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ARTICLE

The Class of Atomic Exponential Basis Functions $\mathbf{EFup_n(x,\omega)}$-Development and Application

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ABSTRACT

The purpose of this paper is to present the class of atomic basis functions (ABFs) which are of exponential type and are denoted by $EFup_n(x,\omega)$. While ABFs of the algebraic type are already represented in the numerical modeling of various problems in mathematical physics and computational mechanics, ABFs of the exponential type have not yet been sufficiently researched. These functions, unlike the ABFs of the algebraic type $Fup_n(x)$, contain the tension parameter ω, which gives them additional approximation properties. Exponential monomials up to the nth degree can be described exactly by the linear combination of the functions $EFup_n(x,\omega)$. The function $EFup_n$ for $n=0$ is called the "mother" ABF of the exponential type, i.e., $EFup_0(x,\omega) \equiv Eup(x,\omega)$. In other words, the functions $EFup_n(x,\omega)$ are elements of the linear vector space EUP_n and retain all the properties of their "mother" function $Eup(x,\omega)$. Thus, this paper, in terms of its content and purpose, can be understood as a sequel of the article by Brajčić Kurbaša et al., which shows the basic properties and application of the basis function $Eup\ (x,\omega)$. This paper presents, in an analogous way, the development and application of the exponential basis functions $EFup_n(x,\omega)$. Here, for the first time, expressions for calculating the values of the functions $EFup_n(x,\omega)$ and their derivatives are given in a form suitable for application in numerical analyses, which is shown in the verification examples of the approximations of known functions.

KEYWORDS

Exponential atomic basis functions; Fourier transform; compact support; tension parameter

1 Introduction

Numerical methods are indispensable for the successful simulation of physical and engineering problems. Many different numerical approaches and methods have been proposed in recent decades. The classical methods are the finite element method (FEM), the finite difference method (FDM), the finite volume method (FVM), the boundary element method (BEM), and the discrete element method (DEM) [1–3]. In addition to traditional mesh-based methods, there are many others, such as various meshless methods [4–6].

The choice of the basis functions plays a key role in all numerical methods. The idea of choosing basis functions that correspond to the class of solutions of the problems we are solving has long been accepted, but, in practice, rarely implemented. Polynomials are fundamental to modeling and numerical methods. They provide canonical local approximations to smooth functions and are used extensively in geometric design. Polynomials not only provide very accurate approximations of smooth functions but also guarantee convergence for any continuous function on a compact interval.

Whereas classical polynomials have dominated in the field of numerical analysis, spline-based basis functions [7] play a crucial role in the field of computational geometry. The true popularity of spline functions for numerical analysis was achieved by the introduction of the concept of isogeometric analysis (Hughes et al. [8] and Cottrell et al. [9]). B-splines play an important role in many areas of applied mathematics, computer science, and engineering. Typical applications arise in the approximation of functions and data, automated design and manufacturing, computer graphics, and numerical simulations. This diversity of areas and techniques involved makes B-splines an extremely interesting research topic, which has attracted a growing number of scientists in universities and industry.

In addition to spline functions, relatively lesser-known atomic basis functions have been used in recent times [10–13]. Atomic basis functions can be placed between classical polynomials and spline functions. However, in practice, their use as basis functions is closer to splines or wavelets (see Beylkin et al. [14]). Rvachev et al. [10], in their pioneering work, called these basis functions "atomic" because they span the vector spaces of all three fundamental functions in mathematics: algebraic, exponential, and trigonometric polynomials. The authors of this article have worked intensively on the development and application of ABFs of algebraic type in solving problems of structural mechanics and have therefore demonstrated their significant potential compared to conventional procedures with finite elements. Gotovac [12] systematized the existing knowledge regarding atomic basis functions of algebraic type and transformed them into a numerically appropriate form, especially Fup basis functions as a typical member of the atomic class of basis functions. Gotovac et al. [15] showed the basic possibilities of using atomic functions in structural mechanics and numerical analysis. The work in [16] gives a generalization of atomic functions to the multivariable case. The use of Fup basis functions, which are atomic functions of the algebraic type, has been shown to solve the problem of signal processing [17], the initial value problem [18], the boundary value problems using the Fup Collocation Method [19], the boundary-initial value problems [20], elasto-plastic analysis of prismatic bars subjected to torsion [21], and modeling of groundwater flow and transport problems [22]. Gotovac et al. [23] presented a true multiresolution approach based on the Adaptive Fup Collocation Method (AFCM). Kamber et al. [24] set the foundation for an efficient adaptive spatial procedure by developing a one-dimensional hierarchical Fup (HF) basis functions. The works in [25,26] gave a brief analysis of the current publications regarding ABFs, from the first publications to current ones.

In the mentioned works, the advantage of atomic basis functions of algebraic type, which significantly improve the quality of numerical solutions in relation to classical basis functions, for example, splines and wavelets, is confirmed. The numerical results thus obtained were the motivation for the development of ABF of the exponential type which are wider than algebraic space, moreover algebraic ABFs space is contained in exponential ABFs space.

The numerical modeling of different physical and engineering problems characterized by large local gradients and singularities often presents a challenge in terms of choosing a numerical approach and basis functions. Classic examples of such are the advection–dispersion equation and the heat conduction equation, which describe the transfer of mass and energy, respectively; beams and plates on a flexible foundation; and special problems of loss of stability. For the simulation of such physical problems, exponential basis functions would be a good choice. Improving the quality of numerical analyzes of problems whose solutions have an exponential form is the main motivation of this paper.

The atomic functions of the exponential type have been developed only at the basic level. In [12], the previous knowledge about ABF of the exponential type was presented, which was later expanded and upgraded in [27]. Reference [28], partly resulted from [27], showed the basic properties and application of the maternal basis function $Eup(x, \omega)$, by which the whole class of atomic functions of the exponential type $EFup_n(x, \omega)$ is generated and given in this article as natural sequel of the [28] to complete "the story" of the ABFs of the exponential type.

The content of this work is focused on the mathematical background, approximation properties, and applications of exponential basis functions $EFup_n(x, \omega)$. There are no articles in the literature that deal with these basis functions. So, this paper is intended to provide novel information for scientists and engineers who are interested in applying the state-of-the-art atomic exponential basis functions to solve real-life problems. The paper presents expressions for the necessary mathematical operations of the ABFs $EFup_n(x, \omega)$ in a simpler, more understandable and more user-friendly way. New expressions have been derived, especially the expression for calculating the value of the function and the desired number of derivatives at an arbitrary point of the basis function support, which is the original contribution of this paper and, most importantly, the rules (elements) for their practical use.

The following section of the article refers to the description of the ABF class of the algebraic type. The procedure used to generate the class of functions $Fup_n(x)$ and the determination of their derivatives are presented, and the basic properties are given in a new and original way, and that is starting from the well-known Fourier transform and the convolution theorem in a way suitable for defining and deriving the ABFs of the exponential type $EFup_n(x, \omega)$, shown in Section 3. The implementation of ABF $EFup_n(x, \omega)$ in the numerical approximations of the given functions is shown in Section 4. Finally, the conclusions are given in Section 5.

2 ABFs of the Algebraic Type: $\mathbf{Fup_n}$ (x)

Atomic Basis Functions (ABFs) are infinitely derivable finite solutions of functional differential equations of the type:

$$Ly(x) = \lambda \sum_{k=1}^{M} C_k y(ax - b_k), \tag{1}$$

where L is a linear differential operator with constant coefficients, λ is a scalar quantity other than zero, C_k are the solution coefficients, $a > 0$ is the support length parameter of the finite function, and b_k are the coefficients that determine the displacements of the finite basis functions [10–12,15].

The type of finite function $y(x)$ from the class of atomic basis functions is determined by choosing the operator L in Eq. (1). Thus, we distinguish the atomic basis functions of the algebraic, exponential, and trigonometric types.

The functions $Fup_n(x)$ are finite ABFs of the algebraic type from the class C^∞ with a compact support, and they are also elements of the universal vector space UP_n. The index n denotes the highest degree of a polynomial that can be accurately represented in the form of a linear combination of basis functions obtained by moving the function $Fup_n(x)$ for the characteristic segment $\Delta x_n = 2^{-n}$. For $n = 0$, it holds that:

$$Fup_0(x) \equiv up(x). \tag{2}$$

The functions $Fup_n(x)$ retain all the good properties of the "maternal" function $up(x)$ [10–12,15], while for the development of a given function, a much smaller number than that of the basis functions obtained by moving the function $up(x)$ is required. For a sufficiently high n function $Fup_n(x)$ has a very small support length, so any function $Fup_k(x), k < n$, including the function $up(x)$, can be expressed using the function $Fup_n(x)$.

Unlike in references [10,11] which define ABFs from Eq. (1), the authors of this paper determine ABFs from their known Fourier transform (FT), and then from their known FT determine everything necessary for their use (e.g., derivatives, integrals, moments, etc.). Namely, we can say that in the "frequency domain" the construction of ABFs becomes more transparent.

The FT of the function $Fup_0(x)$, according to Eq. (2), corresponds exactly to the FT of the function $up(x)$ from [15,28], i.e.,

$$F_0(t) = \prod_{j=1}^{\infty} \frac{\sin\left(t \cdot 2^{-j}\right)}{t \cdot 2^{-j}}. \tag{3}$$

The Fourier transform of the function $Fup_n(x)$ is given by the expression [15,28]:

$$F_n(t) = \left(\frac{\sin\left(t \cdot 2^{-n-1}\right)}{t \cdot 2^{-n-1}}\right)^{n+1} \prod_{j=n+2}^{\infty} \frac{\sin\left(t \cdot 2^{-j}\right)}{t \cdot 2^{-j}}. \tag{4}$$

Thus, according to Eq. (4), the functions $Fup_n(x)$ can be written in integral form:

$$Fup_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \cdot F_n(t) dt. \tag{5}$$

From the known FT, as is shown similarly for the function $up(x)$ in [28], the functions $Fup_n(x)$ can also be generated using the convolution theorem. In Eq. (4), it is seen that the FTs $F_n(t)$ of the basis functions $Fup_n(x)$ are equal to the product of the nth degree B-spline FT compressed on the support of length $(n+1)2^{-n}$ and the function $up(x)$ FT from Eq. (3) compressed on a support of length 2^{-n}. Thus, the functions $Fup_n(x)$ can be written using the convolution theorem in the form:

$$Fup_n(x) = B_n(2^n x) * up\left(2^{n+1} x\right). \tag{6}$$

According to Eq. (6), the support of the function $Fup_n(x)$ is an interval composed of $n+2$ segments of length 2^{-n}, which are called characteristic segments, that is,

$$supp\ Fup_n(x) = \left[-(n+2) \cdot 2^{-n-1}, (n+2) \cdot 2^{-n-1}\right]. \tag{7}$$

The functional differential equations of the basis functions $Fup_n(x)$ are of the following form [12,15]:

$$Fup_n'(x) = 2 \sum_{k=0}^{n+2} \left(C_n^k - C_n^{k-2}\right) \cdot Fup_n\left(2x - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right), \tag{8}$$

where C_n^k are binomial coefficients.

Solving the functional differential Eq. (8), or Eqs. (4) and (5), is not numerically convenient for calculating the values of the function $Fup_n(x)$. Practically, the most convenient possibility to construct the functions $Fup_n(x)$ is in the form of a linear combination of functions $up(x)$ mutually shifted for the characteristic segment 2^{-n}, i.e.,

$$Fup_n(x) = \sum\nolimits_{k=0}^{\infty} C_k(n) \cdot up\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right). \tag{9}$$

The "zero*th*" coefficient follows from Eq. (9) and is

$$C_0 = 2^{C_{n+1}^2} = 2^{n(n+1)/2}. \tag{10}$$

The other coefficients are obtained in the form $C_k(n) = C_0(n) \cdot C'_k(n)$,, where the auxiliary coefficients $C'_k(n)$ are calculated by the recursive formula [15]:

$$\begin{aligned} &C'_0(n) = 1, \quad for\ k = 0; \quad for\ k > 0: \\ &C'_k(n) = (-1)^k C_{n+1}^k - \sum\nolimits_{j=1}^{min\left\{k;2^{n+1}-1\right\}} C'_{k-j}(n) \cdot \delta_{j+1}. \end{aligned} \tag{11}$$

The coefficients from Eq. (11) for $n \le 6$ and $k \le 9$ are given in Table 1.

