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ARTICLE

Some Properties of Degenerate r-Dowling Polynomials and Numbers of the Second Kind

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ABSTRACT

The generating functions of special numbers and polynomials have various applications in many fields as well as mathematics and physics. In recent years, some mathematicians have studied degenerate version of them and obtained many interesting results. With this in mind, in this paper, we introduce the degenerate r-Dowling polynomials and numbers associated with the degenerate r-Whitney numbers of the second kind. We derive many interesting properties and identities for them including generating functions, Dobinski-like formula, integral representations, recurrence relations, differential equation and various explicit expressions. In addition, we explore some expressions for them that can be derived from repeated applications of certain operators to the exponential functions, the derivatives of them and some identities involving them.

KEYWORDS

Dowling lattice; Whitney numbers and polynomials; r-Whitney numbers and polynomials of the second kind; r-Bell polynomials; r-Stirling numbers; dowling numbers and polynomials

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1 Introduction

The Stirling number $S_2(n,k)$ of the second kind counts the number of partitions of the set $\{1,2,\ldots,n\}$ into k-nonempty disjoint set. The Bell polynomials $B_n(x)$ are given by $B_n(x)=\sum_{k=0}^{n} S_2(n,k)x^k$, (see [1]).

When $x=1$, $B_n=B_n(1)$ are called the Bell numbers. The Stirling number $S_1(n,k)$ of the first kind counts the number of having permutations of the set $\{1,2,\ldots,n\}$ having k disjoint cycles.

Dowling [2] constructed a certain lattice for a finite group of order m, called Dowling lattice, and using the Möbius function, he introduced the corresponding Whitney numbers of the first kind $w_m(n,k)$ and Whitney numbers of the second kind $W_m(n,k)$ $(0\le k\le n, m\ge 1)$, which are independent of the group itself, but depend only on its order. For the trivial group, we have $w_1(n,k)=S_1(n+1,k+1)$

and $W_1(n,k) = S_2(n+1,k+1)$. Benoumhani [3,4] gave a detailed description of properties of these numbers.

For $x \in \mathbb{R}$, the falling factorials $(x)_n$ are given by $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ and $(x)_0 = 1$, (see [1]).

As a generalization of the the Whitney numbers $w_m(n,k)$ and $W_m(n,k)$ of the first and second kind associated with $Q_n(G)$, respectively, Mezö [5] introduced r-Whitey numbers of the first and second kind given by

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k,$$

and

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)m^k(x)_n, \tag{1}$$

respectively, for $n \geq k \geq 0$. And $w_{m,r}(0,0) = 1$ and $W_{m,r}(0,0) = 1$.

When $r = 1$, $w_m(n,k) = w_{m,1}(n,k)$ and $W_m(n,k) = W_{m,1}(n,k)$.

We note that

$$w_{1,0}(n,k) = S_1(n,k), \qquad W_{1,0}(n,k) = S_2(n,k)$$
$$w_{1,r}(n,k) = S_1(n+r,k+r), \qquad W_{1,r}(n,k) = S_2(n+r,k+r)$$
$$w_{m,1}(n,k) = w_m(n,k), \qquad W_{m,1}(n,k) = W_m(n,k).$$

Note that the r-Whitney numbers of the second kind are exactly the same numbers defined by Ruciński and Voigt et al. [6] and the (r,β)-Stirling numbers defined by Corcino et al. [7].

The r-Whitey numbers of both kinds and r-Dowling polynomials were studied by several authors. The references [2–5,7–15] provided readers more information. In particular, Cheon et al. [8] and Corcino et al. [11] gave combinatorial interpretations of the r-Whitney numbers of the first and second kind, respectively. In recently years, many mathematicians have been studied the degenerate special polynomials and numbers, and have obtained many interesting results [14,16–24]. In particular, the generating functions of (degenerate) special numbers and polynomials have various applications in many fields as well as mathematics and physics [1–32]. Kim et al. [14] introduced the degenerate Whitney numbers of the first kind and the second kind of Dowling lattice $Q_n(G)$ of rank n over a finite group G of order m, respectively, as follows:

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,\lambda}(n,k)(mx+1)_{k,\lambda}, \quad (n \geq 0). \tag{2}$$

and

$$(mx+1)_{n,\lambda} = \sum_{k=0}^{n} W_{m,\lambda}(n,k)m^k(x)_k, \quad (n \geq 0), \qquad \text{see [14]}. \tag{3}$$

With these in mind, we naturally introduce the degenerate r-Dowling polynomials and numbers associated with the degenerate r-Whitney numbers $W_{m,r}(n,k)$ of the second kind in this paper. We explore various properties and identities for the degenerate r-Dowling polynomials and numbers including generating functions, Dobinski-like formula, integral representations, recurrence relations, various explicit expressions. Furthermore, we investigate several expressions for them that can be derived from repeated applications of certain operators to the exponential functions, the derivatives of them and some identities involving them.

2 Preliminaries

In this section, we introduce the basic definitions and properties of the degenerate r-Dowling polynomials and numbers needed in this paper.

For $x \in \mathbb{R}$, the rising factorials $\langle x\rangle_n$ are given by $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$, $(n \geq 1)$ and $\langle x\rangle_0 = 1$, (see [1]).

Cheon et al. [8] introduced the r-Dowling polynomials associated with the r-Whitney numbers $W_{m,r}(n,k)$ of the second kind are given by

$$D_{m,r}(n,x) = \sum_{k=0}^{n} W_{m,r}(n,k)x^k, \qquad \text{(see [8])}. \tag{4}$$

By (1) and (4), the generating function of r-Dowling polynomials is given by

$$\sum_{k=0}^{\infty} D_{m,r}(n,x)\frac{t^n}{n!} = \exp\left(rt + x\,\frac{e^{mt}-1}{m}\right), \qquad \text{(see [8,11,13])},$$

where $\exp(t) = e^t$.

Corcino et al. [11] studied asymptotic formulas for r-Whitney numbers of the second kind with integer and real parameters. They also obtained the range of validity of each formula.

As is well known, for any $\lambda \in \mathbb{R}$,

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \quad (|\lambda t| < 1), \qquad \text{(see [16–24])}, \tag{5}$$

where $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$ and $(x)_{0,\lambda} = 1$. When $\lambda \to 0$, $e_\lambda^x(t) = e^{xt}$.

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0), \qquad \text{(see [16,19,22])}. \tag{6}$$

Kim et al. studied the unsigned degenerate r-Stirling numbers of the second kind defined by

$$(x+r)_{n,\lambda} = \sum_{j=0}^{n} S_{2,\lambda}^{(r)}(n+r,j+r)(x)_j, \quad (n \geq 0), \qquad \text{(see [22])}. \tag{7}$$

From (7), the generating function of the degenerate r-Stirling numbers of the second kind is given by

$$e_\lambda^r(t)\frac{1}{j!}(e_\lambda(t)-1)^j = \sum_{n=j}^{\infty} S_{2,\lambda}^{(r)}(n+r,j+r)\frac{t^n}{n!}, \qquad \text{(see [22])}. \tag{8}$$

where j is a non-negative integer.

