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Article

The Kemeny's Constant and Spanning Trees of Hexagonal Ring Network

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Abstract: Spanning tree (τ) has an enormous application in computer science and chemistry to determine the geometric and dynamics analysis of compact polymers. In the field of medicines, it is helpful to recognize the epidemiology of hepatitis C virus (HCV) infection. On the other hand, Kemeny's constant (Ω) is a beneficial quantifier characterizing the universal average activities of a Markov chain. This network invariant infers the expressions of the expected number of time-steps required to trace a randomly selected terminus state since a fixed beginning state s_i. Levene and Loizou determined that the Kemeny's constant can also be obtained through eigenvalues. Motivated by Levene and Loizou, we deduced the Kemeny's constant and the number of spanning trees of hexagonal ring network by their normalized Laplacian eigenvalues and the coefficients of the characteristic polynomial. Based on the achieved results, entirely results are obtained for the Möbius hexagonal ring network.

Keywords: Matrix Analysis; hexagonal ring network; Kemeny's constant; Spanning tree

1 Introduction

Obtaining the total number of spanning trees of any network is the central part of exploration in network theory, as spanning trees of any network grow exponentially through a network size. Earlier in the 1960s, researchers around the world explored numerous procedures of fluctuating efficiency methods. It uses various fields of computer science such as image processing, networking, and countless other usages of minimum spanning trees or entirely possible spanning trees of a network.

Another network invariant is entitled Kemeny's constant (Ω). In [1], the Kemeny's constant is proposed by Kemeny and spell. It is motivating to perceive that this unique network invariant is closely related to the analogous Spectrum of the normalized Laplacian (see Lemma 2.2 in the next section). Kemeny's constant is formally defined as the expected number of steps desirable for the transition from a starting node to a terminus node. It is chosen randomly by a stationary distribution of unbiased random walks on network N. In finite ergodic Markov chains, the Ω has an essential property independent of the initial state of the Markov chain [2]. The adjacency matrix $A(N)$ of N is a matrix whose (i,j)-entry is 1 if and only if $ij \in E_N$ and 0, otherwise. Define the Laplacian matrix of N as $L(N) = D(N) - A(N)$, where $D(N)$ is the diagonal matrix whose main diagonal entries are the degrees in N. In recent years, the method of using eigenvalues of normalized Laplacian, $\Gamma(N)$, consisting of the matrix in spectral geometry and random walks [3,4], attracted the researchers due to its numerous applications.

1.1 Preliminaries

All the networks considered in this paper are finite, connected, simple, and undirected. Let $N = (U_N, E_N)$ be any network, where U_N denote the node-set and E_N denote the link set. We represent the order of N as $n = |U_N|$ and its size as $|E_N|$. The traditional notation and terminology not defined in this paper are referred to [2,3].

The adjacency matrix $A(N)$ of N is a matrix whose (i,j)-entry is 1 if and only if $ij \in E_N$ and 0, otherwise. Define the Laplacian matrix of N as $L(N) = D(N) - A(N)$, where $D(N)$ is the diagonal matrix whose main diagonal entries are the degrees in N. We assume that $\mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_n$ be the eigenvalues of $L(N)$. It is obvious that $\mu_1 = 0$ and $\mu_2 > 0$ if and only if N is a connected network. Further, regarding the results on $L(N)$, we recommend the recent work [4] and the references within.

Let M be an $m \times n$ matrix. We assume that $S \subset \{1, 2, \ldots, m\}$ and $T \subset \{1, 2, \ldots, n\}$. Denote $M(S|T)$ for the submatrix of M, which is obtained by deleting the rows of S and the columns of T. Notably, we denote $M(S|T)$ by $M(i|j)$, where $S = \{i\}$ and $T = \{j\}$.

In recent years, the method using eigenvalues of normalized Laplacian, $\Gamma(N)$, which consists of the matrix in spectral geometry and random walks [5,6], has attracted more and more researchers' attention. Defining the normalized Laplacian of nonregular networks also attracted researchers. Furthermore, the normalized Laplacian of any network is defined as:

$$\Gamma(N) = I - D^{-\frac{1}{2}}(N)\, A(N)\, D^{-\frac{1}{2}}(N) = D^{-\frac{1}{2}}(N)\, L(N)\, D^{-\frac{1}{2}}(N).$$

Here, when a degree of the node w_j in N is 0, then $(d_j)^{-\frac{1}{2}} = 0$, see [5]. That is to say

$$(\Gamma(N))_{ij} = \begin{cases} 1, & \text{if } i = j; \\ -\dfrac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise}, \end{cases}$$

The notation $(\Gamma(N))_{ij}$ symbolizes the (i,j)-entry of $\Gamma(N)$, and we assume that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the Spectrum of the normalized Laplacian of N. These eigenvalues are labeled as $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$, with the fact that N is connected if and only if $\lambda_2 > 0$. In [7], Chen and Zhang determined that the resistance distance can also be obtained from eigenvalues expressions and their multiplicities in the sense of normalized Laplacian.

The hexagonal system plays an essential role in theoretical chemistry. Since the hexagonal systems are natural network illustrations of benzenoid hydrocarbon [8]. Therefore, in various fields, hexagonal systems have been widely studied. The perfect matching in random hexagonal chain network is established by Kennedy et al. [9] in 1991. The hexagonal chain for Wiener index and Edge-Szeged index is determined in [10] and [11], respectively. In [12], Lou and Huang gave complete descriptions of the characteristic polynomial of a hexagonal system.

In this paper, motivated by [13–17] and from the normalized Laplacian decomposition theorem, we obtained the explicit closed-form formulations for Ω and τ for Δ_n as well as ∇'_n.

1.2 Definition and Structures of the Two Hexagonal Ring Networks

We denote the linear hexagonal chain with n hexagons by M_n. The hexagonal ring network is denoted by Δ_n and computed from M_n by identifying the opposite boundary links in an ordered way. The Möbius hexagonal ring network ∇_n' obtained by M_n by identifying the opposite boundary links in a reversed way.

In this paper, we focus on two interesting molecular network types: the hexagonal ring network (see Fig. 1) and the Möbius hexagonal ring network (see Fig. 2). The hexagonal ring network Δ_n is the network obtained from the linear hexagonal chain M_n by identifying node 1 with $(2n+1)$ the node $1'$ with $(2n+1)'$, respectively. Similarly, the Möbius hexagonal ring network ∇_n' is the network obtained from the linear hexagonal chain M_n by identifying the node 1 with $(2n+1)'$, the node $1'$ with $(2n+1)$, respectively.