Table 1: Coefficients $\boldsymbol{C'_k(n)}$ for $\boldsymbol{n \le 6}$ and $\boldsymbol{k \le 9}$

C'_kn n	C'_0	C'_1	C'_2	C'_3	C'_4	C'_5	C'_6	C'_7	C'_8	C'_9
0	1	0	0	0	0	0	0	0	0	0
1	1	−1	1	−1	1	−1	1	−1	1	−1
2	1	−2	2	−2	3	−4	4	−4	5	−6
3	1	−3	4	−4	5	−7	8	−8	10	−14
4	1	−4	7	−8	9	−12	15	−16	18	−24
5	1	−5	11	−15	17	−21	27	−31	34	−42
6	1	−6	16	−26	32	−38	48	−58	65	−76

The derivatives of the function $Fup_n(x)$ are obtained by a linear combination of the derivatives of the shifted functions $up(x)$ using the coefficients from Eq. (11), i.e.,

$$Fup_n^{(m)}(x) = \sum\nolimits_{k=0}^{\infty} C_k(n) \cdot up^{(m)}\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right), \tag{12}$$

where m is the order of derivation, and n is the order of the basis function. Fig. 1 shows the function $Fup_2(x)$ and its first three derivatives. The third, and all further derivatives, of the function $Fup_2(x)$ correspond in parts to the compressed function $up(x)$.

The integrals of the function $Fup_n(x)$ are also obtained by a linear combination of the integrals of the shifted functions $up(x)$ using the coefficients from Eq. (11):

$$\int_{-\infty}^{x} Fup_n(x)\,dx = \sum\nolimits_{k=0}^{\infty} C_k \int_{-\infty}^{x} up\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right) dx. \tag{13}$$

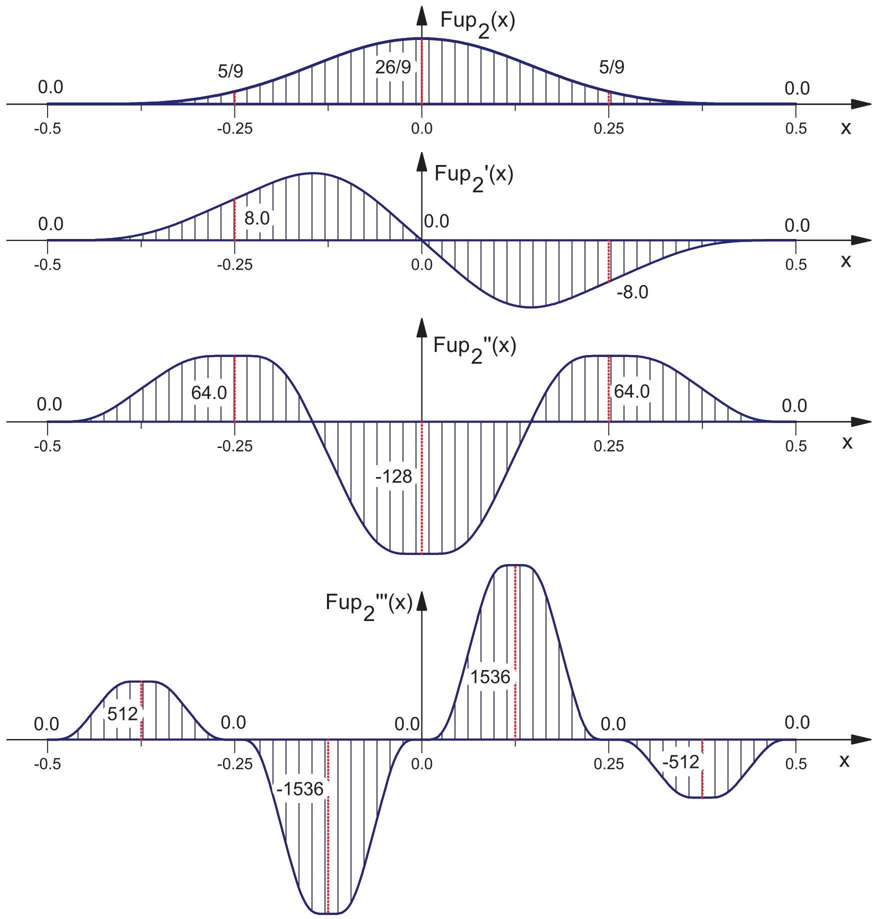


Figure 1: Function $\boldsymbol{Fup_2(x)}$ and it is first three derivatives

3 ABFs of the Exponential Type: $EFup_n$ (x, ω)

The functions $EFup_n(x,\omega)$ are finite functions of class C^∞ with compact support, and are the elements of linear vector space EUP_n [12,27,28], and retain all the properties of their "maternal" basis function $Eup(x,\omega)$. The index 'n' denotes the largest degree of an exponential monomial that can be represented exactly in the form of a linear combination of mutually shifted functions $EFup_n(x,\omega)$ on a characteristic segment of length $\Delta x_n = 2^{-n}$.

3.1 Generating the Fourier Transform of the Function $EFup_n(x, \omega)$

The Fourier transform of the atomic basis function $EFup_n(x,\omega)$ is constructed by a similar procedure applied to the function $Fup_n(x)$ using the so-called "fragmentation process" of the FT as shown below.

The first from the ABF class of the exponential type $EFup_n(x,\omega)$ for $n = 0$ is precisely the "maternal" basis function $Eup(x,\omega)$, i.e.,

$$EFup_0\,(x,\omega) = Eup\,(x,\omega). \tag{14}$$

Thus, according to Eq. (14), the Fourier transform of the function $EFup_0(x, \omega)$ is determined by the expression from [28]:

$$F_0(t) = \prod_{j=1}^{\infty} \frac{\omega}{2sh(\omega/2)} \frac{sh(\omega/2 + i \cdot t/2^j)}{\omega/2 + i \cdot t/2^j}. \tag{15}$$

Writing Eq. (15) in an extended form, applying the basic trigonometric relations, and fragmenting the expression thus obtained (omitting the term $ch\,(\alpha/2)$), after arranging the expression, we obtain the Fourier transform of the finite function $EFup_1(x, \omega)$ of the form:

$$F_1(t) = \left(\frac{\omega}{2sh(\omega/2)} \frac{sh(\omega/2 + i \cdot t/4)}{\omega/2 + i \cdot t/4} \frac{\omega}{4sh(\omega/4)} \frac{sh(\omega/4 + i \cdot t/4)}{\omega/4 + i \cdot t/4}\right) \cdot \prod_{j=3}^{\infty} \frac{\omega}{2sh(\omega/2)} \frac{sh(\omega/2 + i \cdot t/2^j)}{\omega/2 + i \cdot t/2^j}. \tag{16}$$

The expression in parentheses from Eq. (16) represents the FT of the corresponding exponential spline, while the product from Eq. (16) is the FT of the function $Eup(x, \omega)$ from Eq. (15), condensed on the support $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (see Fig. 2b). Continuing the presented procedure and generalizing it, we obtain the class of Fourier transforms of the exponential functions $EFup_n(x, \omega)$ in the form:

$$F_n(t) = \prod_{j=1}^{n+1} \frac{\omega}{2^j sh(\omega/2^j)} \cdot \frac{sh(\omega/2^j + i \cdot t/2^{n+1})}{\omega/2^j + i \cdot t/2^{n+1}} \cdot \prod_{k=n+2}^{\infty} \frac{\omega}{2sh(\omega/2)} \cdot \frac{sh(\omega/2 + i \cdot t/2^k)}{\omega/2 + i \cdot t/2^k}. \tag{17}$$

Thus, analogously to the ABF $Fup_n(x)$, according to Eq. (17), the function $EFup_n(x, \omega)$ can be written using the convolution theorem in the following form:

$$EFup_n(x, \omega) = \left[\varphi_0^0(x, \omega) * \ldots * \varphi_0^n(x, \omega)\right] * 2^{n+1} Eup(2^{n+1} \cdot x, \omega), \tag{18}$$

where

$$\varphi_0^j(x, \omega) = \frac{2^{j-1} \cdot \omega}{sh(\omega/2^{n-j+1})} e^{2^j \omega x}, j = 0, \ldots, n \tag{19}$$

are the zero-degree exponential splines $\varphi_0^j(x, \omega)$ normalized to the support $supp\ \varphi_0^j(x, \omega) = [-2^{-(n+1)}, 2^{-(n+1)}]$. The convolution of splines $\varphi_0^j(x, \omega)$ in square brackets in Eq. (18) represents the corresponding exponential spline of nth degree $f_n(x, \omega)$; thus, Eq. (18) can also be written in the form:

$$EFup_n(x, \omega) = f_n(x, \omega) * 2^{n+1}\, Eup(2^{n+1} \cdot x, \omega). \tag{20}$$

When the parameter ω weighs zero, the exponential ABF turns into an algebraic ABF, so Eqs. (17) and (20) become Eqs. (4) and (6), respectively.

Fig. 2 shows a graphical interpretation of Eq. (18) (or Eq. (20)), i.e., the procedure for generating the function $EFup_n(x, \omega), n = 0, 1, 2$ using the convolution theorem. For example, the function $EFup_2(x, \omega)$ represents the convolution of four functions normalized to the support $\left[-\frac{1}{8}, \frac{1}{8}\right]$: three zero-degree exponential splines and the condensed $Eup(x, \omega)$ function, as shown in Fig. 2c, i.e.,

$$EFup_2(x, \omega) = \frac{\omega \cdot e^{\omega x}}{2 \cdot sh\left(\frac{\omega}{8}\right)} * \frac{\omega \cdot e^{2\omega x}}{sh\left(\frac{\omega}{4}\right)} * \frac{2 \cdot \omega \cdot e^{4\omega x}}{sh\left(\frac{\omega}{2}\right)} * 8 \cdot Eup\,(8x, \omega).$$