In view of (8), the degenerate r-Bell polynomials are given by

$$Bel_n^{(r)}(x|\lambda) = \sum_{j=0}^{n} S_{2,\lambda}^{(r)}(n+r, j+r)x^j, \quad (n \geq 0), \quad \text{(see [22])}. \tag{9}$$

From (9), it is easy to show that the generating function of degenerate r-Bell polynomials is given by

$$e_\lambda^r(t)e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \quad \text{(see [22])}. \tag{10}$$

when $x = 1$, $Bel_n^{(r)}(\lambda) = Bel_n^{(r)}(1|\lambda)$ are called the degenerate r-Bell numbers.

Kim et al. introduced the λ-binomial coefficients defined as

$$\binom{x}{n}_\lambda = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda)}{n!}, \quad (n \geq 1) \text{ and } \binom{x}{0}_\lambda = 1 \quad (\lambda \in \mathbb{R}), \quad \text{(see [20])}. \tag{11}$$

From (11), we easily get

$$\binom{x+y}{n}_\lambda = \sum_{l=0}^{n} \binom{x}{l}_\lambda \binom{y}{n-l}_\lambda, \quad (n \geq 0), \quad \text{(see [20])}. \tag{12}$$

From (12), we note that

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k}(x)_{k,\lambda}(y)_{n-k,\lambda}. \tag{13}$$

3 Degenerate *r*-Dowling Polynomials and Numbers

In this section, we explore various properties for the degenerate r-Dowling polynomials and numbers.

From (1), the degenerate r-Whitney numbers $W_{m,\lambda}^{(r)}(n,k)$ of the second kind are given by

$$(mx+r)_{n,\lambda} = \sum_{k=0}^{n} W_{m,r}(n,k)m^k(x)_n, \quad \text{(see [14])}. \tag{14}$$

Lemma 3.1. [14] For $k \geq 0$, we have the generating function of the degenerate r-Whitney numbers of the second kind as follows:

$$\sum_{n=j}^{\infty} W_{m,r,\lambda}(n,j)\frac{t^n}{n!} = e_\lambda^r(t)\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^j.$$

In Lemma 3.1, when $r = 1$, we have the generating function of the degenerate Whitney numbers of the second kind as follows:

$$\sum_{n=j}^{\infty} W_{m,\lambda}(n,j)\frac{t^n}{n!} = e_\lambda(t)\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^j, \quad \text{(see [16])}.$$

From Lemma 3.1, (6) and (8), we get

$$\begin{aligned} W_{1,r,\lambda}(n,j) &= S_{2,\lambda}^{(r)}(n+r, j+r), \\ W_{1,0,\lambda}(n,j) &= S_{2,\lambda}(n,j), \\ W_{m,1,\lambda}(n,j) &= W_{m,\lambda}(n,j). \end{aligned} \tag{15}$$

The next theorem is a recurrence relation of the degenerate Whitney numbers of the second kind.

Theorem 3.1. For $n \geq 0$, we have

$$W_{m,r,\lambda}(n+1,j) = W_{m,r,\lambda}(n,j-1) + (mj + r - n\lambda)W_{m,r,\lambda}(n,j).$$

Proof. From (14), we observe that

$$\begin{aligned}\sum_{j=0}^{n+1} W_{m,r,\lambda}(n+1,j)m^j(x)_j &= (mx+r)_{n+1,\lambda} = (mx+r-\lambda n)(mx+r)_{n,\lambda} \\ &= \sum_{j=0}^{n} W_{m,r,\lambda}(n,j)m^j(x)_j\{m(x-j)+mj+r-n\lambda\} \\ &= \sum_{j=1}^{n+1} W_{m,r,\lambda}(n,j-1)m^j(x)_j + \sum_{j=0}^{n} W_{m,r,\lambda}(n,j)(mj+r-n\lambda)m^j(x)_j \\ &= \sum_{j=0}^{n+1}\{W_{m,r,\lambda}(n,j-1) + W_{m,r,\lambda}(n,j)(mj+r-n\lambda)\}m^j(x)_j. \end{aligned} \tag{16}$$

By comparing the coefficients of both sides of (16), we get the desired recurrence relation.

The following theorem shows that the degenerate r-Whitney numbers of second kind expresses the finite sum of degenerate falling factorials.

Theorem 3.2. For $n, j \geq 0$, we have

$$\frac{1}{j!\,m^j}\sum_{d=0}^{j}\binom{j}{d}(-1)^{j-d}(dm+r)_{n,\lambda} = \begin{cases} W_{m,r,\lambda}(n,j) & \text{if} \quad n \geq j, \\ 0 & \text{if} \quad otherwise. \end{cases}$$

Proof. By (5) and Lemma 3.1, we observe that

$$\begin{aligned}\sum_{n=j}^{\infty} W_{m,r,\lambda}(n,j)\frac{t^n}{n!} &= e_\lambda^r(t)\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^j \\ &= \frac{1}{j!\,m^j}\sum_{d=0}^{j}\binom{j}{d}(-1)^{j-d}\sum_{n=0}^{\infty}(dm+r)_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{j!\,m^j}\sum_{d=0}^{j}\binom{j}{d}(-1)^{j-d}(dm+r)_{n,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{17}$$

By comparing the coefficients of both sides of (17), we get the desired result.

In Theorem 3.2, when $r = 1$, for $n \geq k \geq 0$, we get

$$\frac{1}{j!\,m^j}\sum_{d=0}^{j}\binom{j}{d}(-1)^{j-d}(dm+1)_{n,\lambda} = \begin{cases} W_{m,\lambda}(n,j) & \text{if} \quad n \geq j, \\ 0 & \text{if} \quad otherwise, \end{cases} \qquad \text{(see [14])}.$$

In this paper, we naturally consider the degenerate r-Dowling polynomials of the second kind given by

$$D_{m,r,\lambda}(n|x) = \sum_{j=0}^{n} W_{m,r,\lambda}(n,j)x^j, \quad (n \geq 0). \tag{18}$$

When $x = 1$, $D_{m,r,\lambda}(n) = D_{m,r,\lambda}(n|1)$ are called the degenerate r-Dowling numbers of the second kind.

When $r = 1$, $D_{m,\lambda}(n,x) = D_{m,1,\lambda}(n|x)$ are the degenerate Dowling polynomials in of the second kind [14].

When $r = 1$, the degenerate r-Dowling polynomials of the second kind are different from the fully degenerate Dowling polynomials in [23].