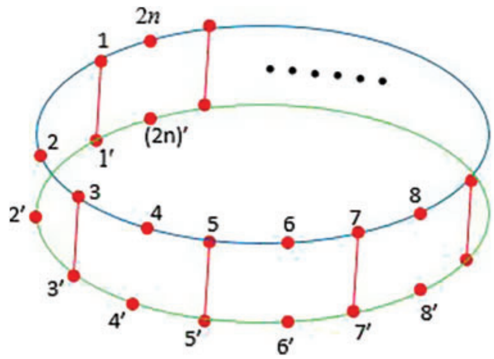

Figure 1: The hexagonal ring network

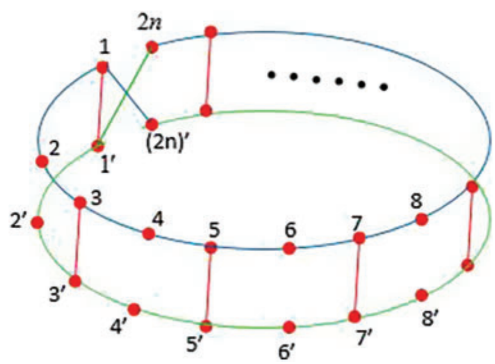

Figure 2: Möbius hexagonal ring network

2 Normalized Laplacian Polynomial Decomposition and Important Lemmas

In this section, we discuss some vital block matrices, characteristic polynomial, and the automorphisms of N, which will be used to prove our main results. We denote $\varphi(B) = \det(xI - B)$ for the characteristic polynomial of a matrix B, where B is a square matrix and I is the corresponding identity matrix. The automorphism of any network N is a permutation π of the nodes of N having the property that uv is a link in the network N, whenever $\pi(u)\pi(v)$ is a link in N. We suppose that the network N is an automorphism in π. Therefore, we write it as a 1-cycle of disjoint product and its transpositions in the form $\pi = (\overline{1})(\overline{2})\cdots(\overline{m})(1,1')(2,2')\cdots(k,k')$. Thus, it is easy to compute that $|U_N| = m + 2k$, and assume that $U_0 = \{\overline{1},\overline{2},\ldots,\overline{m}\}$, $U_1 = \{1,2,\ldots,k\}$ and $U_2 = \{1',2',\ldots,k'\}$. After an appropriate organization of the nodes in N, the normalized Laplacian matrix $\Gamma(N)$ can be arranged in the following way

$$\Gamma(N) = \begin{pmatrix} \Gamma_{U_{00}} & \Gamma_{U_{01}} & \Gamma_{U_{02}} \\ \Gamma_{U_{10}} & \Gamma_{U_{11}} & \Gamma_{U_{12}} \\ \Gamma_{U_{20}} & \Gamma_{U_{21}} & \Gamma_{U_{22}} \end{pmatrix}.$$

The submatrix $\Gamma_{U_{ij}}$ is formed by rows corresponding to nodes of the network N in U_i and columns corresponding to those in U_j, where $i = 0,1,2$ and $j = 0,1,2$. We assume that $T = \begin{pmatrix} I_m & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}I_k & \frac{1}{\sqrt{2}}I_k \\ 0 & \frac{1}{\sqrt{2}}I_k & -\frac{1}{\sqrt{2}}I_k \end{pmatrix}$

is a block matrix in which the dimension of a block is the same as the corresponding blocks of $\Gamma(N)$. Note that the automorphism of N is denoted by π. Hence $\Gamma_{U_{11}} = \Gamma_{U_{22}}$. Considering the unitary transformation $T\Gamma(N)T^T$ yields

$$T\Gamma(N)T^T = \begin{pmatrix} \Gamma_R(N) & 0 \\ 0 & \Gamma_S(N) \end{pmatrix}. \tag{1}$$

where $\Gamma_R(N) = \begin{pmatrix} \Gamma_{U_{00}} & \sqrt{2}\Gamma_{U_{01}} \\ \sqrt{2}\Gamma_{U_{10}} & \Gamma_{U_{11}} + \Gamma_{U_{12}} \end{pmatrix}$, $\Gamma_S(N) = \Gamma_{U_{11}} - \Gamma_{U_{12}}$.

In [15], the first author of this article mentioned the decomposition theorem of the Laplacian polynomial. In the following lemma, it is easy to see that the decomposition theorem for normalized Laplacian polynomial is also existed as:

Lemma 2.1: The matrices $\Gamma(N)$, $\Gamma_R(N)$ and $\Gamma_S(N)$ as defined above are, satisfies that $\varphi(\Gamma(N)) = \varphi(\Gamma_R(N))\varphi(\Gamma_S(N))$.

The following two lemmas are essential to obtain our main results.

Lemma 2.2: [18] The Kemeny's constant of a simple connected network N with n nodes is denoted by Ω and defined as $\Omega(N) = \sum_{i=2}^{n}\frac{1}{\lambda_i}$.

Lemma 2.3: [5] The Spanning trees of a network N with order n and links m are denoted by $\tau(N)$ and defined as $\tau(N) = \frac{1}{2m}\prod_{i=1}^{n}d_i\prod_{k=2}^{n}\lambda_k$.

3 Important Matrices and the Spectrum of $\Gamma(\Delta_n)$

According to Lemma 2.1, we firstly obtain the normalized Laplacian eigenvalues for Δ_n. Then we give the formula for the sum of the normalized Laplacian eigenvalues' reciprocals and the product of the normalized Laplacian eigenvalues, which motivate us to calculate the Ω and the number of spanning trees of Δ_n. We also deduce the corresponding results based on our achieved results. Bearing in mind the labeled nodes of Δ_n as shown in Fig. 1, one can see that $\pi = (1,1')(2,2')\cdots(2n,(2n)')$ is an automorphism of the network Δ_n. That is to say, $U_0 = \emptyset$, $U_1 = \{1,2,\ldots,2n\}$ and $U_2 =$

$\{1', 2', \ldots, (2n)'\}$. From the notation in (1), we may denote $\Gamma_R(\Delta_n)$ and $\Gamma_S(\Delta_n)$ as Γ_R and Γ_S respectively, and we have

$\Gamma_R = \Gamma_{U_{11}} + \Gamma_{U_{12}}, \Gamma_S = \Gamma_{U_{11}} - \Gamma_{U_{12}}.$

The matrices $\Gamma_{U_{11}}$ and $\Gamma_{U_{12}}$ are of order $2n \times 2n$ as given below:

$$\Gamma_{U_{11}} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix},$$

$$\Gamma_{U_{12}} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

$$\text{Hence } \Gamma_R = \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix}_{2n\times 2n}$$

$$\text{and } \Gamma_S = \begin{pmatrix} \frac{4}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{4}{3} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix}_{2n\times 2n}.$$

For the sake of simplicity, we denote eigenvalues of Γ_R and Γ_S are respectively, as $\eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_{2n}$ and $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_{2n}$. Furthermore, the Spectrum of $\Gamma(\Delta_n)$ is exactly $\{\eta_1, \eta_2, \ldots, \eta_{2n}, \xi_1, \xi_2, \ldots, \xi_{2n}\}$, due to Lemma 2.1. Obviously, $\eta_1 = 0$, $\eta_i > 0$ $(i = 2, \ldots, 2n)$ and $\xi_j > 0$ $(j = 1, \ldots, 2n)$. It is easy to calculate that $|U_{\Delta_n}| = |U_{\nabla'_n}| = 4n$ and $|E_{\Delta_n}| = |E_{\nabla'_n}| = 5n$.

