]	✓	✓	×	×
IVPFSWA [29]	✓	✓	×	✓
IVPFSWG [29]	✓	✓	×	✓
IFHSWA [36]	✓	✓	✓	×
IFHSWG [36]	✓	✓	✓	×
PFHSWA [43]	✓	✓	✓	×
PFHSWG [43]	✓	✓	✓	×
PFHSIWA [44]	✓	✓	✓	×

(Continued)

Table 5 (continued)

	Fuzzy information	Aggregated attributes information	Aggregated sub-attributes information of any attribute	Aggregated information in intervals form
PFHSIWG [44]	✓	✓	✓	×
Proposed IVPFHSWA	✓	✓	✓	✓
Proposed IVPFHSWG	✓	✓	✓	✓

5.2 Comparative Analysis

To prove the usefulness of the planned technique, we equate the attained consequences with some prevailing approaches under the setting of IVPFS, IVIFSS, and IVPFSS. A summary of outcomes is specified in Table 6. Wang et al. [41] developed IVIFWA, and Xu et al. [42] presented that IVIFWG operators cannot compute the parametrized values of the alternatives. Furthermore, if any expert considers the MD and NMD whose sum exceeds 1, the AOs mentioned above fail to accommodate the scenario. Zulqarnain et al. [22] established AOs for IVIFSS that cannot accommodate the decision-maker's selection when the sum of upper MD and NMD of the parameters surpasses one. Peng et al.'s [14] interval-valued Pythagorean fuzzy weighted average operator and Rahman et al. [15] interval-valued Pythagorean fuzzy weighted geometric operator cannot handle the parametrized values of the alternatives. Zulqarnain et al. [29] established the interval-valued Pythagorean fuzzy soft weighted average and interval-valued Pythagorean fuzzy soft weighted geometric operators to deal with the parameterized values of alternatives. But, these AOs fail to handle the scenario if any parameter contains a different sub-parameter. Furthermore, if any parameter has any other sub-parameter, the IVPFHSS reduces to the interval-valued Pythagorean fuzzy soft set. Suppose the sum of upper values of MD and NMD is less or equal to 1. Then, IVPFHSS is reduced to IVIFHSS. Thus, IVPFHSS is the most generalized form of interval-valued Pythagorean fuzzy set and IVPFSS. Hence, based on the details mentioned above, the anticipated operators in this paper are more influential, consistent, and prosperous.

Table 6: Comparison of proposed operators with some existing operators

Authors	AO	$\mathfrak{J}^1$	$\mathfrak{J}^2$	$\mathfrak{J}^3$	$\mathfrak{J}^4$	Alternatives ranking	Optimal choice
Wang et al. [41]	IVIFWA	0.4573	0.3509	0.3681	0.2146	$\mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^2 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Xu et al. [42]	IVIFWG	0.3952	0.3104	0.2914	0.2753	$\mathfrak{J}^1 > \mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Peng et al. [14]	IVPFWA	0.0251	0.0154	0.0198	0.0247	$\mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^2$	$\mathfrak{J}^1$

(Continued)

Table 6 (continued)

Authors	AO	$\mathfrak{J}^1$	$\mathfrak{J}^2$	$\mathfrak{J}^3$	$\mathfrak{J}^4$	Alternatives ranking	Optimal choice
Rahman et al. [15]	IVPFWG	0.0856	0.0475	0.0786	0.0302	$\mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^2 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Zulqarnain et al. [22]	IVIFSWA	0.0723	0.0530	0.0584	0.0235	$\mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^2 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Zulqarnain et al. [22]	IVIFSWG	0.7234	0.2365	0.5840	0.6525	$\mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^2$	$\mathfrak{J}^1$
Zulqarnain et al. [29]	IVPFSWA	0.0834	0.0377	0.0121	0.0141	$\mathfrak{J}^1 > \mathfrak{J}^2 > \mathfrak{J}^4 > \mathfrak{J}^3$	$\mathfrak{J}^1$
Zulqarnain et al. [29]	IVPFSWG	0.0754	0.0524	0.0251	0.0114	$\mathfrak{J}^1 > \mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Proposed	IVPFHSWA	0.0599	0.0578	0.0266	−0.0382	$\mathfrak{J}^1 > \mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^1$
Proposed	IVPFHSWG	0.0752	0.0654	0.0242	0.0114	$\mathfrak{J}^1 > \mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^1$

The graphical demonstration of Table 6 is given in the following Fig. 1.

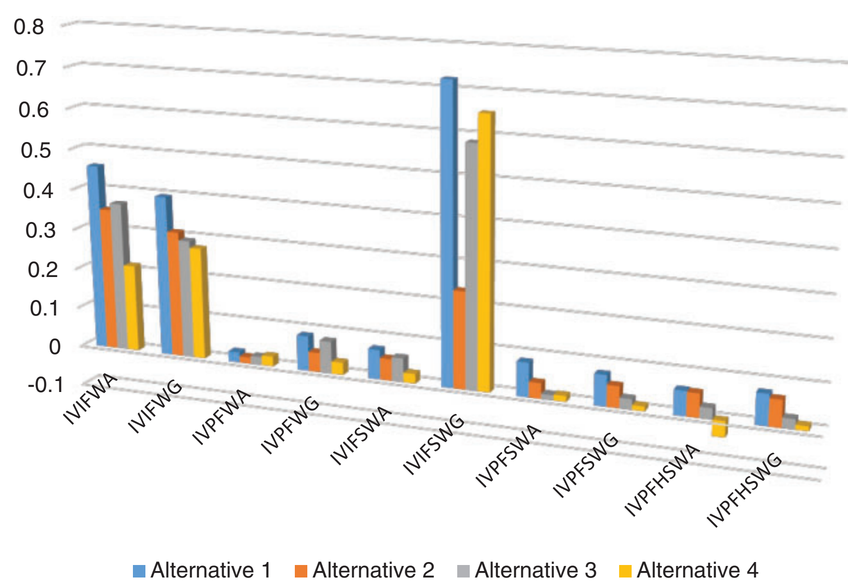

Figure 1: Comparative analysis of the proposed approach with existing models

6 Conclusion

In manufacturing, the refined solidity of manipulation is neutral; authentic materials and fabrication encompass wide-ranging materials. Mathematical demonstration in industrial inventiveness formations exploits all assets while merging design intentions under financial, superior, and safety limitations. Inquiries must be restricted for best judgment, consulting to decision requirements. In genuine DM, the valuation of alternative facts conveyed by the professional is consistently inaccurate, irregular, and impulsive, so IVPFHSNs can be used to comport this uncertain data. The principal objective of this work is to prolong the Pythagorean fuzzy hypersoft sets to interval-valued Pythagorean fuzzy hypersoft sets. Firstly, we introduce the operational laws for the interval-valued Pythagorean fuzzy hypersoft setting. Considering the developed operational laws, we presented the IVPFHSWA and IVPFHSWG operators for IVPFHSS with their desired properties. Also, a DM method has been planned to address MCGDM complications based on the validated operators. To state the stoutness of the developed methodology, we deliver a comprehensive mathematical illustration for MS in manufacturing engineering. A comprehensive analysis of some existing procedures is described to ensure the practicality of the developed approach. Lastly, based on the consequences achieved, it is determined that the method proposed in this study is the most practical and operative way to explain the problem of MCGDM.

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