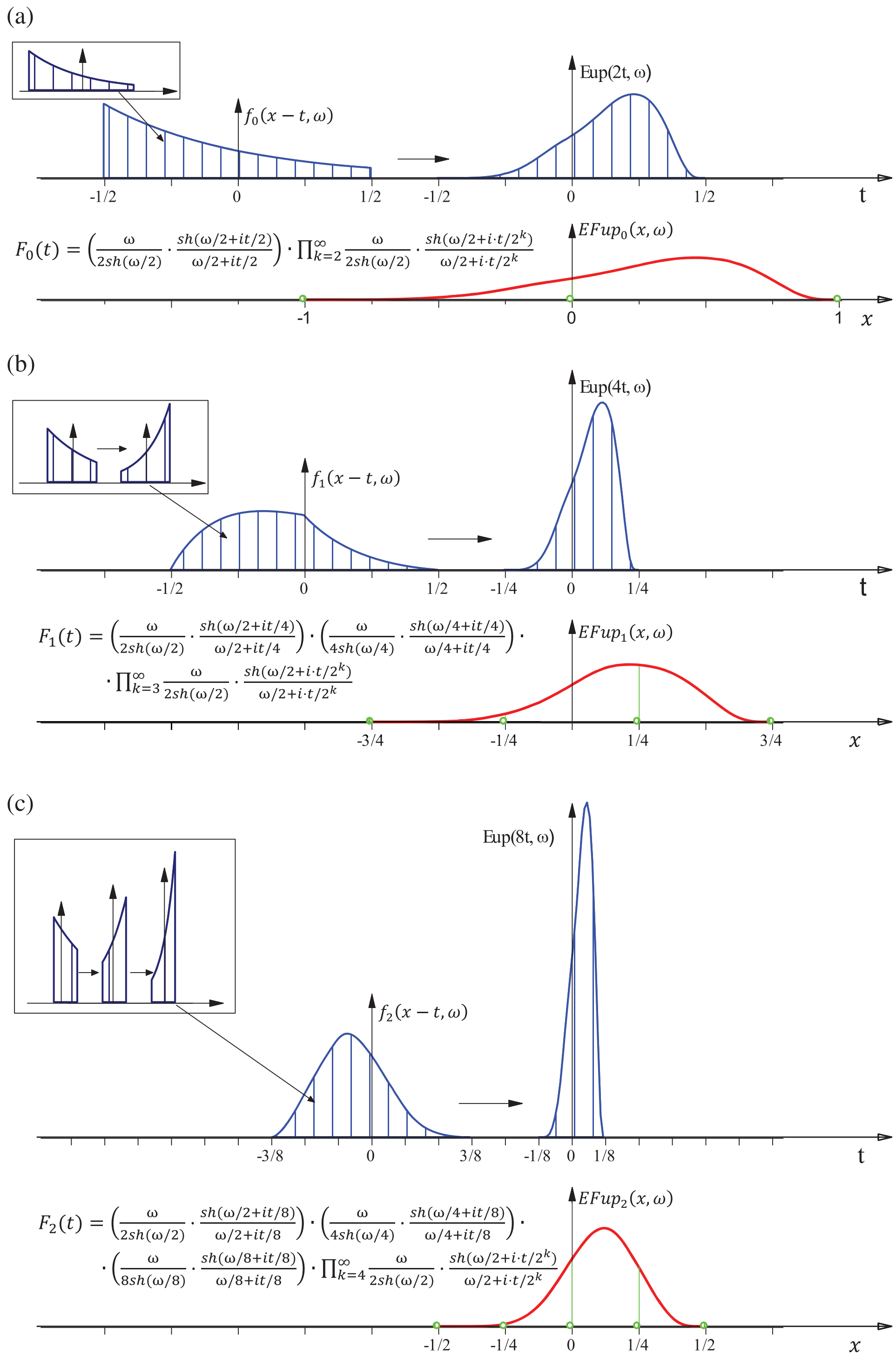
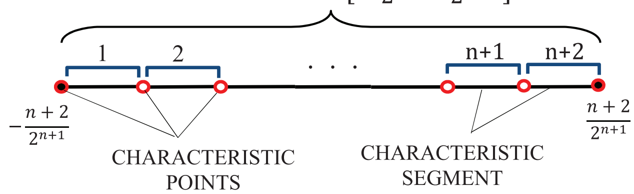


Figure 2: Generating the exponential functions $\boldsymbol{EFup_n}(\boldsymbol{x},\omega)$ for: (a) $\boldsymbol{n=0}$; (b) $\boldsymbol{n=1}$; (c) $\boldsymbol{n=2}$

According to Eq. (18), the support of the function $EFup_n(x,\omega)$ is an interval composed of $(n+2)$ segments of length 2^{-n}. The characteristic points are the boundary points of the characteristic segment.

$$supp\ EFup_n(x,\omega) = \left[-\frac{n+2}{2^{n+1}}, \frac{n+2}{2^{n+1}}\right]$$



The inverse Fourier transform, i.e., the function $EFup_n(x,\omega)$, having satisfied the Paley–Wiener normalization condition, can be expressed in the form:

$$EFup_n(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} F_n(t)dt. \tag{21}$$

By developing Eq. (21) in the Fourier series, the "original" of the function $EFup_n(x,\omega)$ can be determined at arbitrary points. However, as for the algebraic ABFs $Fup_n(x)$, the most favorable possibility of constructing the $EFup_n(x,\omega)$ functions is in the form of a linear combination of shifted $Eup(x,\omega)$ functions, as shown below.

Fig. 3 shows the function $EFup_2(x,\omega)$ for different values of the parameter ω. Similar to the "maternal" function $Eup(x,\omega)$, the function is tilted to the left for negative values of the parameter ω, while for positive ones it is tilted to the right. In the limitary case when $\omega \to 0$, the exponential function $EFup_2(x,\omega)$ is identically equal to the algebraic function $Fup_2(x)$.

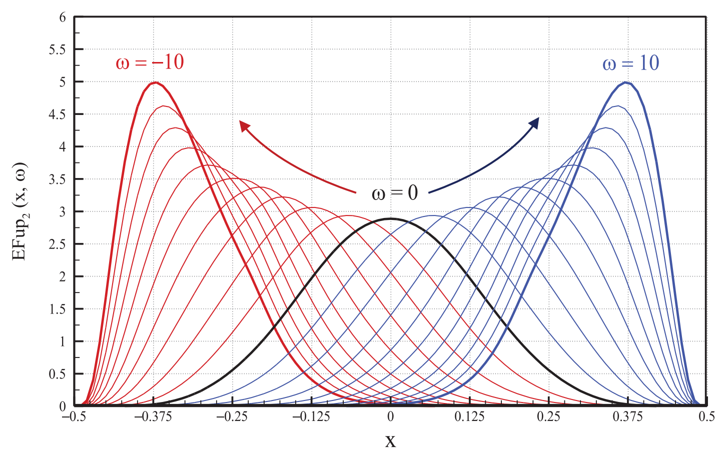


Figure 3: Function $\boldsymbol{EFup_2(x,\omega)}$ for different values of the parameter $\mathbf{-10} \le \omega \le \mathbf{10}$

3.2 *Functional Differential Equation of the Function EFup$_n$(x,ω)*

Analogous to the algebraic ABF $Fup_n(x)$, the functional differential equation of the function $EFup_n(x,\omega)$ is determined from the Fourier transform (17), which can also be written as follows:

$$F_n(t) = \frac{\omega}{2^{n+1}sh(\omega/2^{n+1})} \cdot \frac{sh(\omega/2^{n+1} + i \cdot t/2^{n+1})}{\omega/2^{n+1} + i \cdot t/2^{n+1}} \cdot \prod_{k=2}^{n+1} \frac{ch(\omega/2^k + i \cdot t/2^{n+2})}{ch(\omega/2^k)} \cdot F_n\left(\frac{t}{2}\right). \tag{22}$$

If Eq. (22) is written in the form of exponential functions and multiplied by $(\omega + it)$, arranging the members of the left and right sides gives the functional differential equation of the function $EFup_{\mathrm{n}}(x, \omega)$ of the form:

$$EFup_n'(x,\omega) - \omega \cdot EFup_n(x,\omega) = \sum_{i=1}^{2^{n+1}} \alpha_i \cdot EFup_n(2x + \beta_i, \omega), \tag{23}$$

where the coefficients are

$$\beta_i = \pm \begin{cases} j \cdot 2^{-2s} & \mathit{for}\ n = 2s \\ (2j+1) \cdot 2^{-2s} & \mathit{for}\ n = 2s-1 \end{cases}; \ j = 1, 2, \ldots, 2s \tag{24}$$

and

$$\alpha_i = \frac{2\omega}{e^{\omega} - 1}\left[A_i^n(\omega) - A_{n+2-i}^n(-\omega)\right], \tag{25}$$

where

$$A_i^n(\omega) = \begin{cases} 0 & \mathit{for}\ i > n \\ e^{\omega/2^n} \cdot A_{i-1}^{n-1}(\omega) + A_i^{n-1}(\omega) & \mathit{for}\ i \le n \end{cases}; \ A_0^0(\omega) = e^{-\omega/2}. \tag{26}$$

In particular, when the value of the parameter $\omega = 0$, Eqs. (23)–(26) become equivalent to Eq. (8).

3.3 The Values of the Function $EFup_n(x, \omega)$ in Characteristic Points

The values of the functions $EFup_n(x, \omega)$ at arbitrary discrete points x can be determined by Convolution (20) as a solution of the following integral:

$$EFup_n(x,\omega) = 2^{n+1} \int_{-\infty}^{\infty} f_n(x-t,\omega) \cdot Eup(2^{n+1}t, \omega)dt, \tag{27}$$

where $f_n(x, \omega)$ is the corresponding exponential spline defined as the result of the convolution of zero-degree exponential splines, i.e.,

$$f_n(x,\omega) = \varphi_0^0(x,\omega) * \ldots * \ \varphi_0^n(x,\omega), \tag{28}$$

while $\varphi_0^j(x, \omega)$ are determined by Eq. (19).

However, calculating the integral (27) at arbitrary points x is not a simple or numerically favorable procedure, and therefore solving the integral (27) is used only to determine the values of the basis functions $EFup_n(x_k, \omega)$ at the characteristic points x_k.

Fig. 4 shows the graphical interpretation of the integral (27) for the basis function $EFup_2(x, \omega)$ at the characteristic points $x_k = -\frac{1}{4}, 0, \frac{1}{4}$.

For example, the value of the function $EFup_2(x, \omega)$ at the point $x_k = -1/4$, according to Eq. (27), corresponds to the solution of the following integral:

$$EFup_2(x_k,\omega) = 8 \cdot \int_{-1/8}^{x_k+3/8} f_2(x_k - t) \cdot Eup(8t,\omega)\, dt, \tag{29}$$

which, when written in exponential form and using the appropriate substitutions after arranging, has the final form:

$$EFup_2\left(-\frac{1}{4},\omega\right) = \frac{e^{-3\omega/4} \cdot (e^{3\omega/4} + 3 \cdot e^{\omega/2} + 3 \cdot e^{\omega/4} + 3)}{3 \cdot (e^{\omega/4} + 1) \cdot (e^{\omega/2} + e^{\omega/4} + 1)} \cdot \lambda_0(\omega), \tag{30}$$

where $\lambda_0(\omega)$ is the value of the "maternal" function $Eup\,(x,\omega)$ at the local origin, i.e., the point $x = 0$, see [28].

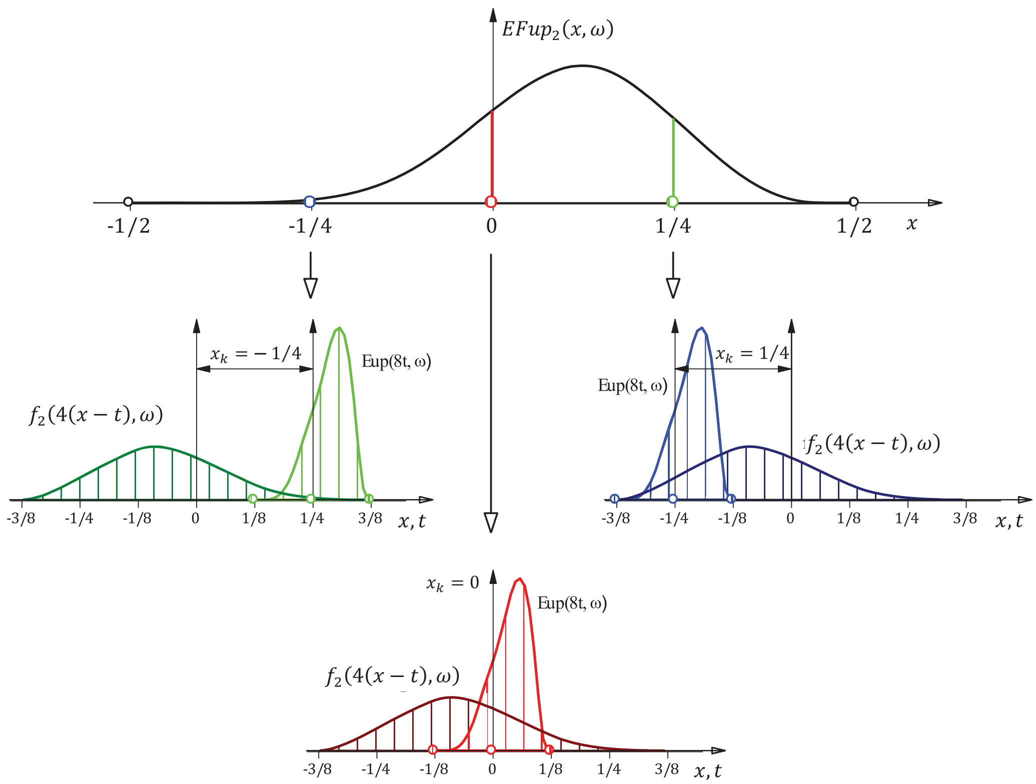


Figure 4: The values of the function $\boldsymbol{EFup_2(x,\omega)}$ at the characteristic points $\boldsymbol{x_k} = -\frac{\mathbf{1}}{\mathbf{4}}, \mathbf{0}, \frac{\mathbf{1}}{\mathbf{4}}$ using the convolution theorem

The values at other characteristic points of the function $EFup_2\,(x,\omega)$ are determined by an analogous procedure:

$$EFup_2\,(0,\omega) = \frac{e^{-\omega/4}\cdot\left(3\cdot e^{\omega}+6\cdot e^{3\cdot\omega/4}+8\cdot e^{\omega/2}+6\cdot e^{\omega/4}+3\right)}{3\cdot\left(e^{\omega/2}+e^{\omega/4}+1\right)}\cdot\lambda_0\,(\omega)\,,$$

$$EFup_2\left(\frac{1}{4},\omega\right) = \frac{e^{3\cdot\omega/4}\cdot\left(3\cdot e^{3\cdot\omega/4}+3\cdot e^{\omega/2}+3\cdot e^{\omega/4}+1\right)}{3\cdot\left(e^{\omega/4}+1\right)\cdot\left(e^{\omega/2}+e^{\omega/4}+1\right)}\cdot\lambda_0\,(\omega)\,, \tag{31}$$

or the values of the basis functions $EFup_n(x,\omega)$ at the characteristic points in general.