Further on, we introduce a matrix named Q, where Q is a matrix constructed from Γ_R with the $(1, 2n)$-entry and the $(2n, 1)$-entry by replacing 0. We consider the i-th order principal submatrix, Q_i (resp. C_i), formed by the first i rows and corresponding columns (resp. the last i rows and corresponding columns) of Q. Put $q_i := \det Q_i$ and $c_i := \det C_i$. Put $q_0 = 1$, $c_0 = 1$ and it is straightforward that $q_i = c_i$ for all even i.

Lemma 3.1: For $0 \leqslant i \leqslant 2n$, $q_i = 6^{-\frac{i}{2}-1}[3 + \sqrt{6} + (-1)^i(3 - \sqrt{6})](i + 1)$.

Proof. Since it is easy to see that $q_1 = \frac{2}{3}$, $q_2 = \frac{1}{2}$, $q_3 = \frac{2}{9}$. For $3 \leqslant i \leqslant 2n$, expanding $\det Q_i$ with respect to its last row yields

$$q_i = \begin{cases} q_{i-1} - \frac{1}{6} q_{i-2}, & \textit{if } i \textit{ is even;} \\ \frac{2}{3} q_{i-1} - \frac{1}{6} q_{i-2}, & \textit{if } i \textit{ is odd.} \end{cases}$$

Let $d_i = q_{2i}$ if $0 \leqslant i \leqslant n$, let $e_i = q_{2i+1}$ if $0 \leqslant i \leqslant n - 1$. Furthermore, $c_0 = 1$, $d_0 = \frac{2}{3}$ and for $i \geqslant 1$, then one has

$$\begin{cases} d_i = e_{i-1} - \frac{1}{6} d_{i-1}, \\ e_i = \frac{2}{3} d_i - \frac{1}{6} e_{i-1}. \end{cases} \tag{2}$$

From the first equation in (2), one has $e_{i-1} = d_i + \frac{1}{6} d_{i-1}$. Hence, $e_i = d_{i+1} + \frac{1}{6} d_i$. Substituting e_{i-1} and e_i into the second equation in (2) yields $d_{i+1} = \frac{1}{3} d_i - \frac{1}{36} d_{i-1}$, $i \geqslant 1$. Keeping the same procedure, one can obtain that $e_{i+1} = \frac{1}{3} e_i - \frac{1}{36} e_{i-1}$, $i \geqslant 1$, and q_i satisfies the below recurrence relation

$$q_i = \frac{1}{3} q_{i-2} - \frac{1}{36} q_{i-4}, q_0 = 1, q_1 = \frac{2}{3}, q_2 = \frac{1}{2}, q_3 = \frac{2}{9}. \tag{3}$$

Then, the characteristic equation of (3) is $x^4 = \frac{1}{3} x^2 - \frac{1}{36}$, the roots of which are $x_1 = x_2 = \frac{1}{\sqrt{6}}$ and $x_3 = x_4 = -\frac{1}{\sqrt{6}}$. The general solution of (3) is given by

$$q_i = \left(\frac{1}{\sqrt{6}}\right)^i (y_1 + iy_2) + \left(-\frac{1}{\sqrt{6}}\right)^i (y_3 + iy_4). \tag{4}$$

Together with the initial conditions of (4), the system of equations yields

$$\begin{cases} y_1 + y_3 = 1, \\ \dfrac{1}{\sqrt{6}}(y_1 + y_2) - \dfrac{1}{\sqrt{6}}(y_3 + y_4) = \dfrac{2}{3}, \\ \left(\dfrac{1}{\sqrt{6}}\right)^2 (y_1 + 2y_2) + \left(-\dfrac{1}{\sqrt{6}}\right)^2 (y_3 + 2y_4) = \dfrac{1}{2}, \\ \left(\dfrac{1}{\sqrt{6}}\right)^3 (y_1 + 3y_2) + \left(-\dfrac{1}{\sqrt{6}}\right)^3 (y_3 + 3y_4) = \dfrac{2}{9}. \end{cases}$$

The unique solution of this system can be found to be $y_1 = \dfrac{3+\sqrt{6}}{6}$, $y_2 = \dfrac{3+\sqrt{6}}{6}$, $y_3 = \dfrac{3-\sqrt{6}}{6}$, $y_4 = \dfrac{3-\sqrt{6}}{6}$. We get our desired result by substituting y_1, y_2, y_3 and y_4 in (4).

Considering the procedure as the proof of Lemma 3.1, it is easy to determine the following results.

Lemma 3.2: For $0 \leqslant i \leqslant 2n$, $c_i = \frac{1}{4} \cdot 6^{-\frac{i}{2}}[2 + \sqrt{6} + (-1)^i(2 - \sqrt{6})](i + 1)$.

Based on Lemmas 3.1 and 3.2, we determine a_{2n-1} and a_{2n-2}. For the sake of simplicity, we denote the entries of Γ_R by l_{ij}, $i, j = 1, 2, \ldots, 2n$.

Lemma 3.3: $-a_{2n-1} = \frac{10n^2}{6^n}$.

Proof. Since the number $-a_{2n-1}$ $(= (-1)^{2n-1}a_{2n-1})$ is the sum of all those principal minors of Γ_R which have $2n - 1$ rows and columns (see [19, P5]); we have

$$-a_{2n-1} = \sum_{i=1}^{2n} det\Gamma_R(i|i) = q_{2n-1} + c_{2n-1} + \sum_{i=2}^{2n-1} det\Gamma_R(i|i).$$

For $2 \leqslant i \leqslant 2n - 1$, one has

$$\Gamma_R(i|i) = \begin{pmatrix} l_{11} & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & l_{22} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & l_{i-1,i-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & l_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & l_{2n-1,2n-1} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & l_{2n,2n} \end{pmatrix}_{(2n-1)\times(2n-1)}.$$

Let $X = \begin{pmatrix} 0 & I_{i-1} \\ I_{2n-i} & 0 \end{pmatrix}$. It is evident that $X^T\Gamma_R(i|i)X = \begin{cases} Q_{2n-1}, & i \text{ is even}; \\ C_{2n-1}, & i \text{ is odd}. \end{cases}$

Thus we have $\det \Gamma_R(i|i) = \begin{cases} q_{2n-1}, & i \text{ is even}; \\ c_{2n-1}, & i \text{ is odd}. \end{cases}$

Therefore,

$$-a_{2n-1} = q_{2n-1} + c_{2n-1} + \sum_{i=2}^{2n-1} det\Gamma_R\left(i|i\right) = nq_{2n-1} + nc_{2n-1} = n\left(\frac{4n}{6^n} + \frac{n}{6^{n-1}}\right) = \frac{10n^2}{6^n}.$$

This completes the proof of Lemma 3.3.

