

Bayesian Analysis in Partially Accelerated Life Tests for Weighted Lomax Distribution

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Abstract: Accelerated life testing has been widely used in product life testing experiments because it can quickly provide information on the lifetime distributions by testing products or materials at higher than basic conditional levels of stress, such as pressure, temperature, vibration, voltage, or load to induce early failures. In this paper, a step stress partially accelerated life test (SS-PALT) is regarded under the progressive type-II censored data with random removals. The removals from the test are considered to have the binomial distribution. The life times of the testing items are assumed to follow length-biased weighted Lomax distribution. The maximum likelihood method is used for estimating the model parameters of length-biased weighted Lomax. The asymptotic confidence interval estimates of the model parameters are evaluated using the Fisher information matrix. The Bayesian estimators cannot be obtained in the explicit form, so the Markov chain Monte Carlo method is employed to address this problem, which ensures both obtaining the Bayesian estimates as well as constructing the credible interval of the involved parameters. The precision of the Bayesian estimates and the maximum likelihood estimates are compared by simulations. In addition, to compare the performance of the considered confidence intervals for different parameter values and sample sizes. The Bootstrap confidence intervals give more accurate results than the approximate confidence intervals since the lengths of the former are less than the lengths of latter, for different sample sizes, observed failures, and censoring schemes, in most cases. Also, the percentile Bootstrap confidence intervals give more accurate results than Bootstrap-t since the lengths of the former are less than the lengths of latter for different sample sizes,



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observed failures, and censoring schemes, in most cases. Further performance comparison is conducted by the experiments with real data.

Keywords: Partially accelerated life testing; progressive type-II censoring; length-biased weighted Lomax; Bayesian and bootstrap confidence intervals

1 Introduction

Technology developments have been continuously improving product manufacturing. Therefore, it is difficult to obtain failure data for high-reliability items under normal operating conditions. This makes the lifetime testing under normal conditions costly and time-consuming. Consequently, under environmental conditions (stresses) that are normal conditions, the accelerated life tests (ALTs) or partially accelerated life tests (PALTs) have been often used. In the ALT, all test items are subjected to higher than usual stress levels, while, in the PALT, items are tested at both accelerated and normal conditions. In [1], the stress was applied in various ways. The constant-stress PALTs and step stress PALTs (SS-PALTs) are frequently used PALTs. In the SS-PALT, a test item is first run under normal conditions, and if it does not fail for a specified stress change time, then it is run under accelerated conditions until the end of the test.

In many cases, when the life data are analyzed, all units in the sample may not fail. This type of data is called censored or incomplete data. The most common censoring schemes are the type-I censored scheme (or time censored scheme) and type-II censored scheme (or failure-censored scheme). These two censoring schemes do not allow for units to be removed from the experiments while they are still alive. In [2], a more general censoring scheme was proposed, which is known as progressive censoring; this scheme allows units to be removed from the test. Progressive censoring is useful in a life-testing experiment because its ability to remove live units from the experiment saves both time and money. The PALTs have been widely studied for step-stress schemes of type-I, type-II, and progressive type-II censoring (PTIIC) schemes [3–12].

In this paper, a PTIIC with random removal is proposed to provide more economical and less time-consuming testing. The PTIIC scheme is designed as follows. Suppose n units are placed on a life test, and an experimenter decides beforehand m , which represents the number of units to be failed. Then, at the time of the first failure, which is denoted as $t(1)$, r_1 of the remaining $(n - 1)$ surviving units are randomly removed from the experiment. The test continues until the m th failure occurs, and at that moment, all the remaining surviving units $r_m = n - m - r_1 - r_2 - \dots - r_{m-1}$ are removed from the experiment. Many authors have studied the PTIIC application using different lifetime distributions [13–15].