As seen in Eqs. (30) and (31), the values of the functions $EFup_n(x,\omega)$ at the characteristic points x_k have a "final" inscription in the form of the product of the corresponding exponential function and the values of the function $Eup(x,\omega)$ at the point $x = 0$, i.e., $\lambda_0(\omega)$ given in [28].

3.4 EFup$_n$(x, ω) as a Linear Combination of Shifted Eup(x, ω) Functions

The values of the basis functions $EFup_n(x,\omega)$ at arbitrary points can be determined, among other methods, by developing Eq. (21) in the Fourier series. However, analogous to the algebraic ABF, the most favorable possibility of constructing the functions $EFup_n(x, \omega)$ is in the form of a linear combination of mutually shifted $Eup\,(x, \omega)$ basis functions:

$$EFup_n(x, \omega) = \sum\nolimits_{k=0}^{\infty} C_k\,(n) \cdot Eup\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}, \omega\right), \tag{32}$$

where $C_k\,(n)$ are the coefficients of the linear combination.

The "zero*th*" coefficient $C_0\,(n)$ is determined in [27] by the expression

$$C_0\,(n) = \prod\nolimits_{i=1}^{n} (e^{\omega/2^{n-i+1}} + 1)^i. \tag{33}$$

The other coefficients of the linear combination $C_k(n), k = 1, \ldots, n+1$ are unknown and are determined as described below.

For example, for the basis function $EFup_2(x, \omega)$, the linear combination (32) has the following form (hereinafter, the functions $Eup\,(x, \omega)$ will be denoted by $y_\omega(x)$ for transparency):

$$EFup_2\,(x, \omega) = C_0\,(2) \cdot y_\omega\left(x - \frac{1}{2}\right) + C_1\,(2) \cdot y_\omega\left(x - \frac{3}{4}\right) + C_2\,(2) \cdot y_\omega\,(x-1) + C_3\,(2) \cdot y_\omega\left(x - \frac{5}{4}\right), \tag{34}$$

or written in characteristic points:

$$\begin{aligned}
&EFup_2\left(-\frac{1}{4}\right) = C_0\,(2) \cdot y_\omega\left(-\frac{3}{4}\right);\\
&EFup_2\,(0) = C_0\,(2) \cdot y_\omega\left(-\frac{1}{2}\right) + C_1\,(2) \cdot y_\omega\left(-\frac{3}{4}\right);\\
&EFup_2\left(\frac{1}{4}\right) = C_0\,(2) \cdot y_\omega\left(-\frac{1}{4}\right) + C_1\,(2) \cdot y_\omega\left(-\frac{1}{2}\right) + C_2\,(2) \cdot y_\omega\left(-\frac{3}{4}\right);\\
&EFup_2\left(\frac{1}{2}\right) = C_0\,(2) \cdot y_\omega\,(0) + C_1\,(2) \cdot y_\omega\left(-\frac{1}{4}\right) + C_2\,(2) \cdot y_\omega\left(-\frac{1}{2}\right) + C_3\,(2) \cdot y_\omega\left(-\frac{3}{4}\right) = 0,
\end{aligned} \tag{35}$$

where the values of the basis function $EFup_2\,(x, \omega)$ at the characteristic points are known and are calculated as shown in Section 3.3.

The expression for the "zero*th*" coefficient follows directly from the first equation in Eq. (35):

$$C_0\,(2) = \frac{EFup_2(-1/4)}{y_\omega(-3/4)}. \tag{36}$$

By including the values from (30) and $y_\omega(-3/4)$ from [28] in Eq. (36), we obtain

$$C_0\,(2) = (e^{\omega/4} + 1) \cdot (e^{\omega/2} + 1)^2, \tag{37}$$

which corresponds to Eq. (33) for $n = 2$.

The "first" coefficient of the linear combination (34) follows from the second equation in Eq. (35) in the form:

$$C_1\,(2) = \frac{EFup_2(0) - C_0(2) \cdot y_\omega(-1/2)}{y_\omega(-3/4)}. \tag{38}$$

By including the coefficient $C_0\,(2)$ and the other required values, we obtain

$$C_1\,(2) = -e^{\omega/4} \cdot (e^{\omega/4} + 1) \cdot (e^{\omega/2} + 1). \tag{39}$$

The expression for the "third" coefficient follows from the third equation in Eq. (35), and so on. By generalizing the presented procedure, a general expression for the coefficients $C_k(n)$ is obtained in the form of a recursive formula:

$$C_k(n) = \frac{EFup_n\left(-\dfrac{n+2}{2^{n+1}} + \dfrac{k+1}{2^n}, \omega\right) - \sum_{i=1}^{k} C_{i-1}(n) \cdot Eup\left(-1 + \dfrac{k+2-i}{2^n}, \omega\right)}{Eup\left(-1 + 2^{-n}, \omega\right)}, \tag{40}$$

where the coefficients $C_0(n)$ are determined by (33).

Thus, to determine the coefficients $C_k(n), k = 0, \ldots, n+1$ of the linear combination (32), it is necessary to know the "zero*th*" coefficient $C_0(n)$ and the values of the functions $Eup(x, \omega)$ and $EFup_n(x, \omega)$ at the characteristic points x_k.

In the limit when $\omega \to 0$, the coefficients $C_k(n)$ for the development of the exponential functions $EFup_n(x, \omega), n = 1, \ldots, 6$ from Eq. (40) become the coefficients $C_k(n)$ for the development of the algebraic basis functions $Fup_n(x)$ in the form of a linear combination of mutually shifted functions $up(x)$ from Eq. (9).

Fig. 5 shows the function $EFup_2(x, \omega)$ in the form of a linear combination of mutually shifted $Eup(x, \omega)$ basis functions.

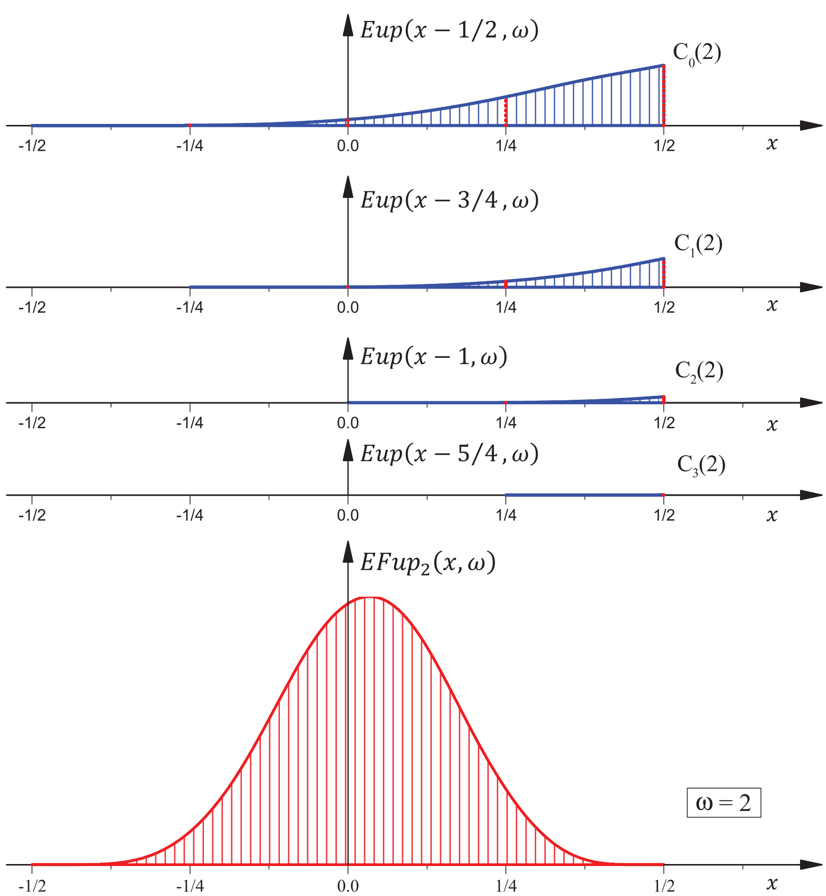


Figure 5: Function $\boldsymbol{EFup_2(x, \omega)}$ as a linear combination of shifted $\boldsymbol{Eup(x, \omega)}$ basis functions

3.5 The Derivatives and Integrals of the Function $EFup_n(x, \omega)$

The derivatives of the function $EFup_n(x, \omega)$ are obtained by a linear combination of the derivatives of the shifted $Eup(x, \omega)$ functions using the coefficients specified in the previous section:

$$EFup_n^{(m)}(x, \omega) = \sum\nolimits_{k=0}^{\infty} C_k(n) \cdot Eup^{(m)}\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}, \omega\right). \tag{41}$$

Fig. 6 shows the basis function $EFup_2(x, \omega)$ and its first three derivatives for the value of the parameter $\omega = 2$.