Lemma 3.4: $a_{2n-2} = \dfrac{25n^4 - 7n^2}{3 \cdot 6^n}$

Proof. Since the number a_{2n-2} $(= (-1)^{2n-2}a_{2n-2})$ is the sum of all those principal minors of Γ_R which have $2n-2$ rows and columns (see [19, P5]), one has

$$a_{2n-2} = \sum_{1\leqslant i<j\leqslant 2n} det\Gamma_R\left(\{i,j\} \mid \{i,j\}\right).$$

We proceed further by considering the below subcases.

Subcase 1: $i = 1, j \in \{2, 3, \ldots, 2n\}$. Let

$$Q = \begin{pmatrix} l_{2,2} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & l_{j-2,j-2} & -\dfrac{1}{\sqrt{6}} \\ 0 & \cdots & -\dfrac{1}{\sqrt{6}} & l_{j-1,j-1} \end{pmatrix}$$

Together with the convention that $\det Q = 1$, whence $j = 2$. Then

$$\Gamma_R\left(\{1,j\} \mid \{1,j\}\right) = \begin{pmatrix} Q & 0 \\ 0 & C_{2n-j} \end{pmatrix}.$$

Hence,

$$\sum_{j=2}^{2n} det\ \Gamma_R\left(\{1,j\} \mid \{1,j\}\right) = \sum_{j=2}^{2n} det\ Q\ detC_{2n-j} = \sum_{j=2}^{2n} c_{j-2}c_{2n-j} = \frac{10n^3 - 4n}{6^n}.$$

Subcase 2: $j = 2n, i \in \{2, 3, \ldots, 2n-1\}$. Let

$$Q' = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & l_{2n-2,2n-2} & -\dfrac{1}{\sqrt{6}} \\ 0 & \cdots & -\dfrac{1}{\sqrt{6}} & l_{2n-1,2n-1} \end{pmatrix}$$

with the convention that $\det Q' = 1$ if $i = 2n-1$. Then

$$\Gamma_R(\{i,2n\} \mid \{i,2n\}) = \begin{pmatrix} Q_{i-1} & 0 \\ 0 & Q' \end{pmatrix}.$$

Hence,

$$\sum_{i=2}^{2n-1} \det \Gamma_R(\{i,2n\} \mid \{i,2n\}) = \sum_{i=2}^{2n-1} \det Q_{i-1} \det Q' = \sum_{i=2}^{2n-1} q_{i-1} q_{2n-1-i} = \frac{2\left(10n^3 - 19n + 9\right)}{3 \cdot 6^n}.$$

Subcase 3: For $1 < i < j < 2n$, we have

$$\Gamma_R(\{i,j\} \mid \{i,j\}) = \begin{pmatrix} Q_{i-1} & 0 & Q_1 \\ 0 & Q_2 & 0 \\ Q_1^T & 0 & C_{2n-j} \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} l_{i+1,i+1} & -\frac{1}{\sqrt{6}} & \cdots & 0 \\ -\frac{1}{\sqrt{6}} & l_{i+2,i+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{j-1,j-1} \end{pmatrix}$$

together with the convention that $\det Q_2 = 1$, whence $j = i+1$. Let $Y = \begin{pmatrix} 0 & I_{i-1} & 0 \\ 0 & 0 & I_{j-i-1} \\ I_{2n-j} & 0 & 0 \end{pmatrix}$, we can see that

$$Y^T \Gamma_R(\{i,j\} \mid \{i,j\})\, Y = \begin{pmatrix} C_{2n-j} & Q_1^T & 0 \\ Q_1 & Q_{i-1} & 0 \\ 0 & 0 & Q_2 \end{pmatrix}.$$

Notice that for even j, $\begin{pmatrix} C_{2n-j} & Q_1^T \\ Q_1 & Q_{i-1} \end{pmatrix} = Q_{2n-1-j+i}$ and $\det Q_2 = q_{j-i-1}$ and thus $\det \Gamma_R(\{i,j\}|\{i,j\}) = q_{2n-1-j+i} q_{j-i-1}$.

Otherwise, it is evident that $\det \begin{pmatrix} C_{2n-j} & Q_1^T \\ Q_1 & Q_{i-1} \end{pmatrix} = c_{2n-1-j+i}$ and $\det Q_2 = c_{j-i-1}$, and thus $\det \Gamma_R(\{i,j\} \mid \{i,j\}) = c_{2n-1-j+i} c_{j-i-1}$.

Therefore,

$$\sum_{1<i<j<2n} \det \Gamma_R(\{i,j\} \mid \{i,j\}) = \sum_{\substack{1<i<j<2n,\\ j\text{is even}}} q_{2n-1-j+i}q_{j-i-1} + \sum_{\substack{1<i<j<2n,\\ j\text{is odd}}} c_{2n-1-j+i}c_{j-i-1}$$

$$= \frac{25n^4 - 50n^3 - 7n^2 + 50n - 18}{3 \cdot 6^n}.$$

Hence, $a_{2n-2} = \sum_{1 \leqslant i<j \leqslant 2n} \det \Gamma_R(\{i,j\}|\{i,j\}) = \dfrac{25n^4 - 7n^2}{3 \cdot 6^n}$.

Thus, we complete proof of Lemma 3.4.

Hence, we introduce a matrix F, where F is a matrix obtained from Γ_S with the $(1, 2n)$-entry and the $(2n, 1)$-entry by replacing 0. We give the detail for i-th order of principal submatrix, F_i (resp. U_i), obtain from the first i rows and corresponding columns (resp. the last i rows and corresponding columns) of F. Put $f_i := \det F_i$, $u_i := \det U_i$ and fixed $f_0 = 1$, $u_0 = 1$. Through the below observations, we proceed further.

Observation 3.5: For $0 \leqslant i \leqslant 2n$, $f_i = \dfrac{1}{12 \cdot \sqrt{6}^i}[((3 + 2\sqrt{3}) + (-1)^i(3 - 2\sqrt{3}))((\sqrt{2} + 1)^{i+1} - (\sqrt{2} - 1)^{i+1})]$.