On another statistical level, the Lomax distribution is an important heavy-tail probability distribution for lifetime analysis, and it has been often used in many different fields, including business, economics, and actuarial modeling [16–21]. The cumulative distribution function (CDF) and the probability density function (PDF) of the Lomax distribution with the shape parameter θ and scale parameter λ are, respectively, given by:

$$G(x; \lambda, \theta) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\theta}, \quad x, \lambda, \theta > 0. \quad (1)$$

$$g(x; \lambda, \theta) = \frac{\theta}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\theta+1)}, \quad x, \lambda, \theta > 0. \quad (2)$$

The weighted distributions arise in the context of unequal probability sampling. The weighted distributions have prominent importance in reliability, biomedicine, ecology, and other fields. The unified concept of the length-biased distribution can be used in the development of proper models for lifetime data. The length-biased distribution represents a special case of the more general form known as weighted distribution. In [22], a length-biased weighted Lomax (LBWL) distribution was introduced, and its CDF and PDF are, respectively, given by:

$$F(x; \lambda, \theta) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\theta} \left(1 + \frac{x\theta}{\lambda}\right), \quad x, \lambda > 0, \theta > 1. \quad (3)$$

$$f(x; \lambda, \theta) = \frac{\theta(\theta - 1)}{\lambda^2} \left(1 + \frac{x}{\lambda}\right)^{-\theta} \left(1 + \frac{x\theta}{\lambda}\right), \quad x, \lambda > 0, \theta > 1. \quad (4)$$

Statistical properties and applications of the LBWL distribution to real data have been presented in [22]. In [23], the estimation of stress strength reliability of the LBWL distribution in the presence of outliers was studied and discussed.

In view of the importance of the weighted distributions and SS-PALT in reliability studies, this paper applies the SS-PALT to items whose lifetimes under design conditions are assumed to follow the LBWL distribution under a PTIIC scheme with random removals. The removals from the test are considered to obey the binomial distribution. The maximum likelihood (ML) and approximate confidence intervals (CIs) of the estimators are presented. The Bayesian estimators, percentile bootstrap CIs, and bootstrap-t CIs are obtained. The Monte Carlo simulations, as well as experiments with real data, are performed to verify the theoretical analysis results.

The rest of the paper is organized as follows. Test procedure and the assumptions of the SS-PALT model are presented in Section 2. In Section 3, the ML and approximate CI estimators of the model parameters are provided. The Bayesian estimators of the model parameters are introduced in Section 4. The bootstrap CI estimates for ML and Bayesian estimation are given in Section 5. In Section 6, the simulation results are presented to illustrate accuracy of the estimates. The experiment with real data is introduced in Section 7. Finally, concluding remarks are given in Section 8.

2 Model Description and Test Procedure

The following assumptions are adopted in this work:

- Assume n identical and independent units obey the LBWL distribution in the test.
- Progressive sample $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ are generated, where m is prefixed and $m \leq n$.
- Each of n units is first run under normal operating conditions. If it does not fail or is removed from the test by a pre-specified time τ , it is placed under accelerated conditions (stress).
- At the i th failure, a random number of the surviving units r_i , where $i = 1, 2, \dots, m-1$, are randomly removed from the test. Finally, at the m th failure, the remaining surviving units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all removed from the test, and the test is terminated.
- Suppose that an individual unit removed from the test is independent of the other units but has the same removal probability p . Then, the number of removed units at each failure time follows a binomial distribution, which is expressed as:

$$R_i = \begin{cases} R_1 = \text{binomial}(n-m, p) \\ R_i = \text{binomial}\left(n-m - \sum_{j=1}^{i-1} R_j\right); & i = 2, \dots, m-1. \\ R_m = n-m - \sum_{i=1}^{m-1} R_i. \end{cases} \quad (5)$$