Theorem 3.3. For $m \in \mathbb{N}$, the generating function of degenerate r-Dowling polynomials of the second kind is

$$e_\lambda^r(t)e^{x(\frac{e_\lambda^m(t)-1}{m})} = \sum_{n=0}^{\infty} D_{m,r,\lambda}(n|x)\frac{t^n}{n!}$$

Proof. From Lemma 3.1 and (18), we observe that

$$\sum_{n=0}^{\infty} D_{m,r,\lambda}(n|x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} W_{m,r,\lambda}(n,j)x^j\right)\frac{t^n}{n!}$$

$$= e_\lambda^r(t)\sum_{j=0}^{\infty} x^j\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right) = e_\lambda^r(t)e^{x(\frac{e_\lambda^m(t)-1}{m})}. \tag{19}$$

By (19), we have the generating function of degenerate r-Dowling polynomials of the second kind.

When $m = 1$, from Theorem 3.3, (10) and (15), we observe that

$$D_{1,r,\lambda}(n) = \sum_{j=0}^{n} W_{1,r,\lambda}(n,j) = \sum_{j=0}^{n} S_{2,\lambda}^{(r)}(n+r,j+r) = Bel_n^{(r)}(\lambda).$$

When $m = 1$, $r = 1$ and $\lambda \to 0$, we note that

$$D_{1,1}(n) = \sum_{j=0}^{n} W_{1,1}(n,j) = \sum_{j=0}^{n} S_2(n+1,j+1).$$

Theorem 3.4. (Dobinski-like formula)

For $n \geq 0$, we have

$$D_{m,r,\lambda}(n|x) = e^{-\frac{x}{m}}\sum_{j=0}^{\infty}\frac{(mj+r)_{n,\lambda}}{j!\ m^j}\ x^j,$$

When $r = 1$, we have

$$D_{m,\lambda}(n|x) = e^{-\frac{1}{m}}\sum_{j=0}^{\infty}\frac{(mj+1)_{n,\lambda}}{j!\,m^j}\ x^j.$$

Proof. From (5) and Theorem 3.3, we note that

$$\sum_{n=0}^{\infty} D_{m,r,\lambda}(n|x)\frac{t^n}{n!} = e_\lambda^r(t)e^{x(\frac{e_\lambda^m(t)-1}{m})} = e^{-\frac{x}{m}}\sum_{j=0}^{\infty} x^j\frac{1}{m^j j!}e_\lambda^{mj+r}(t)$$

$$= e^{-\frac{x}{m}}\sum_{j=0}^{\infty}\frac{x^j}{m^j j!}\sum_{n=0}^{\infty}(mj+r)_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(e^{-\frac{x}{m}}\sum_{j=0}^{\infty}\frac{x^j}{m^j j!}(mj+r)_{n,\lambda}\right)\frac{t^n}{n!}. \tag{20}$$

By comparing the coefficients of both sides of (20), we have Dobinski-like formula for the degenerate r-Dowling polynomials.

In the following theorem and corollary, we have integral representations of the degenerate r-Whitney numbers and the degenerate r-Dowling polynomials, respectively.

Theorem 3.5. For $n, l \in \mathbb{Z}$ with $n \geq l \geq 0$, we have

$$W_{m,r,\lambda}(n,l) = \frac{n!}{\pi}Im\int_0^{2\pi}\frac{1}{l!}e_\lambda^r(e^{i\theta})\left(\frac{e_\lambda^m(e^{i\theta})-1}{m}\right)^l \sin(n\theta)d\theta,$$

where $i = \sqrt{-1}$.

Proof. From Lemma 3.1, we get

$$\int_0^{2\pi}\frac{1}{l!}e_\lambda^r(e^{i\theta})\left(\frac{e_\lambda^m(e^{i\theta})-1}{m}\right)^l \sin(n\theta)d\theta = \sum_{j=l}^{\infty} W_{m,r,\lambda}(j,l)\frac{1}{j!}\int_0^{2\pi} e^{ij\theta}\sin(n\theta)d\theta$$

$$= i\sum_{j=l}^{\infty} W_{m,r,\lambda}(j,l)\frac{1}{j!}\int_0^{2\pi}\sin(j\theta)\sin(n\theta)d\theta = \frac{i\pi}{n!}W_{m,r,\lambda}(n,l). \tag{21}$$

Therefore, by (21) we have the desired result.

Corollary 3.1. For $n \geq 0$, we have

$$\frac{n!}{\pi}Im\int_0^{2\pi} e_\lambda^r(e^{i\theta})\exp\left(\frac{e_\lambda^m(e^{i\theta})-1}{m}\right)\sin(n\theta)d\theta = D_{m,r,\lambda}(n).$$

Proof. By Lemma 3.1 and Theorem 3.5, we have

$$\int_0^{2\pi} e_\lambda^r(e^{i\theta})\exp\left(\frac{e_\lambda^m(e^{i\theta})-1}{m}\right)\sin(n\theta)d\theta = \sum_{l=0}^{\infty}\int_0^{2\pi} e_\lambda^r(e^{i\theta})\frac{1}{l!}\left(\frac{e_\lambda^m(e^{i\theta})-1}{m}\right)^l \sin(n\theta)d\theta$$

$$= \sum_{l=0}^{\infty}\sum_{j=l}^{\infty} W_{m,r,\lambda}(j,l)\frac{1}{j!}\int_0^{2\pi} e^{ji\theta}\sin(n\theta)d\theta$$

$$= i\sum_{j=0}^{\infty}\sum_{l=0}^{j}\frac{1}{j!}W_{m,r,\lambda}(j,l)\int_0^{2\pi}\sin(j\theta)\sin(n\theta)d\theta = \frac{i\pi}{n!}D_{m,r,\lambda}(n). \tag{22}$$

From (22), we get the desired identity.

Lemma 3.2. For $n \geq j \geq 0$ and $r, m \in \mathbb{N}$, we have

$$W_{m+1,r,\lambda}(n,j) = \frac{1}{(m+1)^j m^{n-j}}\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}(m+1)^s < r >_{n-s,m\lambda} W_{m,r,\frac{m}{m+1}\lambda}(s,j).$$

Proof. From Theorem 3.2 and (13), we get

$$W_{m+1,r,\lambda}(n,j)=\frac{1}{j!\,(m+1)^j}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}(l(m+1)+r)_{n,\lambda}$$

$$=\frac{(m+1)^{n-j}}{j!}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\left(l+\frac{r}{m}-\frac{r}{m(m+1)}\right)_{n,\frac{\lambda}{m+1}}$$

$$=(m+1)^n\sum_{l=0}^{j}\binom{j}{l}\frac{(-1)^{j-l}}{(m+1)^j j!}\sum_{s=0}^{n}\binom{n}{s}\left(l+\frac{r}{m}\right)_{n-s,\frac{\lambda}{m+1}}\left(\frac{-r}{m(m+1)}\right)_{s,\frac{\lambda}{m+1}}$$

$$=(m+1)^n\sum_{s=0}^{n}\binom{n}{s}\frac{m^j}{(m+1)^j m^{n-s}}\left(-\frac{r}{m(m+1)}\right)_{s,\frac{\lambda}{m+1}}\times\frac{1}{j!\,m^j}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}(lm+r)_{n-s,\frac{m}{m+1}\lambda}$$

$$=(m+1)^n\sum_{s=0}^{n}\binom{n}{s}\frac{m^j}{(m+1)^j m^{n-s}}\frac{(-1)^s}{(m(m+1))^s}<r>_{s,m\lambda}W_{m,r,\frac{m}{m+1}\lambda}(n-s,j)$$

$$=\frac{1}{(m+1)^j m^{n-j}}\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}(m+1)^s<r>_{n-s,m\lambda}W_{m,r,\frac{m}{m+1}\lambda}(s,j). \tag{23}$$

By (23), we obtain the desired result.