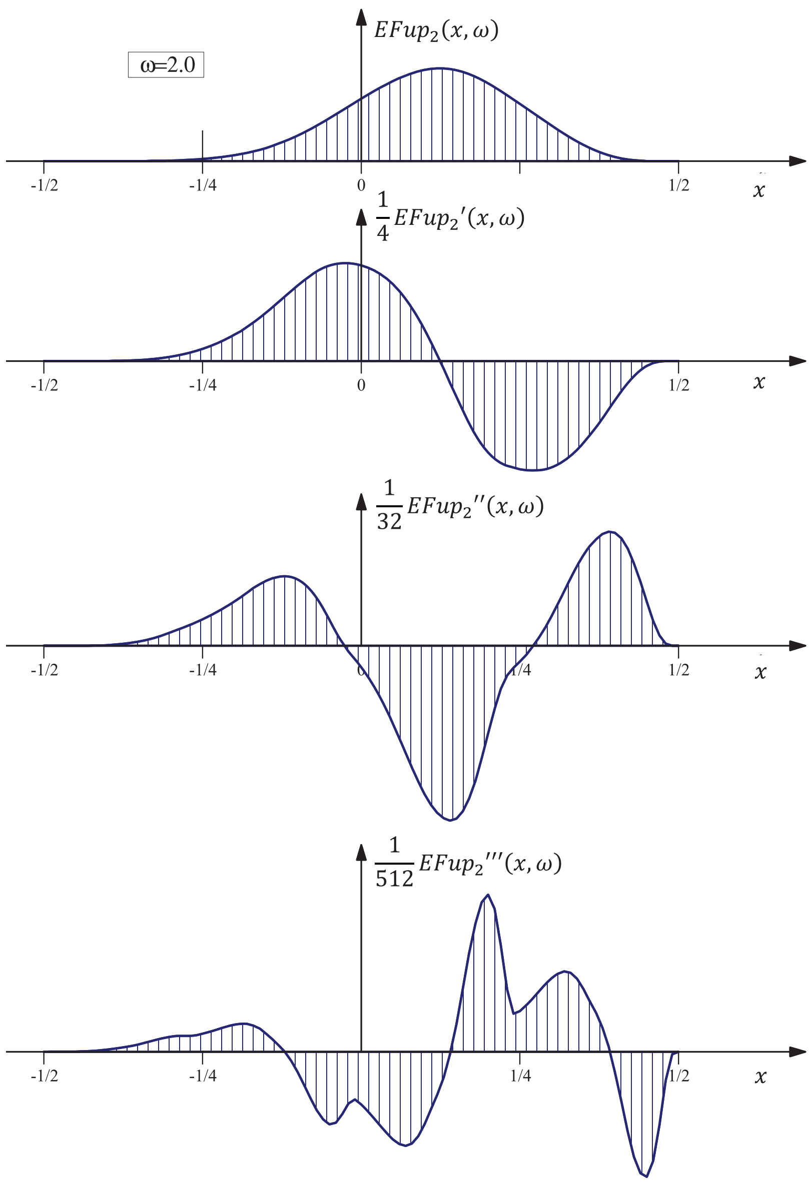


Figure 6: Function $EFup_2(x, \omega)$ and the first three derivatives

The integrals of the function $EFup_n(x,\omega)$ are also obtained by a linear combination of the integrals of the shifted $Eup(x,\omega)$ functions:

$$\int_{-\infty}^{x} EFup_n\left(x,\omega\right) dx = \sum\nolimits_{k=0}^{\infty} C_k\left(n\right) \int_{-\infty}^{x} Eup\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}, \omega\right) dx. \tag{42}$$

3.6 The Connection between the Function EFup$_n$(x, ω) and the Exponential Monomials

Similar to the function $Eup(x,\omega)$ [28], which is only a special case of the function $EFup_n(x,\omega)$ for $n = 0$, a connection between the functions $EFup_n(x,\omega)$ and exponential monomials $e^{2^n\omega\cdot x}$ can be established.

For a linear combination of the basis functions $EFup_n(x,\omega)$,

$$\varphi\left(x\right) = \sum\nolimits_{k=-\infty}^{\infty} C_k^{(n)} \cdot EFup_n\left(x - k\cdot 2^{-n}, \omega\right), \tag{43}$$

(offset from each other for the characteristic section $\Delta x_n = 2^{-n}$) to represent an exponential monomial of degree n, it is necessary and sufficient that by the action of the differential operator from [12,28] for a given $n \in N$

$$L_n = \prod\nolimits_{j=1}^{n} \left(d/dx - 2^{j-1}\omega\right) \tag{44}$$

on Eq. (43), the linear combination on the right is annuled.

For example, we show the calculation of the coefficients in the case of the basis function $EFup_2\left(x,\omega\right)$. Fig. 7 shows the disposition of the basis functions $EFup_2\left(x,\omega\right)$. Such an disposition of the basis functions accurately develops the exponential monomials up to and including the second degree, as well as the exponential polynomials formed by their combination. By the action of the operator from Eq. (44) on Eq. (43) for $n = 2$, the following recursion is obtained:

$$a_1 \cdot C_{k-1}^{(2)} + a_2 \cdot C_k^{(2)} + a_3 \cdot C_{k+1}^{(2)} + a_4 \cdot C_{k+2}^{(2)} = 0, \tag{45}$$

where

$$a_1 = 64 \cdot \omega^3 \cdot e^{7\cdot\omega/4}/N$$

$$a_2 = -64 \cdot \omega^3 \cdot e^{3\cdot\omega/4} \cdot \left(e^{3\cdot\omega/4} + e^{\omega/2} + 1\right)/N$$

$$a_3 = 64 \cdot \omega^3 \cdot e^{\omega/4} \cdot \left(e^{3\cdot\omega/4} + e^{\omega/4} + 1\right)/N$$

$$a_4 = -64 \cdot \omega^3/N$$

$$N = \left(e^{\omega/4} + 1\right)^2 \cdot \left(1 - e^{\omega/4}\right)^3 \cdot \left(e^{\omega/2} + 1\right)$$

or, after reordering:

$$e^{7\cdot\omega/4} \cdot C_{k-1}^{(2)} + e^{3\cdot\omega/4} \cdot \left(e^{3\cdot\omega/4} + e^{\omega/2} + 1\right) \cdot C_k^{(2)} + e^{\omega/4} \cdot \left(e^{3\cdot\omega/4} + e^{\omega/4} + 1\right) \cdot C_{k+1}^{(2)} + C_{k+2}^{(2)} = 0. \tag{46}$$

By introducing the substitution $C_k^{(2)} = \lambda^k$ in Eq. (46), we obtain a characteristic equation whose roots are

$$\lambda_0 = \mathrm{e}^{\omega/4};\ \lambda_1 = \mathrm{e}^{\omega/2};\ \lambda_2 = \mathrm{e}^{\omega}. \tag{47}$$

"Recompositioning" the roots (47) gives the general form of the coefficients $C_k^{(n)}$ for $n = 2$:

$$C_k^{(2)} = A_0^{(2)} e^{(\omega/4)k} + A_1^{(2)} e^{(\omega/2)k} + A_2^{(2)} e^{\omega k}, \tag{48}$$

or

$$C_k^{(2)} = \sum_{j=0}^{n} A_j^{(2)} e^{\omega k/2^{n-j}}. \tag{49}$$

The coefficients $A_j^{(2)}$ from Eq. (49) are calculated from the following system of equations:

$$C_{k-1}^{(2)} \cdot EFup_2\,(1/4,\omega) + C_k^{(2)} \cdot EFup_2\,(0,\omega) + C_{k+1}^{(2)} \cdot EFup_2\,(-1/4,\omega) = e^{\omega \mathrm{k}\cdot\Delta \mathrm{x}}$$
$$C_k^{(2)} \cdot EFup_2\,(1/4,\omega) + C_{k+1}^{(2)} \cdot EFup_2\,(0,\omega) + C_{k+2}^{(2)} \cdot EFup_2\,(-1/4,\omega) = e^{\omega (\mathrm{k}+1)\cdot\Delta \mathrm{x}}$$
$$C_{k+1}^{(2)} \cdot EFup_2\,(1/4,\omega) + C_{k+2}^{(2)} \cdot EFup_2\,(0,\omega) + C_{k+3}^{(2)} \cdot EFup_2\,(-1/4,\omega) = e^{\omega (\mathrm{k}+2)\cdot\Delta \mathrm{x}}$$

For $k = 0$ and $\Delta\mathrm{x} = 1/4$, we obtain

$$A_0^{(2)} = \frac{1}{e^{-\omega/4} \cdot EFup_2\,(1/4,\omega) + EFup_2\,(0,\omega) + e^{\omega/4} \cdot EFup_2\,(-1/4,\omega)},\ A_1^{(2)} = 0,\ A_2^{(2)} = 0.$$

In general, the exponential monomial $e^{2^m\omega x}, m = 0, 1, \ldots, n, n \in N$ on a segment of length 2^{-n} can be accurately represented by the linear combination of the $(n+2)\cdot 2^n$ basis functions $EFup_n\,(x,\omega)$ offset from each other by 2^{-n} in the form:

$$e^{2^m\omega x} = \sum\nolimits_{k=-\infty}^{\infty} \frac{e^{2^m\cdot\omega\cdot k\cdot\Delta x_n}}{A_n^{(m)}} \cdot EFup_n(x - k\cdot\Delta x_n, 2^n\cdot\omega\cdot\Delta x_n)\ ;\ m = 0, 1, \ldots, n, \tag{50}$$

where the coefficients $A_n^{(m)}$ (calculated from Eq. (49) for $x = 0$) are of the following form:

$$A_n^{(m)} = \sum\nolimits_{i=-(n+2)\cdot 2^{n-1}}^{(n+2)\cdot 2^{n-1}} e^{2^m\cdot\omega\cdot i\cdot\Delta x_n} \cdot EFup_n\left(-\frac{i}{\Delta x_n}, 2^n\cdot\omega\cdot\Delta x_n\right). \tag{51}$$

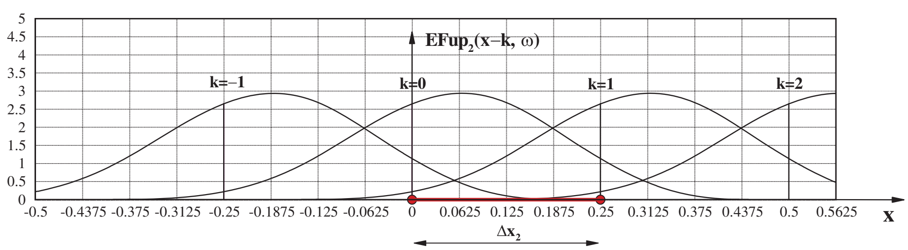


Figure 7: Composition of the $\boldsymbol{EFup_2}(\boldsymbol{x},\omega)$ functions in a linear combination to obtain monomials $\boldsymbol{\varphi}\,(\boldsymbol{x}) = \boldsymbol{e}^{\omega\cdot x\cdot 2^n}$

4 Practical Use of the Exponential ABF $EFup_n$ (x, ω)

For practical use we created the efupnM module to calculate the values of the functions $EFup_n\,(x,\omega)$ and their derivatives at arbitrary points. The use of the software modules comes down to simply describing a function in a similar way to that of, for example, the trigonometric function sine: $sin(\omega x + \varphi) \rightarrow$ sine (omega, xpoint, fi). Fig. 8 shows a graphical interpretation of the variables that need to be specified when using the efupnM module.

dummy = efupnM (NFUP, OMEGA, VERTEX, DELTAX, XPOINT, KOD, NMAX),

where:

NFUP = n - the order of the function $EFup_n(x,\omega)$;

OMEGA - frequency or tension parameter;

VERTEX - x-local coordinate system coordinates (located in the center of the support);

DELTAX - the real length of the characteristic segment;

XPOINT - the real x-coordinate of the arbitrary point at which the value of the function $EFup_n(x, \omega)$ is sought;

KOD - the order of derivation of the function;

NMAX - accuracy parameter (depends on computer characteristics).

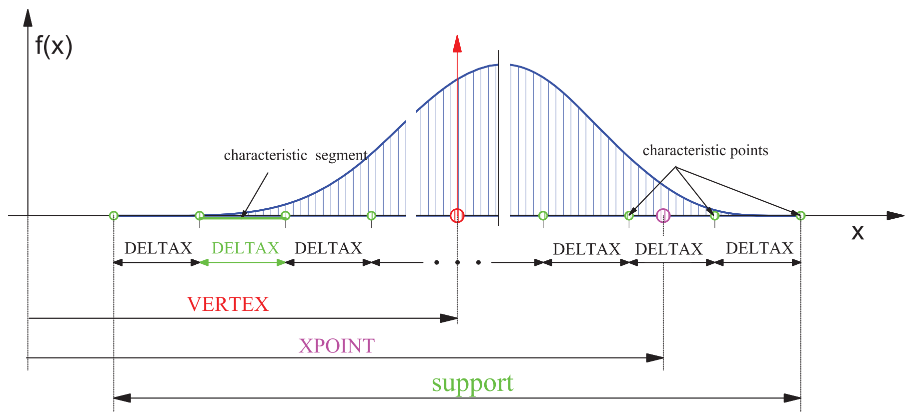


Figure 8: Using the efupnM software module to calculate the values of the basis functions $\boldsymbol{Fup_n(x)}$ and $\boldsymbol{EFup_n(x, \omega)}$

4.1 Determination of the Best Frequency in the Function Approximation

The basis functions of the exponential type, such as trigonometric functions, exponential splines, or ABFs of the exponential type, contain the parameter ω that provides them additional approximation properties. However, their application in numerical analysis is limited by the fact that the value of the parameter ω is, in most cases, unknown, and there is no universal criterion for choosing its value.