➤ The lifetime of a unit under the SS-PALT denoted as X can be expressed as:

$$X = \begin{cases} T & T \leq \tau \\ T + \frac{T-\tau}{\beta} & T > \tau \end{cases}, \quad (6)$$

where T denotes the lifetime of an item under normal conditions, and β represents the acceleration factor. Thus, from the assumptions of the SS-PALT model, the PDF (3) of a total lifetime of the test item can be expressed as:

$$f(x) = \begin{cases} f_1(x) = \frac{\theta(\theta-1)}{\lambda^2} x \left(1 + \frac{x}{\lambda}\right)^{-(\theta+1)}, & 0 < x \leq \tau \\ f_2(x) = \frac{\theta\beta(\theta-1)}{\lambda^2} (\tau + \beta(x-\tau)) \left(1 + \frac{\tau + \beta(x-\tau)}{\lambda}\right)^{-(\theta+1)}, & x > \tau \end{cases}, \quad (7)$$

where $f_2(x)$ is obtained by using the transformation-variable technique. The number of units removed at each failure time follows a binomial distribution $R_1 \sim \text{binomial}(n-m, p)$. For $i = 1, 2, \dots, (m-1)$, the probability mass function is expressed as:

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \dots, R_1 = r_1) = \binom{n-m - \sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m - \sum_{j=1}^i r_j}, \quad (8)$$

and $R_m = n-m - \sum_{j=1}^{m-1} R_j$. i.e., $R_i \sim \text{binomial}\left(n-m - \sum_{j=1}^{i-1} r_j, p\right)$.

3 Maximum Likelihood Estimation

This section introduces the ML estimators of the population parameters and acceleration factor based on the PTIIC data with binomial removal. Moreover, approximate CIs (ACIs) of the population parameters and acceleration factor are also presented.

Let $(x_i, r_i, \delta_{1i}, \delta_{2i})$, $i = 1, 2, \dots, m$, denote the observation obtained from a PTIC sample under a SS-PALT. For simplicity, in the following text, when referring to $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(m)}$, x_i will be used instead of $x_{(i)}$. The conditional likelihood function of observations $x = \{(x_i, r_i, \delta_{1i}, \delta_{2i}), i = 1, 2, \dots, m\}$ for a $x = \{(x_i, r_i, u_{1i}, u_{2i}), i = 1, 2, \dots, m\}$ pre-determined number of removals $R = R_1 = r_1, R_2 = r_2, \dots, R_{m-1} = r_{m-1}$, can be expressed as:

$$L_1(x_i, \beta, \lambda, \theta, \delta_{1i}, \delta_{2i} | R = r) = \prod_{i=1}^m \{f_1(x_i) (S_1(x_i))^{r_i}\}^{\delta_{1i}} \{f_2(x_i) (S_2(x_i))^{r_i}\}^{\delta_{2i}}, \tag{9}$$

$$S_1(x) = \left(1 + \frac{x}{\lambda}\right)^{-\theta} \left(1 + \frac{x\theta}{\lambda}\right), \quad S_2(x) = \left(1 + \frac{\tau + \beta(x - \tau)}{\lambda}\right)^{-\theta} \left(1 + \frac{(\tau + \beta(x - \tau))\theta}{\lambda}\right).$$

Suppose that R is independent of X for all i ; then, the likelihood function can be expressed as follows:

$$L(x_i, \beta, \lambda, \theta, p) = L_1(x_i, \beta, \lambda, \theta, \delta_{1i}, \delta_{2i} | R = r) P(R = r), \tag{10}$$

where

$$L_1(x_i, \beta, \lambda, \theta, \delta_{1i}, \delta_{2i} | R = r) = \theta^m (\theta - 1)^m \beta^{m_a} \lambda^{-2m} \times \prod_{i=1}^{m_u} (x_i D_i^{-\theta(1+r_i)-1} k_i^{r_i}) \prod_{i=1}^{m_a} [(\tau + \beta(x_i - \tau)) W_i^{-\theta(1+r_i)-1} h_i^{r_i}], \tag{11}$$

$$D_i = \left(1 + \frac{x_i}{\lambda}\right), \quad k_i = \left(1 + \frac{x_i\theta}{\lambda}\right), \quad W_i = \left(1 + \frac{\tau + \beta(x_i - \tau)}{\lambda}\right), \quad h_i = \left(1 + \frac{(\tau + \beta(x_i - \tau))\theta}{\lambda}\right),$$

$$\begin{aligned} P(R = r) &= P(R_1 = r_1, R_2 = r_2, \dots, R_{m-1} = r_{m-1}) \\ &= P(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \dots, R_1 = r_1) \\ &\times P(R_{m-2} = r_{m-2} | R_{m-3} = r_{m-3}, \dots, R_1 = r_1) \\ &\times \dots P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1). \end{aligned} \tag{12}$$