Proof of Observation 3.5: It is routine to check that $f_1 = \dfrac{4}{3}, f_2 = \dfrac{7}{6}, f_3 = \dfrac{4}{3}$. For $3 \leqslant i \leqslant 2n$, expanding $\det F_i$ respect to its last row yields

$$f_i = \begin{cases} f_{i-1} - \dfrac{1}{6}f_{i-2}, & \textit{if } i \textit{ is even};\\ \dfrac{4}{3}f_{i-1} - \dfrac{1}{6}f_{i-2}, & \textit{if } i \textit{ is odd}. \end{cases}$$

Let $s_i = f_{2i}$, if $0 \leqslant i \leqslant n$ and let $t_i = f_{2i+1}$, if $0 \leqslant i \leqslant n-1$. For $i \geqslant 1$, we set that $s_0 = 1$ and $t_0 = \frac{4}{3}$, then one has

$$\begin{cases} s_i &= t_{i-1} - \dfrac{1}{6}s_{i-1},\\ t_i &= \dfrac{4}{3}s_i - \dfrac{1}{6}t_{i-1}. \end{cases} \tag{5}$$

From the first equation of (5), we have $t_{i-1} = s_i + \dfrac{1}{6}s_{i-1}$, replace $i-1$ by i, we have $t_i = s_{i+1} + \dfrac{1}{6}s_i$.

Putting the values of t_{i-1} and t_i into the second equation in (5) gives $s_{i+1} = s_i - \dfrac{1}{36}s_{i-1}, i \geqslant 1$. By keeping the same procedure, we have $t_{i+1} = t_i - \dfrac{1}{36}t_{i-1}, i \geqslant 1$. Therefore, f_i satisfies the below recurrence relation

$$f_i = f_{i-2} - \frac{1}{36}f_{i-4},\ f_0 = 1,\ f_1 = \frac{4}{3},\ f_2 = \frac{7}{6}, f_3 = \frac{4}{3}. \tag{6}$$

Then in (6) the characteristic equation is $x^4 = x^2 - \frac{1}{36}$ and its roots are $x_1 = \dfrac{1+\sqrt{2}}{\sqrt{6}}$, $x_2 = -\dfrac{1+\sqrt{2}}{\sqrt{6}}$, $x_3 = \dfrac{\sqrt{2}-1}{\sqrt{6}}$ and $x_4 = -\dfrac{\sqrt{2}-1}{\sqrt{6}}$. The general solution of (6) is given by

$$f_i = \left(\frac{1+\sqrt{2}}{\sqrt{6}}\right)^i y_1 + \left(-\frac{1+\sqrt{2}}{\sqrt{6}}\right)^i y_2 + \left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)^i y_3 + \left(-\frac{\sqrt{2}-1}{\sqrt{6}}\right)^i y_4. \tag{7}$$

Together with the initial conditions in (6) gives the following system of equations

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = 1, \\ \dfrac{1+\sqrt{2}}{\sqrt{6}}(y_1 - y_2) + \dfrac{\sqrt{2}-1}{\sqrt{6}}(y_3 - y_4) = \dfrac{4}{3}, \\ \left(\dfrac{1+\sqrt{2}}{\sqrt{6}}\right)^2 (y_1 + y_2) + \left(\dfrac{\sqrt{2}-1}{\sqrt{6}}\right)^2 (y_3 + y_4) = \dfrac{7}{6}, \\ \left(\dfrac{1+\sqrt{2}}{\sqrt{6}}\right)^3 (y_1 - y_2) + \left(\dfrac{\sqrt{2}-1}{\sqrt{6}}\right)^3 (y_3 - y_4) = \dfrac{4}{3}. \end{cases}$$

The unique solution of this system of equations is $y_1 = \dfrac{(1+\sqrt{2})(3+2\sqrt{3})}{12}$, $y_2 = \dfrac{(1+\sqrt{2})(3-2\sqrt{3})}{12}$, $y_3 = \dfrac{(1-\sqrt{2})(3+2\sqrt{3})}{12}$ and $y_4 = \dfrac{(1-\sqrt{2})(3-2\sqrt{3})}{12}$. By putting the values of y_1, y_2, y_3 and y_4 in (7), it is easy to get the desired result.

We give the below observation with the same procedure as in the proof of Observation 3.5.

Observation 3.6: For $0 \leqslant i \leqslant 2n$, $u_i = \frac{1}{8\cdot\sqrt{6}^i}[((2+\sqrt{3}) + (-1)^i(2-\sqrt{3}))((\sqrt{2}+1)^{i+1} - (\sqrt{2}-1)^{i+1})]$.

By using the expansion to determine $\det \Gamma_S$ with regards to its last row, one has

$$\det \Gamma_S = \det F_{2n-1} + \frac{1}{\sqrt{6}}\left[\left(-\frac{1}{\sqrt{6}}\right)^{2n-1} - \frac{1}{\sqrt{6}} \det U_{2n-2}\right] + \frac{1}{\sqrt{6}}\left[\left(-\frac{1}{\sqrt{6}}\right)^{2n-1} - \frac{1}{\sqrt{6}} \det F_{2n-2}\right]$$
$$= f_{2n-1} - \frac{1}{6}(f_{2n-2} + u_{2n-2}) - \frac{2}{6^n}.$$

Together with Observations 3.5 and 3.6, we obtain the following observation.

Observation 3.7: $det\Gamma_S = \dfrac{[(\sqrt{2}+1)^n - (\sqrt{2}-1)^n]^2}{6^n}$.

Now, we are ready to determine h_{2n-1}. For the sake of simplicity, we denote the entries of Γ_S with k_{ij}, $i,j = 1,2,\ldots,2n$.

Observation 3.8: $h_{2n-1} = -\dfrac{7\sqrt{2}n[(\sqrt{2}+1)^{2n} - (\sqrt{2}-1)^{2n}]}{4\cdot 6^n}$.

Proof of Observation 3.8: Since $-h_{2n-1}$ $(= (-1)^{2n-1}h_{2n-1})$ is the sum of all those principal minors of Γ_S which have $2n-1$ rows and columns (see also in [19, P5]), one has $-h_{2n-1} = \sum_{i=1}^{2n} \det \Gamma_S(i|i)$. For $2 \leqslant i \leqslant 2n-1$, one has

$$\Gamma_S(i|i) = \begin{pmatrix} k_{11} & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & k_{22} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & k_{i-1,i-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & k_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & k_{2n-1,2n-1} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & k_{2n,2n} \end{pmatrix}_{(2n-1)\times(2n-1)}.$$

By the same procedure as in the detail of $\det \Gamma_R(i|i)(2 \leqslant i \leqslant 2n-1)$ in Lemma 3.3, we have

$$-h_{2n-1} = f_{2n-1} + u_{2n-1} + \sum_{i=2}^{2n-1} \det \Gamma_S(i|i) = nf_{2n-1} + nu_{2n-1} = \frac{7\sqrt{2}n\left[\left(\sqrt{2}+1\right)^{2n} - \left(\sqrt{2}-1\right)^{2n}\right]}{4 \cdot 6^n}.$$

This completes the proof of Observation 3.8.

The below proposition is a direct consequence of Lemma 2.2.

Proposition 3.9: Let Δ_n be a zig-zag polyhex network with n hexagons. Then

$$Ke(N) = \sum_{i=2}^{2n} \frac{1}{\eta_i} + \sum_{j=1}^{2n} \frac{1}{\xi_j}. \tag{8}$$

The eigenvalues of Γ_R are characterized as $0 = \eta_1 < \eta_2 \leqslant \cdots \leqslant \eta_{2n}$ and the eigenvalues of Γ_S are $0 < \xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_{2n}$.