In this paper, the value of the parameter ω is determined using the least squares method by adopting the value of the parameter that gives the smallest deviation between a given function and its approximation at each characteristic segment of length Δx. This method proved to be simple and efficient, and is shown in the example of the exponential function below.

Let there be given a function at the section $\overline{AB} = [0, 1]$ in the form:

$$f(x) = e^{10(x-1)}, x \in [0, 1]. \tag{52}$$

Using the two characteristic segments of the length $\Delta x = 0.5$ and the formation of the basis functions according to Fig. 7, the corresponding approximations are determined using the basis functions $Fup_2(x)$ and $EFup_2(x, \omega)$.

As previously shown in Section 3.6, the linear combination of the basis functions $EFup_n(x, \omega)$ identically approximates the exponential monomials (as well as their linear combination), i.e., the given function (52), for any number of basis functions or characteristic segments Δx in the region $\overline{AB}$, as shown in Fig. 9. On the other hand, the approximation of the function (52) using the basis functions

$Fup_2(x)$ shows a significant deviation from the given function on a small number of segments, as shown in Fig. 9.

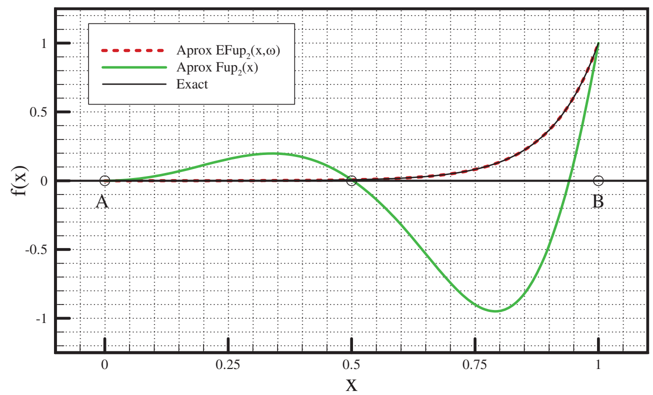


Figure 9: Comparison of the given function (52) and approximations obtained using algebraic and exponential (frequency $\omega = 10$) ABFs for $\boldsymbol{n} = \mathbf{2}$

Thus, the criterion for choosing the parameter's value is in terms of least squares:

$$snk(\omega) = \sum_{k=1}^{ns} \int \left[\mathrm{f}(\mathrm{x}) - \tilde{\mathrm{f}}(\mathrm{x}) \right]^2 \mathrm{dx} = \min, \tag{53}$$

where $f(x)$ is a given function, $\tilde{f}(x)$ is an approximation of a given function, and ns is the number of characteristic segments Δx in the domain $\overline{AB}$.

Fig. 10 shows the values of the least squares sum (53) for the approximations obtained by the values of the parameter ω in the interval $[0, 60]$ with a step $\Delta\omega = 0.1$.

Since the frequency of the given function (52) is known and is $\omega^* = 10$, it is to be expected that, for the given value of ω^*, the least squares sum (53) for approximation by the exponential basis functions $EFup_2(x, \omega^*)$ will be equal to zero; however, according to Section 3.6, for the values of the parameter $\omega_i = 2^m \cdot \omega^*, m = 0, 1, 2$ also, which is confirmed in Fig. 10.

This confirms that the least squares method is a reliable, simple, and optimal choice of criteria for the determination of the value of the parameter ω.

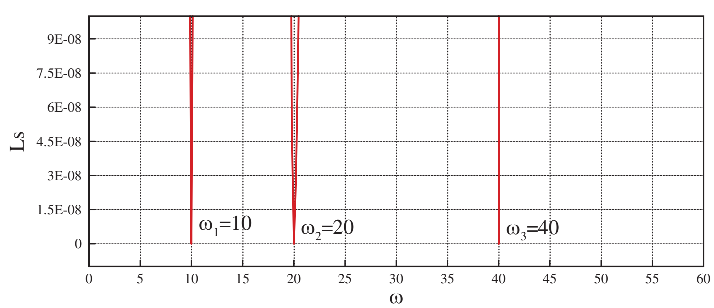


Figure 10: Values of the least squares sum of the approximation obtained by the basis functions $\boldsymbol{EFup_2}(\boldsymbol{x}, \boldsymbol{\omega_i})$ for the different values of the parameter $\boldsymbol{\omega}$ on the interval $\boldsymbol{I} = [\mathbf{0}, \mathbf{60}]$

4.2 Example 1: Approximation of a High-Degree Polynomial

An algebraic polynomial of degree 12 is approximated

$$f(x) = 858 \cdot x^2 \cdot (1 - x)^{10}, x \in [0, 1]. \tag{54}$$

Unlike the previous example where the value of the parameter ω followed from the function itself, here we have a "problem" of choosing the value of the parameter ω. The procedure for determining the value of the parameter ω is reduced to the simultaneous direct solution of the linear system of Eq. (43) using the point collocation method for different values of the parameter ω. Of all the numerical solutions thus obtained, the one that gives the minimum of the least squares function (53) for a given number ns of characteristic segments Δx is adopted.

Fig. 11 shows the values of the least squares sum of the approximation obtained by the $EFup_2(x, \omega)$ basis functions on two characteristic segments for the values of the parameter ω on the interval $I = [-30, 10]$ with the step $\Delta\omega = 0.1$.

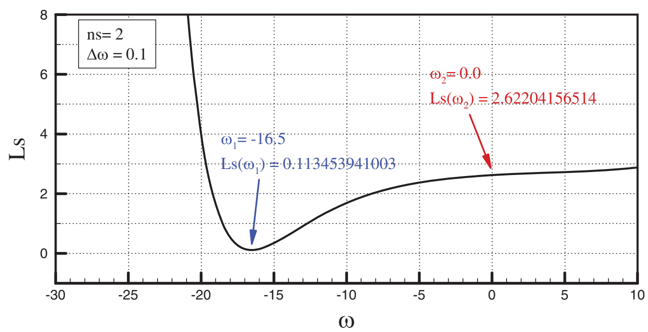


Figure 11: Values of the least squares sum of the approximation obtained by the basis functions $\boldsymbol{EFup_2(x, \omega_i)}$ for the values of the parameter $\boldsymbol{\omega}$ on the interval $\boldsymbol{I = [-30, 10]}$

The minimum value of the least squares sum for two characteristic segments was obtained for the approximation using the exponential basis functions $EFup_2(x, \omega)$, when the value of the parameter $\omega = -16.5$, and is $Ls = 0.113453941003$, while the value of the least squares sum of the approximation obtained by the algebraic basis functions $Fup_2(x)$, i.e., when the value of the parameter $\omega = 0.0$, for the same number of sections is 2.62204156514.

Fig. 12 shows a comparison of the given function (54) with the approximations obtained by the algebraic basis functions $Fup_2(x)$ and the exponential basis functions $EFup_2(x, \omega)$ for four different segment lengths $\Delta x = 1/ns$, where $ns = 2, 4, 8, 16$.

In Figs. 12a and 12b, it can be seen that for a small number of characteristic segments, the approximation by the exponential basis functions $EFup_2(x, \omega)$ gives a significantly better approximation to the given function (54) than the approximation obtained by the algebraic functions $Fup_2(x)$, while as the number of segments (ns) increases, this difference in approximations decreases, as shown in Figs. 12c and 12d.

In order to draw a conclusion regarding the character of the convergence of the mentioned numerical approximations to a given function, it is necessary to perform a calculation by increasing the number of segments to a certain desired accuracy of the results. From Fig. 12, it can be seen that the best approximation is achieved using the exponential basis functions with different parameterω values depending on the number of segments in the area (ns). Fig. 13 shows the values of the parameter ω

obtained by the least squares method in relation to the number of characteristic segments in the area. It can be seen that the value of the parameter ω is sensitive to discretization of the domain only for a small number of sections up to $ns = 16$, while when the number of sections is greater than 16, the parameter ω has a constant value of -19.0. Therefore, the convergence diagram of the numerical solution for the exponential basis functions $EFup_2(x, \omega)$ is obtained using the values $\omega = -19.0$.

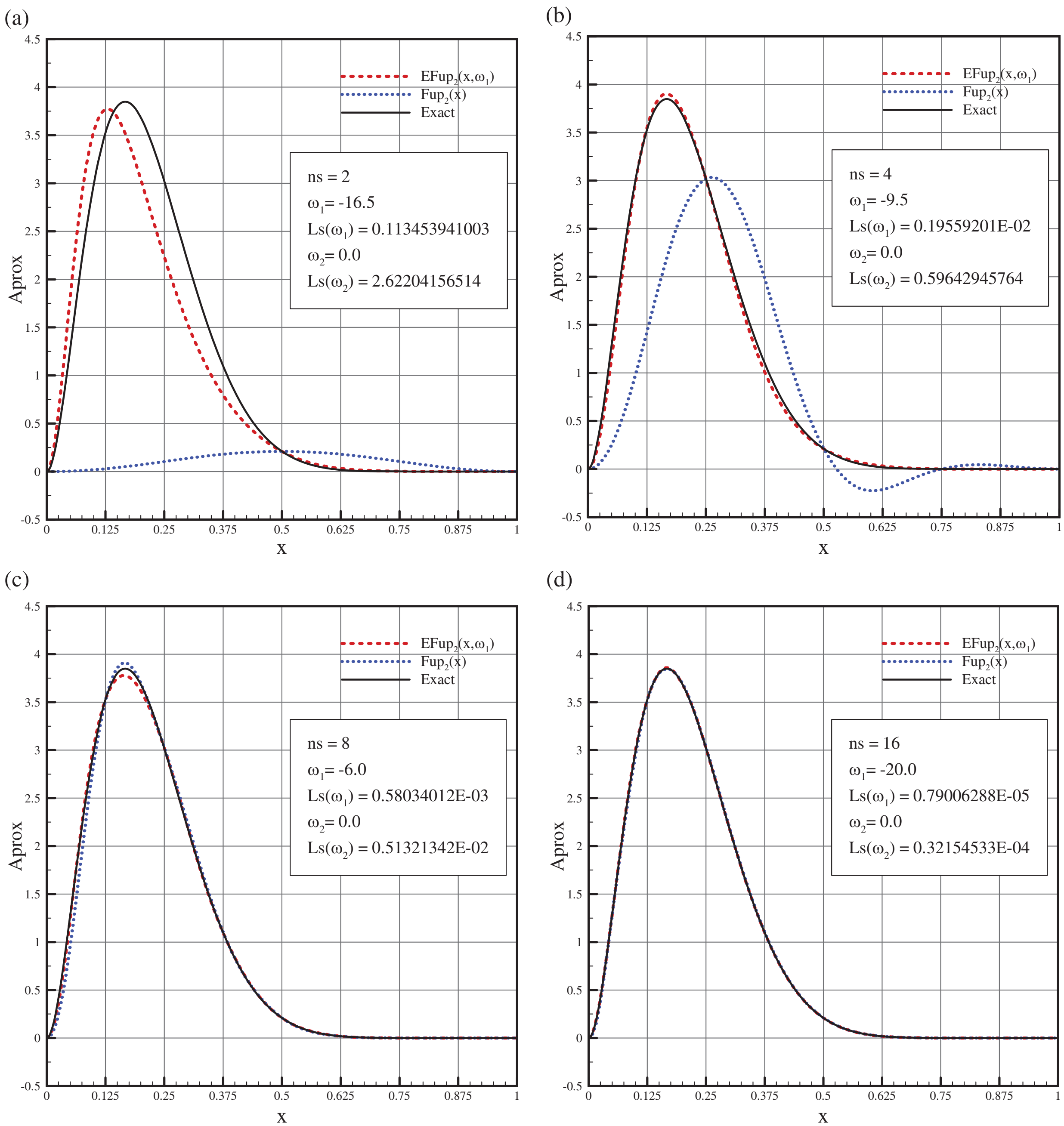


Figure 12: Comparison of the approximations with a given function (54) using the algebraic and exponential ABFs for: (a) $\boldsymbol{ns} = \mathbf{2}$, (b) $\boldsymbol{ns} = \mathbf{4}$, (c) $\boldsymbol{ns} = \mathbf{8}$, and (d) $\boldsymbol{ns} = \mathbf{16}$