The next theorem is a recurrence relation of degenerate r-Dowling polynomials.

Theorem 3.6. For $n\geq 0$, we have

$$D_{m+1,r,\lambda}(n|x)=\frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j<r>_{n-j,m\lambda}D_{m,r,\frac{m}{m+1}\lambda}\left(j,\frac{m}{m+1}x\right),$$

Proof. From (18) and Lemma 3.2, we have

$$D_{m+1,r,\lambda}(n|x)=\sum_{j=0}^{n}W_{m+1,r,\lambda}(n,j)x^j$$

$$=\sum_{j=0}^{n}\left(\frac{1}{(m+1)^j m^{n-j}}\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}(m+1)^s<r>_{n-s,m\lambda}W_{m,r,\frac{m}{m+1}\lambda}(s,j)\right)x^j$$

$$=\frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j<r>_{n-j,m\lambda}D_{m,r,\frac{m}{m+1}\lambda}\left(j,\frac{m}{m+1}x\right). \tag{24}$$

Here $W_{m,r,\lambda}(j,d)=0$, if $d\geq j$. Thus, by (24), we get what we want.

Theorem 3.7. For $n\geq j\geq 0$, we have the recursion formula for $W_{m,r,\lambda}(n,j)$ as follows:

$$W_{m,r,\lambda}(n+1,j)=n!\sum_{d=j-1}^{n}\left(rW_{m,r,\lambda}(d,j)+\sum_{c=j-1}^{d}\binom{d}{c}W_{m,r,\lambda}(c,j-1)(m)_{d-c,\lambda}\right)\frac{(-\lambda)^{n-d}}{d!}.$$

Proof. For $j\geq 1$, from (5) and Lemma 3.1, we observe that

$$\sum\nolimits_{n=j-1}^{\infty}W_{m,r,\lambda}(n+1,j)\frac{t^n}{n!}=\frac{d}{dt}\frac{1}{j!}e_\lambda^r(t)\left(\frac{e_\lambda^m(t)-1}{m}\right)^j$$

$$= \left(\frac{r}{j!} e_\lambda^r(t) \left(\frac{e_\lambda^m(t)-1}{m} \right)^j + \frac{1}{(j-1)!} e_\lambda^r(t) \left(\frac{e_\lambda^m(t)-1}{m} \right)^{j-1} e_\lambda^m(t) \right) \frac{1}{1+\lambda t}$$

$$= (\sum_{b=j}^{\infty} r W_{m,r,\lambda}(b,j) \frac{t^b}{b!} + \sum_{l=j-1}^{\infty} \sum_{c=j-1}^{l} \binom{l}{c} W_{m,r,\lambda}(c,j-1)(m)_{l-c,\lambda} \frac{t^l}{l!}) \sum_{i=0}^{\infty} (-1)^i \lambda^i t^i$$

$$= \sum_{n=j}^{\infty} \sum_{b=j}^{n} \binom{n}{b} r W_{m,r,\lambda}(b,j)(n-b)!\,(-\lambda)^{n-b} \frac{t^n}{n!}$$

$$+ \sum_{n=j-1}^{\infty} \sum_{l=j-1}^{n} \sum_{c=j-1}^{l} \binom{n}{l} \binom{l}{c} W_{m,r,\lambda}(c,j-1)(m)_{l-c,\lambda}(n-l)!\,(-\lambda)^{n-l} \frac{t^n}{n!}$$

$$= \sum_{n=j-1}^{\infty} \left(n! \sum_{b=j-1}^{n} \left(r W_{m,r,\lambda}(b,j) + \sum_{c=j-1}^{b} \binom{b}{c} W_{m,r,\lambda}(c,j-1)(m)_{b-c,\lambda} \right) \frac{(-\lambda)^{n-b}}{b!} \right) \frac{t^n}{n!}. \tag{25}$$

By comparing the coefficients of both sides of (25), we get what we want.

The following theorem is another recurrence relation of degenerate r-Dowling polynomials.

Theorem 3.8. For $n \geq 0$, we have the recurrence formula of $D_{m,r,\lambda}$ as follows:

$$D_{m,r,\lambda}(n+1|u) = \sum_{l=0}^{n} \binom{n}{l} \{ r(-\lambda)^{n-l}(n-l)! + u(m-\lambda)_{n-l,\lambda} \} D_{m,r,\lambda}(l|u).$$

Proof. From Theorem 3.3, we note that

$$\frac{\partial}{\partial t} e_\lambda^r(t) \exp\left(u \frac{e_\lambda^m(t)-1}{m} \right) = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} D_{m,r,\lambda}(n|u) \frac{t^n}{n!} = \sum_{n=0}^{\infty} D_{m,r,\lambda}(n+1|u) \frac{t^n}{n!}. \tag{26}$$

On the other hand, by (26), we get

$$\frac{\partial}{\partial t} e_\lambda^r(t) \exp\left(u \frac{e_\lambda^m(t)-1}{m} \right) = r e_\lambda^{r-\lambda}(t) \exp\left(u \frac{e_\lambda^m(t)-1}{m} \right) + u e_\lambda^r(t) \exp\left(u \frac{e_\lambda^m(t)-1}{m} \right) e_\lambda^{m-\lambda}(t)$$

$$= \left(r \sum_{i=0}^{\infty} (-\lambda)^i i!\, \frac{t^i}{i!} + u \sum_{j=0}^{\infty} (m-\lambda)_{j,\lambda} \frac{t^j}{j!} \right) \sum_{l=0}^{\infty} D_{m,r,\lambda}(l|u) \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \left(r(-\lambda)^{n-l}(n-l)!\, D_{m,r,\lambda}(l|u) + u(m-\lambda)_{n-l,\lambda} D_{m,r,\lambda}(l|u) \right) \frac{t^u}{u!}. \tag{27}$$

By comparing the coefficients of (26) with (27), we get the desired identity.