That is,

$$P(R = r) = \frac{(n - m)!}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i} p^{\sum_{i=1}^{m-1} r_i} (1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i}. \tag{13}$$

Based on (11), the natural logarithm of the conditional likelihood function, denoted as $\ln L_1$, is given by:

$$\begin{aligned} \ln L_1 = & m \ln \theta (1 - \theta) - 2m \ln \lambda + m_a \ln \beta + \sum_{i=1}^{m_u} \ln x_i - \sum_{i=1}^{m_u} \ln (D_i) - \theta \sum_{i=1}^{m_u} (1 + r_i) \ln (D_i) \\ & + \sum_{i=1}^{m_u} r_i \ln (k_i) + \sum_{i=1}^{m_a} \ln (\tau + \beta (x_i - \tau)) - \theta \sum_{i=1}^{m_a} (1 + r_i) \ln (W_i) - \sum_{i=1}^{m_a} \ln (W_i) + \sum_{i=1}^{m_a} r_i \ln (h_i), \end{aligned} \tag{14}$$

where m_u and m_a are the numbers of unites under normal and accelerated conditions, respectively, and $m = \sum_{i=1}^{m_u} \delta_{1i} + \sum_{i=1}^{m_a} \delta_{2i}$.

Since $P(R=r)$ does not involve parameters θ , λ and β their ML estimators, denoted as $\hat{\theta}$, $\hat{\lambda}$ and $\hat{\beta}$ can be directly obtained by maximizing $\ln L_1$. Thus, the ML estimators of the population parameters can be derived by solving the following non-linear equations:

$$\frac{\partial \ln L_1}{\partial \theta} = \frac{m(1 - 2\theta)}{\theta(1 - \theta)} - \sum_{i=1}^{m_u} (1 + r_i) \ln (D_i) + \sum_{i=1}^{m_u} \frac{r_i x_i}{\lambda} - \sum_{i=1}^{m_a} (1 + r_i) \ln (W_i) + \sum_{i=1}^{m_a} \frac{\beta r_i (x_i - \tau)}{h_i \lambda}, \tag{15}$$

$$\begin{aligned} \frac{\partial \ln L_1}{\partial \lambda} = & \frac{-2m}{\lambda} + \sum_{i=1}^{m_u} [\theta (1 + r_i) + 1] \frac{x_i}{\lambda^2 D_i} - \sum_{i=1}^{m_u} r_i \frac{\theta x_i}{\lambda^2 k_i} - \sum_{i=1}^{m_a} r_i \frac{\theta (\tau + \beta (x_i - \tau))}{h_i \lambda^2} \\ & + \sum_{i=1}^{m_a} [\theta (1 + r_i) + 1] \frac{(\tau + \beta (x_i - \tau))}{\lambda^2 W_i}, \end{aligned} \tag{16}$$

$$\frac{\partial \ln L_1}{\partial \beta} = \frac{m_a}{\beta} + \sum_{i=1}^{m_a} \frac{(x_i - \tau)}{(\tau + \beta (x_i - \tau))} - \sum_{i=1}^{m_a} [\theta (1 + r_i) + 1] \frac{(x_i - \tau)}{\lambda W_i} + \sum_{i=1}^{m_a} r_i \frac{\theta (x_i - \tau)}{\lambda h_i}. \tag{17}$$

Eqs. (15)–(17) have no closed-form solutions, so an iterative technique has to be employed to obtain the ML estimators of the parameters.