In the following propositions, we derived the expressions $\sum_{i=2}^{2n} \frac{1}{\eta_i}$ and $\sum_{j=1}^{2n} \frac{1}{\xi_j}$.Based on the relationship between roots and coefficients of $\varphi(\Gamma_R)$ and $\varphi(\Gamma_S)$

Proposition 3.10: Let $0 = \eta_1 < \eta_2 \leqslant \cdots \leqslant \eta_{2n}$ be eigenvalues of Γ_R. Then

$$\sum_{i=2}^{2n} \frac{1}{\eta_i} = \frac{25n^2 - 7}{30}.$$

Proof. Let $\varphi(\Gamma_R) = x^{2n} + a_1x^{2n-1} + \cdots + a_{2n-2}x^2 + a_{2n-1}x = x(x^{2n-1} + a_1x^{2n-2} + \cdots + a_{2n-2}x + a_{2n-1})$, where $a_{2n-1} \neq 0$. Then $\eta_2, \eta_3, \ldots, \eta_{2n}$ are the roots of the following equation

$$x^{2n-1} + a_1x^{2n-2} + \cdots + a_{2n-2}x + a_{2n-1} = 0.$$

That is to say, $\frac{1}{\eta_2}, \frac{1}{\eta_3}, \ldots, \frac{1}{\eta_{2n}}$ are the roots of $a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \cdots + a_1x + 1 = 0$.

From the Vieta's Theorem, one has

$$\sum_{i=2}^{2n} \frac{1}{\eta_i} = -\frac{a_{2n-2}}{a_{2n-1}}. \tag{9}$$

Putting Lemmas 3.3 and 3.4 into (9) yields $\sum_{i=2}^{2n} \frac{1}{\eta_i} = \frac{25n^2 - 7}{30}$, as desired.

Proposition 3.11: Let $\xi_1, \xi_2, \ldots, \xi_{2n}$ be eigenvalues of Γ_S. Then

$$\sum_{i=1}^{2n} \frac{1}{\xi_i} = \frac{7\sqrt{2}n}{4} \cdot \frac{\left(\sqrt{2}+1\right)^n - \left(\sqrt{2}-1\right)^n}{\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n}.$$

Proof. Let $\varphi(\Gamma_S) = x^{2n} + h_1 x^{2n-1} + \cdots + h_{2n-1}x + h_{2n}$,

where $h_{2n} \neq 0$. Then $\xi_1, \xi_2, \ldots, \xi_{2n}$ are the roots of the following equation

$$x^{2n} + h_1 x^{2n-1} + \cdots + h_{2n-1}x + h_{2n} = 0.$$

That is to say, $\frac{1}{\xi_1}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_{2n}}$ are the roots of

$$h_{2n}x^{2n} + h_{2n-1}x^{2n-1} + \cdots + h_1 x + 1 = 0.$$

Bear in mind the Vieta's Theorem; we have

$$\sum_{i=1}^{2n} \frac{1}{\xi_i} = -\frac{h_{2n-1}}{h_{2n}} = -\frac{h_{2n-1}}{\det \Gamma_S}. \tag{10}$$

To obtain the expression $\sum_{i=1}^{2n} \frac{1}{\xi_i}$, it is enough to obtain h_{2n-1} and $\det \Gamma_S$ in (10). In view of (10), Observations 3.7 and 3.8, Proposition 3.11 follows directly.

Now, we will calculate some significant invariants related to Δ_n (resp. ∇'_n) by the expression of the eigenvalues of $\Gamma(N)$. We also contribute closed-form formulae of Ω and τ for Δ_n (resp. ∇'_n) in the subsequent section.

4 Main Results

4.1 The Kemeny's Constant and the Number of Spanning Trees of Δ_n

Let Δ_n is a hexagonal ring network with n hexagons. Then

Theorem 4.1: $\Omega(\Delta_n) = \frac{25n^2 - 7}{30} + \frac{7\sqrt{2}n}{4} \cdot \frac{(\sqrt{2}+1)^n + (\sqrt{2}-1)^n}{(\sqrt{2}+1)^n - (\sqrt{2}-1)^n}.$

Proof. Based on Propositions 3.1–3.3 and $|E_{\Delta_n}| = 5n.$, we obtain the result immediately.

Theorem 4.2: $\tau(\Delta_n) = n[(\sqrt{2}+1)^n - (\sqrt{2}-1)^n]^2.$

Proof. Based on the proof of Proposition 3.10, it is easy to see that $\eta_2, \ldots, \eta_{2n}$ are the roots of an equation $x^{2n-1} + a_1 x^{2n-2} + \cdots + a_{2n-2}x + a_{2n-1} = 0$. Thereby, one has $\prod_{i=2}^{2n} \eta_i = -a_{2n-1}$. By Lemma 3.3, we have

$$\prod_{i=2}^{2n} \eta_i = \frac{10n^2}{6^n}.$$

Similarly,

$$\prod_{j=1}^{2n} \xi_j = \frac{\left[\left(\sqrt{2}+1\right)^n - \left(\sqrt{2}-1\right)^n\right]^2}{6^n}.$$

Note that

$$\prod_{v \in U_{\Delta_n}} d_{\Delta_n}(v) = 2^{2n} 3^{2n}, |E_{\Delta_n}| = 5n.$$

Together with Lemma 2.3, we get our desired result.

4.2 The Kemeny's Constant and the Number of Spanning Trees of ∇'_n

Now, we devote our attention to determining the Ω and the τ for the Möbius hexagonal ring network ∇'_n. Based on the labeled nodes of ∇'_n as shown in Fig. 2, one has $\pi = (1, 1')(2, 2') \cdots (2n, (2n)')$ is an automorphism of ∇'_n. From Fig. 2, we have $U_0 = \varnothing$, $U_1 = \{1, 2, \ldots, 2n\}$ and $U_2 = \{1', 2', \ldots, (2n)'\}$. For the sake of simplicity, we denote $\Gamma_R(\nabla'_n)$ and $\Gamma_S(\nabla'_n)$ to Γ'_R and Γ'_S, respectively. Thereby, it is easy to see that $\Gamma'_R = \Gamma_R$ and

$$\Gamma'_S = \begin{pmatrix} \frac{4}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & \frac{4}{3} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix}_{2n \times 2n}.$$

Note that $\eta_1, \eta_2, \ldots, \eta_{2n}$ are the spectrums of Γ'_R and suppose that $\beta'_j (1 \leqslant i \leqslant 2n)$ are the spectra of Γ'_S. Due to Lemma 2.1, we have the normalized Laplacian eigenvalues of ∇'_n is $\{\eta_1, \ldots, \eta_{2n}, \beta'_1, \ldots, \beta'_{2n}\}$. In the following theorem, we give the formula for Ω and the τ for ∇'_n.