Remark. When $u = 1$, we have

$$D_{m,r,\lambda}(n+1) = \sum_{l=0}^{n} \binom{n}{l} \{ r(-\lambda)^{n-l}(n-l)! + (m-\lambda)_{n-l,\lambda} \} D_{m,r,\lambda}(l). \tag{28}$$

Next, we explore two identities including degenerate r-Dowling polynomials that can be derived from repeated applications of certain operators to the degenerate exponential functions.

Theorem 3.9. For $n \geq 0$, we have

$$\frac{\partial^n}{\partial t^n} e_\lambda^r(t) \exp\left(\frac{x}{m} e_\lambda^m(t)\right) = e_\lambda^{r-n\lambda}(t) \exp\left(\frac{x}{m} e_\lambda^m(t)\right) D_{m,r,\lambda}(n|xe_\lambda^m(t)).$$

Proof. First, we observe that

$$\frac{\partial}{\partial t} e_\lambda^{mj+r}(t) = \frac{\partial}{\partial t}(1+\lambda t)^{\frac{mj+r}{\lambda}} = (mj+r)(1+\lambda t)^{\frac{(mj+r)-\lambda}{\lambda}},$$

$$\frac{\partial^2}{\partial t^2} e_\lambda^{mj+r}(t) = (mj+r)(mj+r-\lambda)(1+\lambda t)^{\frac{(mj+r)-2\lambda}{\lambda}},$$

$$\vdots$$

$$\frac{\partial^n}{\partial t^n} e_\lambda^{mj+r}(t) = (mj+r)_{n,\lambda} e_\lambda^{-n\lambda}(t) e_\lambda^{mj+r}(t). \tag{29}$$

By (29) and Theorem 3.4,

$$\begin{aligned}\frac{\partial^n}{\partial t^n} e_\lambda^r(t) e^{\frac{x}{m}(e_\lambda^m(t))} &= \frac{\partial^n}{\partial t^n}\left(e_\lambda^r(t)\sum_{j=0}^{\infty}\frac{x^j}{m^j j!} e_\lambda^{mj}(t)\right) = \sum_{j=0}^{\infty}\frac{x^j}{m^j j!}\left(\frac{\partial^n}{\partial t^n} e_\lambda^{mj+r}(t)\right)\\ &= \sum_{j=0}^{\infty}\frac{x^j}{m^j j!}(mj+r)_{n,\lambda} e_\lambda^{-n\lambda}(t) e_\lambda^{mj+r}(t)\\ &= e_\lambda^{r-n\lambda}(t)\sum_{j=0}^{\infty}\frac{(mj+r)_{n,\lambda}}{m^j j!}(xe_\lambda^m(t))^j = e_\lambda^{r-n\lambda}(t) e^{\frac{xe_\lambda^m(t)}{m}} D_{m,\lambda}(n|xe_\lambda^m(t)).\end{aligned} \tag{30}$$

From (30), we have what we want.

Let $A_{n,\lambda} = \sum_{j=0}^{\infty}\frac{(mj+r)_{n,\lambda}}{j! m^j}$, $n = 0, 1, 2, \ldots$. From Theorem 3.4, we have $D_{m,r,\lambda}(n) = e^{-\frac{1}{m}} A_{n,\lambda}$.

By Theorem 3.3, we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_{n,\lambda}\frac{t^n}{n!} &= e^{\frac{1}{m}}\sum_{n=0}^{\infty} D_{m,r,\lambda}(n)\frac{t^n}{n!}\\ &= e^{\frac{1}{m}} e_\lambda^r(t)\exp\left(\frac{e_\lambda^m(t)-1}{m}\right) = e_\lambda^r(t)\exp\left(\frac{e_\lambda^m(t)}{m}\right).\end{aligned} \tag{31}$$

From (31), the generating function of $A_{n,\lambda}$ is

$$e_\lambda^r(t)\exp\left(\frac{e_\lambda^m(t)}{m}\right) = \sum_{n=0}^{\infty} A_{n,\lambda}\frac{t^n}{n!}. \tag{32}$$

Theorem 3.10. For $n \geq 0$, we have

$$\left(mu^{1-\frac{\lambda}{m}}\frac{d}{du}\right)^n u^{\frac{r}{m}} e^{\frac{ux}{m}} = u^{\frac{r-n\lambda}{m}} e^{\frac{ux}{m}} D_{m,r,\lambda}(n|u).$$

Proof. Let $e_\lambda^m(t) = u$, Then we have

$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = (me_\lambda^{m-\lambda}(t))\frac{d}{du} = (mu^{\frac{m-\lambda}{m}})\frac{d}{du}. \tag{33}$$

By (33), we get

$$(mu^{\frac{m-\lambda}{m}}\frac{d}{du})^n u^{\frac{r}{m}} \exp\left(\frac{x}{m}u\right) = u^{\frac{r-n\lambda}{m}} \exp\left(\frac{x}{m}u\right) D_{m,r,\lambda}(n|xu), (n \geq 0). \tag{34}$$

By (34), we attain the desired result.

Remark. When $x = 1$, we have

$$\left(mu^{1-\frac{\lambda}{m}}\frac{d}{du}\right)^n u^{\frac{r}{m}} e^{\frac{u}{m}} = u^{\frac{r-n\lambda}{m}} e^{\frac{u}{m}} D_{m,r,\lambda}(n|u).$$

In Theorem 3.1, when $u = 1$ we observe that

$$\left(mu^{\frac{m-\lambda}{m}}\frac{d}{du}\right)^n u^{\frac{r}{m}} \exp\left(\frac{u}{m}\right)\Big|_{u=1} = e^{\frac{1}{m}} D_{n,r,\lambda}(n) = A_{n,\lambda}. \tag{35}$$

From (35), we obtain

$A_{0,\lambda} = e^{\frac{1}{m}}$ and $D_{m,r,\lambda}(0) = e^{-\frac{1}{m}} A_{0,\lambda} = 1$.

In (35), when $n = 1$, we get

$$\left(mu^{\frac{m-\lambda}{m}}\frac{d}{du}\right)^1 u^{\frac{r}{m}} \exp\left(\frac{u}{m}\right)$$

$$= mu^{\frac{m-\lambda}{m}}\left(\frac{r}{m}u^{\frac{r}{m}-1}\exp\left(\frac{u}{m}\right) + \frac{1}{m}u^{\frac{r}{m}}\exp\left(\frac{u}{m}\right)\right) = (r+u)u^{\frac{r-\lambda}{m}} e^{\frac{u}{m}}. \tag{36}$$

From (36), $A_{1,\lambda} = (r+1)e^{\frac{1}{m}}$ and $e^{-\frac{1}{m}} A_{1,\lambda} = (r+1) = D_{m,r,\lambda}(1)$.