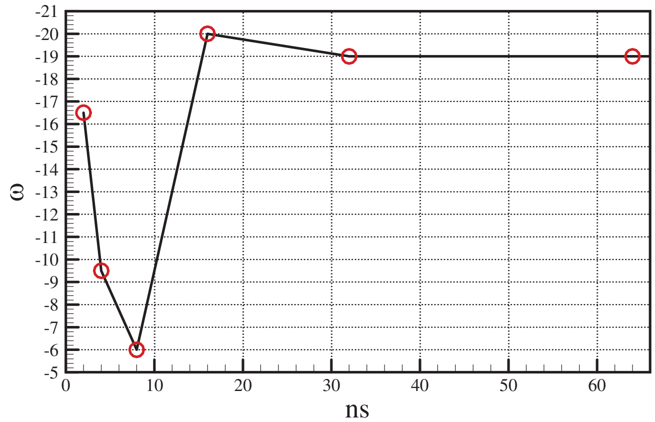


Figure 13: Dependence of the parameter's ω value on the number of characteristic segments in the approximation of the given function (54) using the $\boldsymbol{EFup_2(x,\omega)}$ basis functions

The diagrams in Fig. 14 show, on a logarithmic scale, the relationship between the error expressed over the L_2-norm and the segment length Δx for the approximations obtained by the basis functions $Fup_2(x)$ and $EFup_2(x,\omega)$. It can be observed that the approximation obtained by the exponential ABFs achieves greater accuracy compared to the approximation obtained by the algebraic ABFs. Both diagrams show that the expected convergence rate is achieved, which, for the problem of the approximation of a given function, is $p = n + 1$.

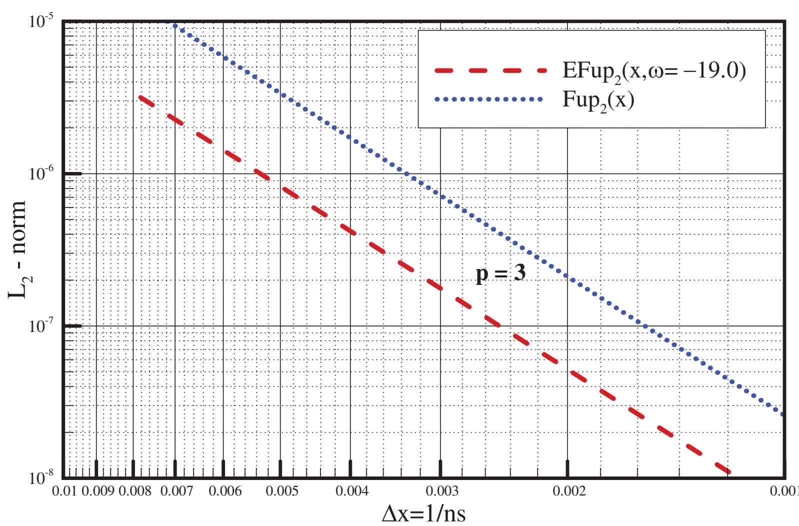


Figure 14: Convergence diagrams of the accuracy of the numerical approximations obtained by the $\boldsymbol{Fup_2(x)}$ and $\boldsymbol{EFup_2(x,\omega)}$ basis functions

4.3 Example 2: Approximation of a Sudden Jump Function

The following function on the interval [0, 1] is analyzed:

$$f(x) = -TANH((x-0.5)/0.02). \tag{55}$$

Fig. 15 shows a comparison of the given function (55) with the approximations obtained by the algebraic basis functions $Fup_2(x)$ and the exponential basis functions $EFup_2(x, \omega)$ for the segments of the lengths $\Delta x = 1/ns$ where $ns = 2, 4, 8, 16$. It can be observed that exponential ABFs better describe the given function near the jump, while in the parts of the domain where the given function has a constant value, the approximation obtained by $EFup_2(x, \omega)$ shows higher oscillations compared to the approximation obtained by $Fup_2\,(x)$ function. Fig. 15 also shows that, for this example of the function with a sudden jump, the exponential basis functions $EFup_2\,(x, \omega)$ achieve a better approximation than the algebraic $Fup_2\,(x)$ basis functions for a smaller number of segments in the domain, while for a larger number of segments, the accuracy of the approximation equates that obtained using the functions $Fup_2\,(x)$.

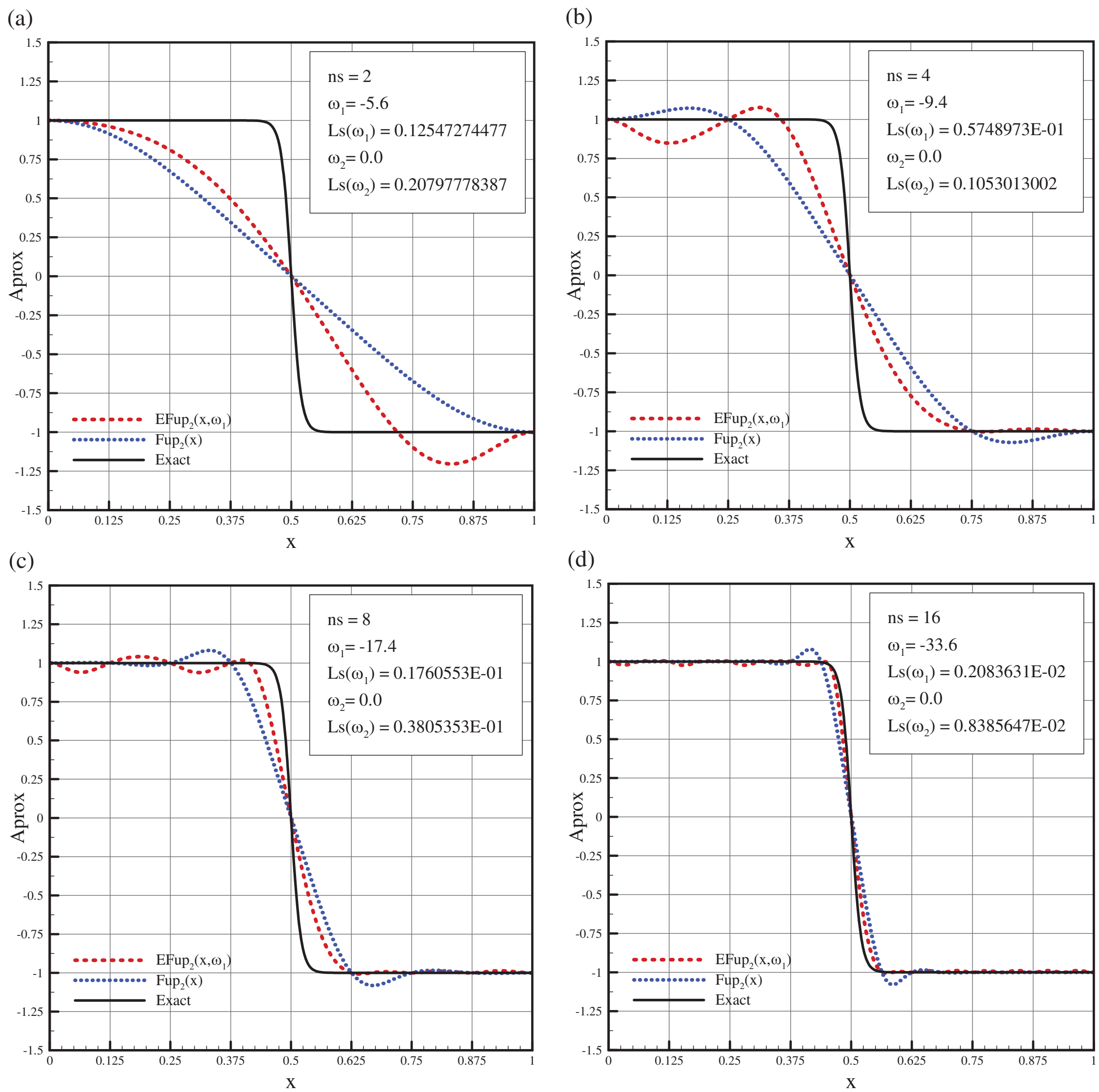


Figure 15: Comparison of the approximations with a given function (55) using the algebraic and exponential basis functions for: (a) $\boldsymbol{ns} = \mathbf{2}$, (b) $\boldsymbol{ns} = \mathbf{4}$, (c) $\boldsymbol{ns} = \mathbf{8}$, and (d) $\boldsymbol{ns} = \mathbf{16}$

4.4 Example 3: Solving the Differential Equation of Conduction

Let a given differential equation of conduction with corresponding boundary conditions be

$$\frac{d^2u}{dx^2} + \omega \cdot \frac{du}{dx} = 0; \; u(0) = 0; \; u(1) = 1. \tag{56}$$

with known exact solution of the form

$$u(x) = \frac{e^{\omega \cdot x} - 1}{e^{\omega} - 1}. \tag{57}$$

Fig. 16 shows the dependence of the solution of Eq. (56) on the parameter ω, and how, for high values of ω, the solution function shifts to the right boundary.

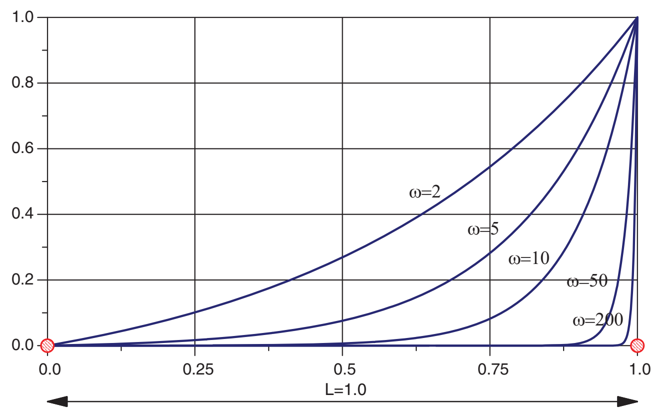


Figure 16: Dependence of the exact solution of the conduction problem (56) on the frequency $\boldsymbol{\omega}$

Fig. 17 compares the exact solution (57) of the conduction problem (56) with the solutions obtained using the basis functions $Fup_2(x)$ and $EFup_2(x, \omega)$ with the point collocation method for the characteristic segment $\Delta x = 0.25$, i.e., with a total of seven basis functions on the domain.

Approximation using the basis functions of the algebraic type is limited by the Peclet number $Pe = \Delta x \cdot \omega < 2$ because, at high values of the Pe, there is a numerical error and oscillation in the approximate solution. For the atomic basis functions of the exponential type there is no such a restriction.

In Fig. 17a for $\omega = 2$ and $Pe = 0.5 \ll 2$, the solutions coincide with the exact solution. In Fig. 17b for $\omega = 5$ and $Pe = 1.25 < 2$, the solution obtained with $EFup_2(x, \omega)$ fully corresponds to the correct solution, while the solution obtained with $Fup_2(x)$ shows a deviation from the exact one, but still does not oscillate. In Fig. 17c for $\omega = 10$ and $Pe = 2.5 > 2$, the solution obtained with $EFup_2(x, \omega)$ still fully corresponds to the exact solution, while the solution obtained with $Fup_2(x)$ begins to oscillate significantly around the exact solutions. In Fig. 17d, for $\omega = 10000$ and $Pe = 2500 \gg 2$, the solution obtained with $EFup_2(x, \omega)$ corresponds to the exact solution, while the solution obtained with $Fup_2(x)$ satisfies the boundary conditions and the differential equation at the collocation points but is completely unusable.