Theorem 4.3: $\Omega(\nabla'_n) = \dfrac{25n^2 - 7}{30} + \dfrac{7\sqrt{2}n}{4} \cdot \dfrac{(\sqrt{2}+1)^n - (\sqrt{2}-1)^n}{(\sqrt{2}+1)^n + (\sqrt{2}-1)^n}.$

Proof. We denote h'_{2n-1} as the coefficient of x in $\det(xI_n - \Gamma'_S)$, and expand $\det \Gamma'_S$ with regards to the last row, we have

$$\det \Gamma'_S = w_{2n-1} - \frac{1}{6}(w_{2n-2} + u_{2n-2}) + \frac{2}{6^n} = \frac{\left[\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n\right]^2}{6^n}.$$

Similar to the method applied in Proposition 3.11, we have $h'_{2n-1} = h_{2n-1}$. Hence

$$\Omega\left(\nabla'_n\right) = \sum_{i=2}^{2n} \frac{1}{\eta_i} + \sum_{j=1}^{2n} \frac{1}{\beta'_j} = \frac{25n^2-7}{30} - \frac{h'_{2n-1}}{\det \Gamma'_S} = \frac{25n^2-7}{30} + \frac{7\sqrt{2}n}{4} \cdot \frac{\left(\sqrt{2}+1\right)^n - \left(\sqrt{2}-1\right)^n}{\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n}.$$

Theorem 4.4: $\tau(\nabla'_n) = n[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]^2$.

Proof. By keeping the same discussion as in the proof of Theorem 4.2, one has

$$\prod_{i=2}^{2n} \eta_i = \frac{10n^2}{6^n}, \prod_{j=1}^{2n} \beta'_j = \frac{\left[\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n\right]^2}{6^n}.$$

Note that

$$\prod_{v \in U_{\nabla_n'}} d_{\nabla'_n}(v) = 2^{2n}3^{2n}, |E_{\nabla'_n}| = 5n.$$

With the above expressions and Lemma 2.3, we get our desired result.

5 Comparison and Discussion

Theorem 4.1 and Theorem 4.2 implies that the $\Omega(\Delta_n)$ and $\tau(\Delta_n)$ of our considered network scales linearly and in direct proportion with n. We have verified our precise results in Figs. 3 and 4, which show that when we increase n, consequently $\Omega(\Delta_n)$ and $\tau(\Delta_n)$ also increases. Our findings give some new insights that can readily distinguish the structure of significant categories in our network. Similarly, Theorem 4.3 and Theorem 4.4 implies that the $\Omega(\nabla'_n)$ and $\tau(\nabla'_n)$ of our considered network scales linearly and in direct proportion with n. We have verified our precise results in Figs. 5 and 6, which show that when we increase n, consequently $\Omega(\nabla'_n)$ and $\tau(\nabla'_n)$ also increases.

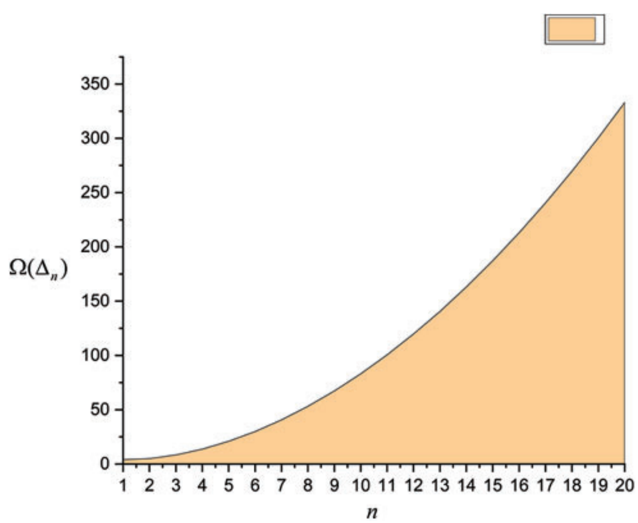

Figure 3: Spanning trees and n in Ω_n

Figs. 7 and 8 reflect the relationship between $\Omega(\Delta_n)$, $\tau(\Delta_n)$ and $\Omega(\nabla'_n)$, $\tau(\nabla'_n)$ respectively.

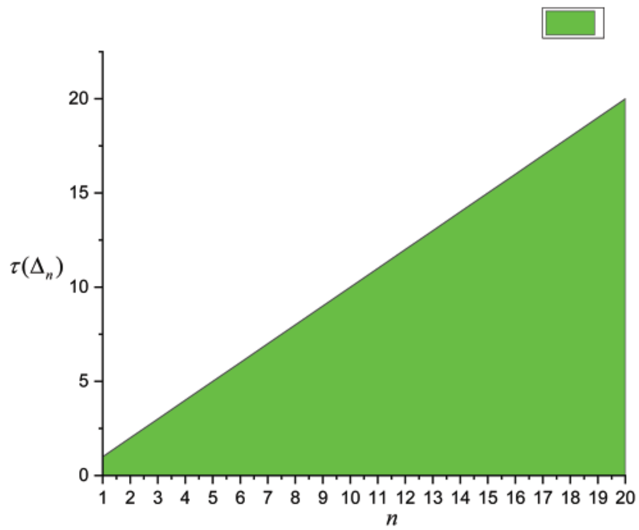

Figure 4: Spanning trees and n in Δ_n

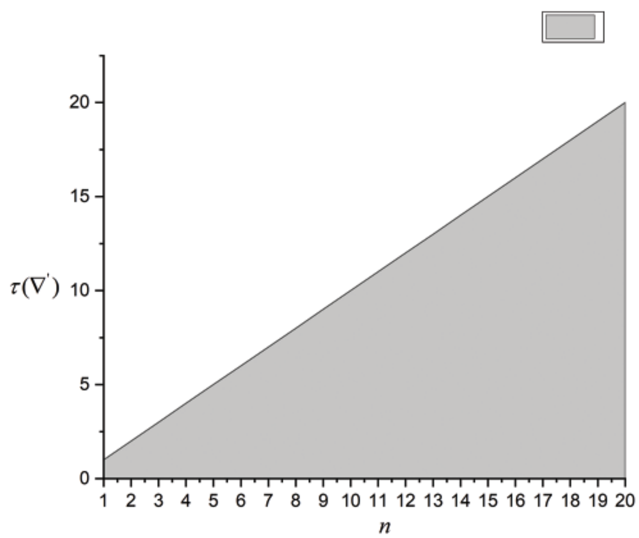

Figure 5: Kemeney's constant and n in ∇'_n

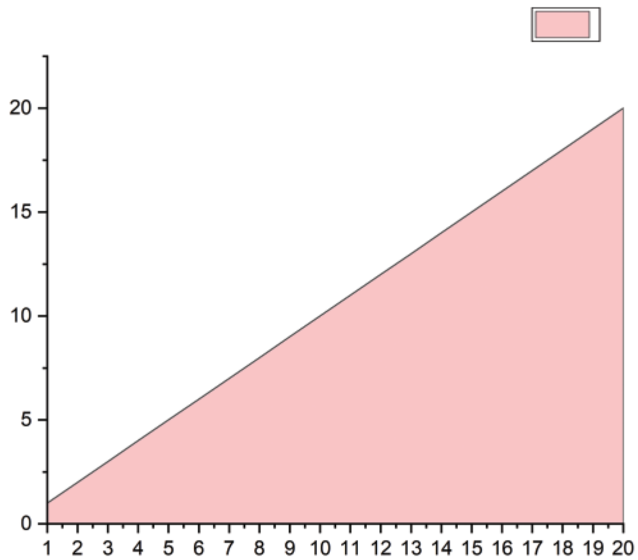

Figure 6: Spanning trees and n in ∇_n'