In (35), when $n = 2$, we observe that

$$\left(mu^{\frac{m-\lambda}{m}}\frac{d}{du}\right)^2 u^{\frac{r}{m}} e^{\frac{u}{m}} = \left(mu^{\frac{m-\lambda}{m}}\frac{d}{du}\right)(r+u)u^{\frac{r-\lambda}{m}} e^{\frac{u}{m}}$$

$$= mu^{\frac{m-\lambda}{m}}\left\{u^{\frac{r-\lambda}{m}} e^{\frac{u}{m}} + \frac{r-\lambda}{m}u^{\frac{r-\lambda-m}{m}}(r+u)e^{\frac{u}{m}} + \frac{1}{m}(r+u)u^{\frac{r-\lambda}{m}} e^{\frac{u}{m}}\right\}$$

$$= u^{\frac{m-r-2\lambda}{m}} e^{\frac{u}{m}}\left\{m + (r-\lambda)u^{-1}(r+u) + (r+u)\right\}. \tag{37}$$

From (37), we get

$$A_{2,\lambda} = e^{\frac{1}{m}}\{m + (r-\lambda)(r+1) + (r+1)\} = e^{\frac{1}{m}}\{m + (r+1)(r-\lambda+1)\}. \tag{38}$$

Thus, by (38), we have $D_{m,r,\lambda}(2) = m + (r+1)(r-\lambda+1)$.

In the same way, we get

$$D_{m,r,\lambda}(3) = 2r\lambda^2 + (m-\lambda)(m-2\lambda) + (r+1)\{3m + (r+1)(r+1-3\lambda)\}. \tag{39}$$

By continuous this process, we get all the r-Dowling numbers $D_{m,r,\lambda}(n)$, for $n \in \mathbb{N}$.

As you can see from (39), the larger n, the more difficult it is to calculate by hand. Here we use Mathematica and Fortran language to find these values.

In Fig. 1, when $m = 5$, we can see the change of $D_{5,r,0.1}(2)$ and $D_{5,r,0.5}(2)$ depending on r by using Mathematica (x-axis is the numbers of r, y-axis $D(2)$ is the numbers of $D_{5,r,0.1}(2)$ and $D_{5,r,0.5}(2)$, respectively).

In Fig. 2, when $m = 5$, we can see the change of $D_{5,r,0.1}(3)$ and $D_{5,r,0.5}(3)$ depending on r by using Mathematica (x-axis is the numbers of r, y-axis $D(3)$ is the numbers of $D_{5,r,0.1}(3)$ and $D_{5,r,0.5}(3)$, respectively).

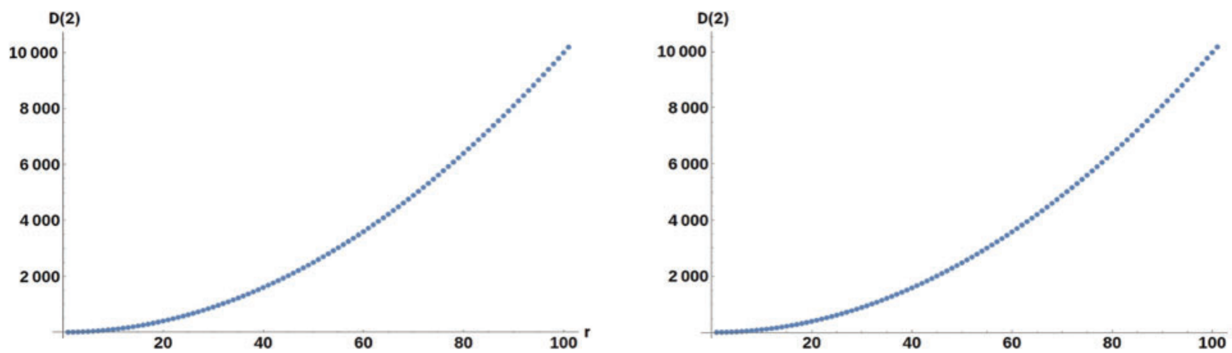

Figure 1: $D(2) = D_{5,r,\lambda}(2)$, when $\lambda = 0.1$ and 0.5, respectively

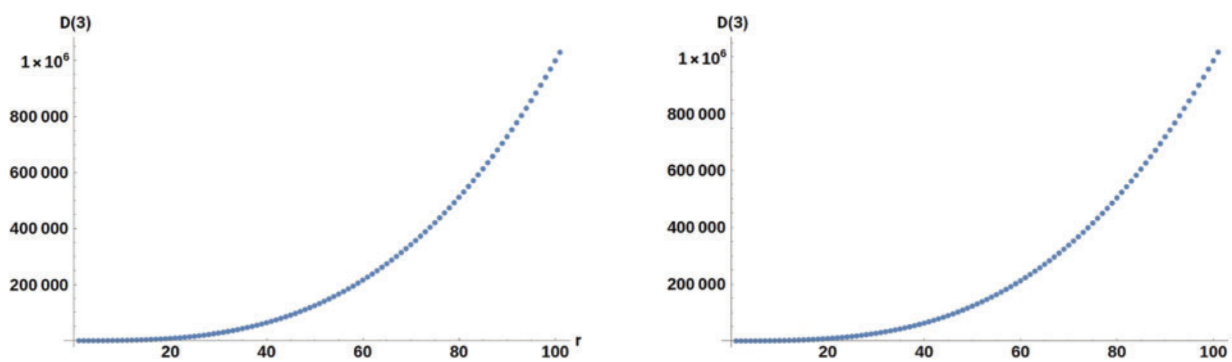

Figure 2: $D(3) = D_{5,r,\lambda}(3)$, when $\lambda = 0.1$ and 0.5, respectively

In Fig. 3, when $\lambda = 0.1$, we can see the change of $D_{10,1,0.1}(n)$ and $D_{50,1,0.1}(n)$, respectively, by using Fortran language (x-axis is the numbers of n, y-axis $\log_{10}(D(n))$ is the value of $\log_{10}$ (numbers of $D_{10,1,0.1}(n)$ and $D_{50,1,0.1}(n)$, respectively).

In Fig. 4, when $\lambda = 0.5$, we can see the change of $D_{10,1,0.5}(n)$ and $D_{50,1,0.5}(n)$, respectively, by using Fortran language (x-axis is the numbers of n, y-axis $\log_{10}(D(n))$ is the value of $\log_{10}$ (numbers of $D_{10,1,0.5}(n)$ and $D_{50,1,0.5}(n)$), respectively).