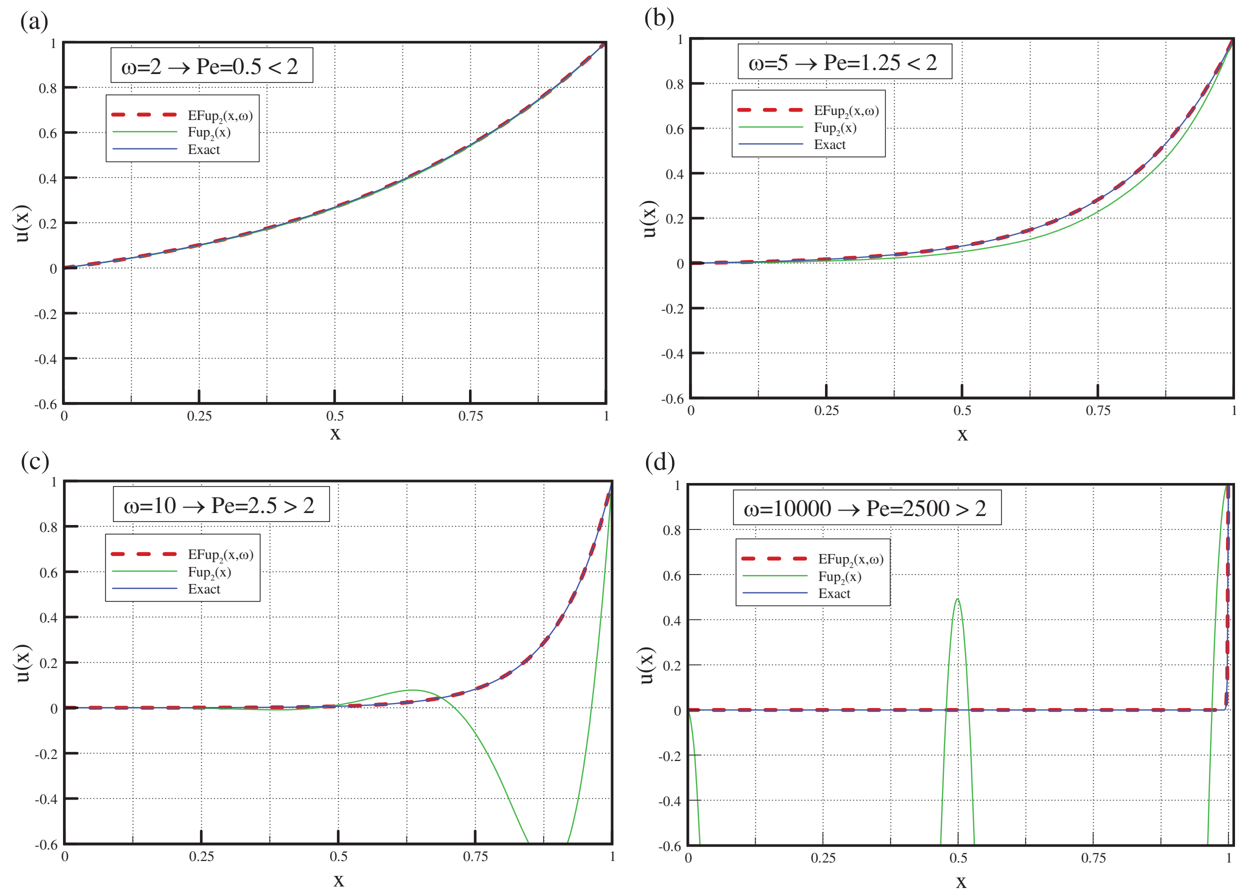


Figure 17: Comparison of the numerical solutions of Eq. (56) obtained by the $\boldsymbol{Fup_2}$ $(\boldsymbol{x})$ and $\boldsymbol{EFup_2}$ $(\boldsymbol{x}, \boldsymbol{\omega})$ basis functions with the exact solution

5 Conclusion

The current knowledge regarding algebraic atomic basis functions $Fup_n(x)$ is synthesized in the paper. Their basic properties are described, and the expressions for the necessary mathematical operations are presented in a simpler, more understandable, and user-friendly way. Very little was known about the ABFs of the exponential type, and they were developed only at the basic level in [12,27,28]. In this paper, the basic properties of the functions $EFup_n(x, \omega)$ are shown using the same approach as that for the ABFs of the algebraic type. The expressions for calculating the values of the functions and the desired number of derivatives at arbitrary points of the basis function support and, most importantly, the rules (elements) for their practical use are derived. The EFupnM software module for the practical application of these functions is also shown.

In the examples of the approximations of given functions, namely, a high-degree algebraic polynomial representing an asymmetric function and functions with a sudden jump, the exponential basis functions $EFup_n(x, \omega)$ show better properties compared to the basis functions of the algebraic type $Fup_n(x)$. This is especially evident in approximations that use a smaller number of basis functions. As the number of basis functions in the region increases, the approximation properties $EFup_n(x, \omega)$ of the functions are equated with the properties of the functions $Fup_n(x)$. The advantage of the

$EFup_n(x, \omega)$ function comes to expression especially when solving a differential equation of conduction that has an exact solution in the form of an exponential-type function. The exponential basis functions give a better approximate solution of high accuracy with the absence of the oscillations of the numerical solution.

Algebraic atomic basis functions have been used for many years to solve various numerical problems, and their advantage over other basis functions has become unquestionable. The ABFs of the exponential type show even better approximation properties, as demonstrated in this paper. The only question that still remains open is the choice of the value of the tension parameter ω. As with exponential splines, this complex issue requires further research both in one-dimensional problems and in the higher dimensions of space. In this paper, for the parameter selection criterion, we used the least squares sum, which proved to be simple and reliable. However, a disadvantage was the additional CPU time required to simultaneously solve the system of equations for the purpose of obtaining the approximations for different values of the parameter ω. This could be reduced by reducing the search interval of the parameter values according to the properties of a given numerical problem, i.e., whether it is an approximation of a given function or solving a differential equation.

Our further research should include an improvement of the procedure for finding the optimal value of the tension parameter. The natural sequence of development and application of the ABF of the exponential type leads to 2D and 3D numerical analysis. The advantage of ABF of the exponential type can be suitable for the application of adaptive procedures in the problems of computational mechanics.

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References

1. Zienkiewicz, O. C., Taylor, R. L. (2002). *The finite element method*. Oxford: Butterworth-Heinemann.
2. Sauter, S. A., Schwab, C. (2011). *Boundary element methods*. Berlin, Heidelberg: Springer.
3. Tavarez, F. A., Plesha, M. E. (2007). Discrete element method for modelling solid and particulate materials. *International Journal for Numerical Methods in Engineering, 70(4),* 379–404. DOI 10.1002/(ISSN)1097-0207.
4. Belytschko, T., Krongauz, Y., Organ, D., Fleming, M. A., Krysl, P. (1996). Meshless methods: An overview and recent developments. *Computer Methods in Applied Mechanics and Engineering, 139(1–2),* 3–47. DOI 10.1016/S0045-7825(96)01078-X.
5. Liu, G. R., Gu, Y. T. (2005). *An introduction to meshfree methods and their programming*. Dordrecht: Springer.
6. Fries, T. P., Matthies, H. G. (2004). *Classification and overview of meshfree methods*. Brunswick: Institute of Scientic Computing Technical University Braunschweig.
7. Hölling, K., Hörner, J. (2013). *Approximation and Modeling with B-splines*. Philadelphia: Siam.
8. Hughes, T., Cottrell, J., Bazilevs, Y. (2005). Isogeometric analysis: CAD, finite elements, nurbs, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering, 194(39),* 4135–4195. DOI 10.1016/j.cma.2004.10.008.
9. Cottrell, J. A., Hughes, T. J. R., Bazilevs, Y. (2009). *Isogeometric analysis toward intergration of CAD and FEA*. Wiley.

10. Rvachev, V. L., Rvachev, V. A. (1971). On a finite function. *Doklady Akademii Nauk Ukrainian SSR, A(6),* 705–707.
11. Rvachev, V. L., Rvachev, V. A. (1979). *Non-classical methods for approximate solution of boundary-value problems*. Kiev: Naukova dumka.
12. Gotovac, B. (1986). *Numerical modelling of engineering problems by smooth finite functions (Ph.D. Thesis)*. University of Split, Split (in Croatian).
13. Kravchenko, V. F. (2003). *Lectures on the theory of atomic functions and their some applications*. Moscow: Radiotechnika.
14. Beylkin, G., Keiser, J. M. (1997). On the adaptive numerical solution of nonlinear partial differential equations in wavelet bases. *Journal of Computational Physics, 132(2),* 233– 259. DOI 10.1006/jcph.1996.5562.
15. Gotovac, B., Kozulić, V. (1999). On a selection of basis functions in numerical analyses of engineering problems. *International Journal for Engineering Modelling, 12(1–4),* 25–41.
16. Kolodiazhny, V. M., Rvachev, V. A. (2007). Atomic functions: Generalization to the multivariable case and promising applications. *Cybernetics and Systems Analysis, 43(6),* 893–911. DOI 10.1007/s10559-007-0114-y.
17. Kravchenko, V. F., Basarab, M. A., Perez-Meana, H. (2001). Spectral properties of atomic functions used in digital signal processing. *Journal of Communications Technology and Electronics, 46,* 494–511.
18. Gotovac, B., Kozulić, V. (2002). Numerical solving of initial-value problems by Rbf basis functions. *Structural Engineering and Mechanics, 14(3),* 263–285. DOI 10.12989/sem.2002.14.3.263.
19. Kozulić, V., Gotovac, B. (2000). Numerical analyses of 2D problems using Fupn(x,y) basis functions. *International Journal for Engineering Modelling, 13(1–2),* 7–18.
20. Gotovac, H., Kozulic, V., Gotovac, B. (2010). Space-time adaptive fup multi-resolution approach for boundary-initial value problems. *Computers, Materials & Continua, 15(3),* 173–198. DOI 10.3970/cmc.2010.015.173.
21. Kozulic, V., Gotovac, B. (2011). Elasto-plastic analysis of structural problems using atomic basis functions. *Computer Modeling in Engineering & Sciences, 80(4),* 251–274. DOI 10.3970/cmes.2011.080.251.
22. Gotovac, H., Andricevic, R., Gotovac, B., Kozulic, V., Vranjes, M. (2003). An improved collocation method for solving the Henry problem. *Journal of Contaminant Hydrology, 64(1),* 129–149. DOI 10.1016/S0169-7722(02)00055-4.
23. Gotovac, H., Cvetkovic, V., Andricevic, R. (2009). Adaptive Fup multi-resolution approach to flow and advective transport in highly heterogeneous porous media: Methodology, accuracy and convergence. *Advances in Water Resources, 32(6),* 885–905. DOI 10.1016/j.advwatres.2009.02.013.
24. Kamber, G., Gotovac, H., Kozulić, V., Malenica, L., Gotovac, B. (2020). Adaptive numerical modeling using the hierarchical Fup basis functions and control volume isogeometric analysis. *International Journal for Numerical Methods in Fluids, 92(10),* 1437–1461. DOI 10.1002/fld.4830.
25. Kravchenko, V. F., Kravchenko, O. V., Konovalov, Y. Y., Budunova, K. A. (2020). Atomic functions theory: History and modern results: Dedicated to the pioneer of atomic functions theory V. L. Rvachev invited paper. *IEEE Ukrainian Microwave Week, 43,* 619–623. DOI 10.1109/UkrMW49653.2020.9252684.
26. Kravchenko, V. F., Kravchenko, O. V., Pustovoit, V. I., Churikov, D. V., Volosyuk, V. K. et al. (2016). Atomic functions theory: 45 years behind. *9th International Kharkiv Symposium on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves (MSMW)*, pp. 1–4. DOI 10.1109/MSMW.2016.7538216.
27. Brajčić Kurbaša, N. (2016). *Exponential atomic basis functions: Development and application (Ph.D. Thesis)*. University of Split, Split (in Croatian).
28. Brajčić Kurbaša, N., Gotovac, B., Kozulić, V. (2016). Atomic exponential basis function Eup(x,ω)-development and application. *Computer Modeling in Engineering & Sciences, 111(6),* 493–530. DOI 10.3970/cmes.2016.111.493.