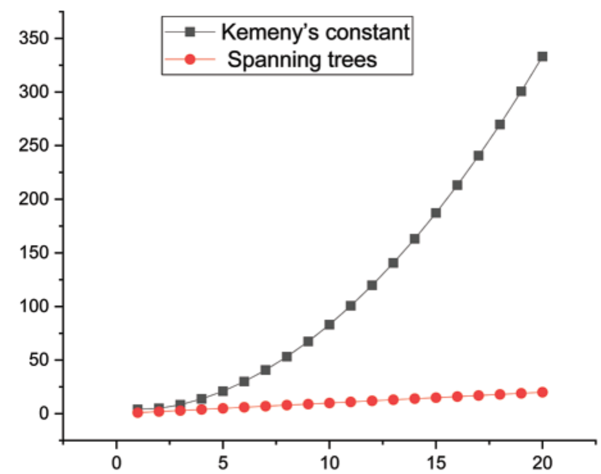

Figure 7: Comparison of Ω and τ in Δ_n

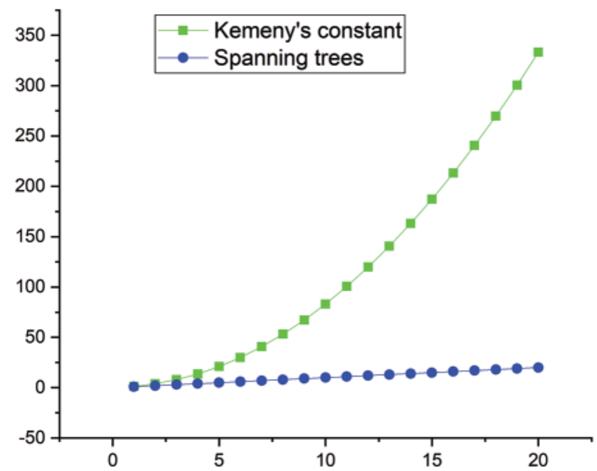

Figure 8: Comparison of Ω and τ in ∇_n'

6 Concluding Remarks

In this paper, bear in mind the spectrums of normalized Laplacian; we identified the explicit closed-form formulae of Ω and τ for Δ_n and ∇_n', respectively. It is natural and exciting to study the hitting times of random walk for the hexagonal ring network and the Möbius hexagonal ring network. We will do it shortly.

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References

[1] J. G. Kemeny and J. L. Snell, *Finite Markov Chains*. New York, Berlin, Heidelberg, Tokyo: Springer-Verlag, 1960.

[2] J. J. Hunter, "The role of Kemeny's constant in properties of Markov chains," *Communications in Statistics-Theory and Methods*, vol. 43, pp. 1309–1321, 2014.

[3] S. Zaman and A. Ali, "On connected graphs having the maximum connective eccentricity index," *Journal of Applied Mathematics and Computing*, vol. 67, no. 1, pp. 131–142, 2021.

[4] S. Kirkland, "The group inverse of the Laplacian matrix of a graph, Combinatorial matrix theory," in *Adv. Courses Math., CRM Barcelona*, Cham: Birkhäuser/Springer, pp. 131–171, 2018.

[5] F. R. K. Chung, *Spectral Graph Theory*, RI: American Mathematical Society Providence, 1997.

[6] L. Lovász, "Random walks on graphs: A survey, in combinatorics, Paul Erdös is Eighty," *Bolyai Society Mathematical Studies*, vol. 2, no. 1, pp. 1–46, 1993.

[7] L. H. Feng, I. Gutman and G. H. Yu, "Degree Kirchhoff index of unicyclic graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 69, pp. 629–648, 2013.

[8] I. Gutman and S. J. Cyvin, *Introduction to the Theory of Benzenoid Hydrocarbon*, Belin, Heidelberg: Springer-Verlag, 1989.

[9] J. W. Kennedy and L. V. Quintas, "Perfect mathchings in random hexagonal chain graphs," *Journal of Mathematical Chemistry*, vol. 6, no. 1, pp. 377–383, 1991.

[10] A. A. Dobrymin, I. Gutman, S. Klavžar and P. Žigert, "Wiener index of hexagonal systems," *Acta Applicandae Mathematicae*, vol. 72, no. 3, pp. 247–294, 2002.
[11] S. Z. Wang and B. L. Liu, "A method of calcutaing the edge-szeged index of hexagonal chain," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 68, pp. 91–96, 2012.
[12] Z. Z. Lou and Q. X. Huang, "On the characteristic polynomial of a hexagonal system and its application," *Journal of Mathematical Research with Applications*, vol. 34, pp. 265–277, 2014.
[13] D. Q. Li and Y. P. Hou, "The normalized Laplacian spectrum of quadrilateral graphs and its applications," *Applied Mathematics and Computation*, vol. 297, no. 4, pp. 180–188, 2017.
[14] Y. G. Pan and J. P. Li, "Kirchhoff index, multiplicative degree-Kirchhoff index and spanning trees of the linear crossed hexagonal chains," *International Journal of Quantum Chemistry*, vol. 118, pp. e25787, 2018.
[15] S. Zaman, "Spectral analysis of three invariants associated to random walks on rounded networks with 2n-pentagons," *International Journal of Computer Mathematics*, vol. 99, no. 3, pp. 465–486, 2022.
[16] S. Zaman, F. A. Abolaban, A. Ahmad and M. A. Asim, "Maximum H-index of bipartite network with some given parameters," *AIMS Mathematics*, vol. 6, no. 5, pp. 5165–5175, 2021.
[17] S. Zaman, "Cacti with maximal general sum-connectivity index," *Journal of Applied Mathematics and Computing*, vol. 65, no. 1–2, pp. 147–160, 2021.
[18] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky and P. Tiwari, "The electrical resistance of a graph captures its commute and cover times," *Computational Complexity*, vol. 6, no. 4, pp. 312–340, 1996.
[19] R. B. Bapat, *Graphs and Matrices*, New York: Springer, 2010.