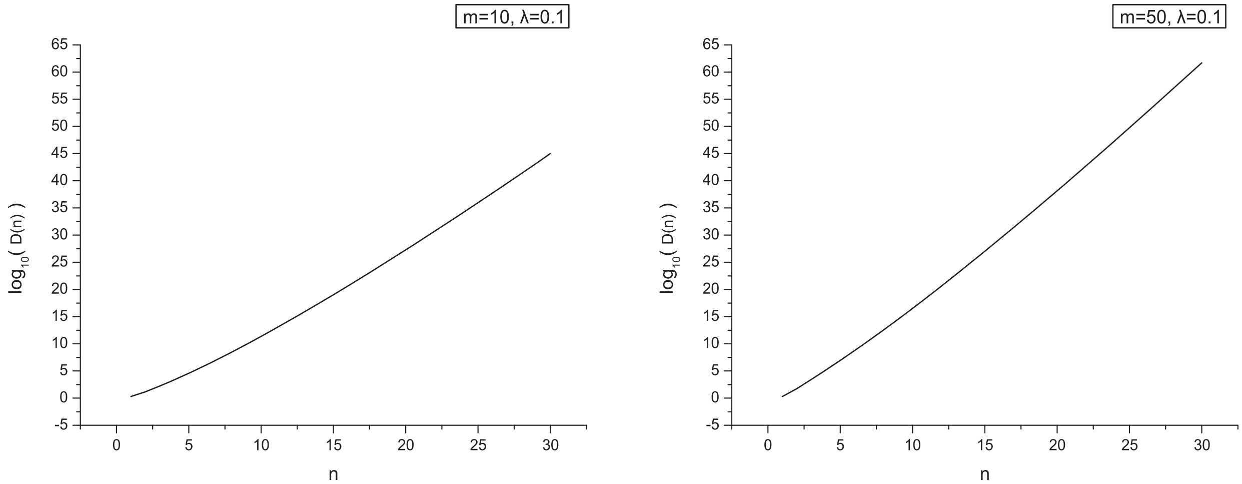

Figure 3: $\log_{10}(D(n)) = \log_{10}(D_{m,1,0.1}(n))$ when $m = 10$ and 50, respectively

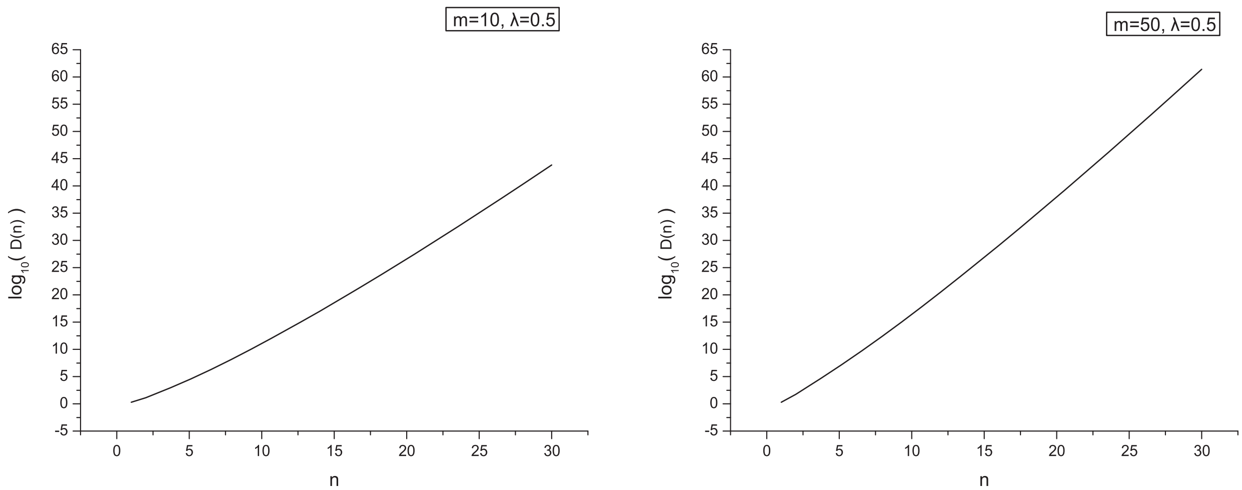

Figure 4: $\log_{10}(D(n)) = \log_{10}(D_{m,1,0.5}(n))$ when $m = 10$ and 50, respectively

Next, we can get differential equation for degenerate r-Dowling polynomials as follows:

Theorem 3.11. For $n \geq 0$, we have

$$D_{m,r,\lambda}(n+1|u) = (u + (r - n\lambda))D_{m,r,\lambda}(n|u) + mu\frac{d}{du}D_{m,r,\lambda}(n|u).$$

Proof. By using Theorem 3.4, we observe

$$\frac{d}{du}(u^{\frac{r-n\lambda}{m}} D_{m,r,\lambda}(n|u)) = \frac{d}{du}\left(u^{\frac{r-n\lambda}{m}} \exp(-\frac{u}{m}) \sum_{j=0}^{\infty} \frac{(mj+r)_{n,\lambda}}{j!\, m^j} u^j\right)$$

$$= \frac{d}{du}\left(\exp\left(-\frac{u}{m}\right) \sum_{j=0}^{\infty} \frac{(mj+r)_{n,\lambda}}{j!\, m^j} u^{\frac{(mj+r)-n\lambda}{m}}\right)$$

$$= -\frac{1}{m}\exp\left(-\frac{u}{m}\right)\sum_{j=0}^{\infty}\frac{(mj+r)_{n,\lambda}}{j!\,m^{j}}u^{\frac{(mj+r)-n\lambda}{m}} + exp\left(-\frac{u}{m}\right)\sum_{j=0}^{\infty}\frac{(mj+r)_{n+1,\lambda}}{j!\,m^{j+1}}u^{\frac{(mj+r)-n\lambda}{m}}u^{-1}$$

$$= -\frac{1}{m}\left\{u^{\frac{r-n\lambda}{m}}D_{m,r,\lambda}(n|u) - u^{\frac{(r-m)-n\lambda}{m}}D_{m,r,\lambda}(n+1|u)\right\}. \tag{40}$$

On the other hand, we have

$$\frac{d}{du}(u^{\frac{r-n\lambda}{m}}D_{m,r,\lambda}(n|u)) = \frac{r-n\lambda}{m}u^{\frac{r-n\lambda-m}{m}}D_{m,r,\lambda}(n|u) + u^{\frac{r-n\lambda}{m}}\frac{d}{du}D_{m,r,\lambda}(n|u). \tag{41}$$

By (40) and (41), we have

$$u^{\frac{(r-m)-n\lambda}{m}}D_{m,r,\lambda}(n+1|u) = u^{\frac{r-n\lambda}{m}}D_{m,r,\lambda}(n|u)\left(1+\frac{r-n\lambda}{u}\right) + u^{\frac{r-n\lambda}{m}}m\frac{d}{du}D_{n,r,\lambda}(n|u). \tag{42}$$

From (42), we get

$$\frac{1}{u}D_{m,r,\lambda}(n+1|u) = \left(1+\frac{r-n\lambda}{u}\right)D_{m,r,\lambda}(n|u) + m\frac{d}{du}D_{m,r,\lambda}(n|u). \tag{43}$$

By (43), we obtain the desire result.

Now, we study the derivative of degenerate r-Dowling polynomials $D_{m,r,\lambda}(n|x)$.

Theorem 3.12. For $n \geq 1$, we have

$$\frac{d}{du}D_{m,r,\lambda}(n|u) = \frac{1}{m}\sum_{l=0}^{n-1}\binom{n}{l}(m)_{n-l,\lambda}D_{m,r,\lambda}(l|u).$$

Proof. From (5) and Theorem 3.3, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{d}{du}D_{m,r,\lambda}(n|u)\frac{t^n}{n!} &= \frac{\partial}{\partial u}\left(e_{\lambda}^{r}(t)\exp\left(u\frac{e_{\lambda}^{m}(t)-1}{m}\right)\right)\\
&= e_{\lambda}^{r}(t)\frac{e_{\lambda}^{m}(t)-1}{m}\exp\left(u\frac{e_{\lambda}^{m}(t)-1}{m}\right)\\
&= \frac{e_{\lambda}^{m}(t)-1}{m}\sum_{l=0}^{\infty}D_{m,r,\lambda}(l|u)\frac{t^l}{l!}\\
&= \frac{1}{m}\left(\sum_{i=0}^{\infty}(m)_{i,\lambda}\frac{t^i}{i!}-1\right)\sum_{l=0}^{\infty}D_{m,r,\lambda}(l|u)\frac{t^l}{l!}\\
&= \frac{1}{m}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(m)_{n-l,\lambda}D_{m,r,\lambda}(l|u) - D_{m,r,\lambda}(n|u)\right)\frac{t^n}{n!}\\
&= \frac{1}{m}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}(m)_{n-l,\lambda}D_{m,r,\lambda}(l|u)\right)\frac{t^n}{n!}.
\end{aligned} \tag{44}$$

By comparing the coefficients on both sides of (44), we attain the desired identity.

Theorem 3.13. For $n \geq 1$, we have

$$D_{m,r,\lambda}(n|u) = (r)_{n,\lambda} + \sum_{l=0}^{n-1} \binom{n}{l} (n)_{n-l,\lambda} \int_0^u D_{m,r,\lambda}(l|u)\ du.$$

Proof. From (5) and Theorem 3.3, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \int_0^u D_{m,r,\lambda}(n|u) du \frac{t^n}{n!} &= \int_0^u e_\lambda^r(t) \exp(u \frac{e_\lambda^m(t) - 1}{m}) du \\ &= e_\lambda^r(t) \int_0^u \exp\left(u \frac{e_\lambda^m(t) - 1}{m}\right) du \\ &= e_\lambda^r(t) \frac{m}{e_\lambda^m(t) - 1} \left[\exp\left(u \frac{e_\lambda^m(t) - 1}{m}\right)\right]_0^u. \end{aligned} \tag{45}$$

From (45), we observe that

$$(e_\lambda^m(t) - 1) \sum_{l=0}^{\infty} \int_0^u D_{m,r,\lambda}(l|u)\ du \frac{t^l}{l!} = e_\lambda^r(t) \left\{\exp\left(u \frac{e_\lambda^m(t) - 1}{m}\right) - 1\right\}. \tag{46}$$

By (46), we have

$$\sum_{j=1}^{\infty} (m)_{j,\lambda} \frac{t^j}{j!} \sum_{l=0}^{\infty} \int_0^u D_{m,r,\lambda}(l|u)\ du \frac{t^l}{l!} = \sum_{n=0}^{\infty} D_{m,r,\lambda}(n|u) \frac{t^n}{n!} - \sum_{n=0}^{\infty} (r)_{n,\lambda} \frac{t^n}{n!}. \tag{47}$$

From (47), we obtain

$$\sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \binom{n}{l} (m)_{n-l,\lambda} \int_0^u D_{m,r,\lambda}(l|u)\ du \frac{t^n}{n!} = \sum_{n=0}^{\infty} \{D_{m,r,\lambda}(n|u) - (r)_{n,\lambda}\} \frac{t^n}{n!}. \tag{48}$$

By comparing the coefficients of both sides of (45), we have the desired identity.

If we put $y = u^q$ $(q \in \mathbb{N} \cup \{0\})$ and apply the next theorem, we get another interesting identity depending on the variable q different from Theorem 3.10.

Theorem 3.14. For $n \geq 0$, we have the operational formula as follows:

$$\left(mu^{1-\frac{\lambda}{m}} \frac{d}{du}\right)^n u^{q\frac{r}{m}} \exp\left(\frac{u^q}{m}\right) = q^n u^{\frac{rq-n\lambda}{m}} \exp\left(\frac{u^q}{m}\right) D_{m,r,\frac{\lambda}{q}}(n|u^q).$$

Proof. Let $y = u^q$ $(q \in \mathbb{N} \cup \{0\})$. Then we have

$$mu^{1-\frac{\lambda}{m}} \frac{d}{du} = mu^{1-\frac{\lambda}{m}} qu^{q-1} \frac{d}{dy} = my^{\frac{m-\lambda}{mq}} qy^{\frac{q-1}{q}} \frac{d}{dy} = mqy^{\frac{mq-\lambda}{mq}} \frac{d}{dy} = mqy^{1-\frac{\lambda}{mq}} \frac{d}{dy}. \tag{49}$$

Thus, by (49), we have

$$\left(mu^{1-\frac{\lambda}{m}} \frac{d}{du}\right)^n u^{q\frac{r}{m}} \exp\left(\frac{u^q}{m}\right) = q^n \left(my^{1-\frac{\lambda}{mq}} \frac{d}{dy}\right)^n y^{\frac{r}{m}} \exp\left(\frac{y}{m}\right)$$

$$= q^n y^{\frac{r-n\frac{\lambda}{q}}{m}} \exp\left(\frac{y}{m}\right) D_{m,r,\frac{\lambda}{q}}(n|y) = q^n u^{\frac{rq-n\lambda}{m}} \exp\left(\frac{u^q}{m}\right) D_{m,r,\frac{\lambda}{q}}(n|u^q). \tag{50}$$

From (50), we attain the desired formula.

4 Conclusion

In this paper, we studied many interesting properties for the degenerate r-Dowling polynomials and numbers associated with the degenerate r-Whitney numbers of the second kind. Among these identity expressions, we obtained the generating function in Theorem 3.3, Dobinski-like formula in Theorem 3.4, recurrence relations in Theorem 3.6 and 3.8, differential equation in Theorem 3.11, the derivatives of them in Theorem 3.12 for r-Dowling polynomials of the second kind. In particular, we obtained some expressions for them that can be derived from repeated applications of certain operators to the exponential functions in Theorem 3.9, 3.10 and 3.14, and some identities involving integration in Theorem 3.13. Furthermore, we found that all exact values of all r-Dowling numbers of the second kind can be obtained using (28). As a follow-up study of this paper, we can explore truncated degenerate r-Dowling polynomials and degenerate r-Dowling polynomials arising from λ-Sheffer sequences. Hence, for future projects, we would like to conduct research into some potential applications of r-Dowling polynomials of the first and second kind, respectively.

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