Similarly, since $L_1(x_i, \beta, \lambda, \theta | R=r)$ does not involve the binomial parameter p , the ML estimator of p can be directly obtained by maximizing (13). Thus, the ML estimator of p can be obtained by solving the following equation:

$$\hat{p} = \left(\sum_{i=1}^{m-1} r_i \right) / \left[(m - 1)(n - m) - \sum_{i=1}^{m-1} (m - i) r_i \right]. \tag{18}$$

Furthermore, the ACIs of the parameters using the PTIC data based on the asymptotic properties of the ML estimators can be obtained. The ACIs can be approximated by numerically inverting Fisher’s information matrix. Accordingly, the approximate $100(1 - \alpha)\%$ and the two-sided ACIs for θ , λ , and β can be, respectively, obtained as:

$$\hat{\theta} \pm z_{\alpha/2} \sigma_{\hat{\theta}}, \quad \hat{\lambda} \pm z_{\alpha/2} \sigma_{\hat{\lambda}}, \quad \hat{\beta} \pm z_{\alpha/2} \sigma_{\hat{\beta}}, \tag{19}$$

where $z_{\alpha/2}$ is the $[100(1 - \alpha)/2]^{\text{th}}$ standard normal quantile, and σ is the standard deviation of the ML estimators.

4 Bayesian Estimation

In this section, the Bayesian estimation of the population parameters and the acceleration factor of the LBWL distribution based on the PTIC data with binomial removal are presented. The Bayesian estimation is considered under the squared error loss function (SELF).

Assume that the prior of θ, λ , and β has an independent gamma distribution. Then, the joint prior density of θ, λ , and β can be expressed as:

$$\pi(\theta, \lambda, \beta) \propto \theta^{a_1-1} e^{-b_1\theta} \lambda^{a_2-1} e^{-b_2\lambda} \beta^{a_3-1} e^{-b_3\beta}; \quad a_1, b_1, a_2, b_2, a_3, b_3 > 0. \tag{20}$$

To elicit the hyper-parameters of the informative priors, the same procedure as that presented in [24] is conducted. Hence, these hyper-parameters of the informative priors are obtained from the ML estimates of (θ, λ, β) by equating their values with the mean and variance using the inverse Fisher information matrix of $(\hat{\theta}, \hat{\lambda}, \hat{\beta})$.

$$V(\theta, \lambda, \beta) = \left[- \begin{matrix} \left[\begin{matrix} \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \theta^2} & \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \theta \partial \lambda} & \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \theta \partial \beta} \\ \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \lambda^2} & \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l(\theta, \lambda, \beta)}{\partial \beta^2} \end{matrix} \right]^{-1} \end{matrix} \right]_{\theta=\hat{\theta}, \lambda=\hat{\lambda}, \beta=\hat{\beta}}. \tag{21}$$

By equating the mean and variance of $(\hat{\theta}, \hat{\lambda}, \hat{\beta})$ for gamma priors, we get:

$$\hat{\theta} = \frac{a_1}{b_1} \quad \& \quad V_{11} = \frac{a_1}{b_1^2}, \quad \hat{\lambda} = \frac{a_2}{b_2} \quad \& \quad V_{22} = \frac{a_2}{b_2^2} \quad \text{and} \quad \hat{\beta} = \frac{a_3}{b_3} \quad \& \quad V_{33} = \frac{a_3}{b_3^2}. \tag{22}$$

Hence, the estimated hyper-parameters can be expressed as:

$$a_1 = \frac{\hat{\theta}^2}{V_{11}} \quad \& \quad b_1 = \frac{\hat{\theta}}{V_{11}}, \quad a_2 = \frac{\hat{\lambda}^2}{V_{22}} \quad \& \quad b_2 = \frac{\hat{\lambda}}{V_{22}} \quad \text{and} \quad a_3 = \frac{\hat{\beta}^2}{V_{33}} \quad \& \quad b_3 = \frac{\hat{\beta}}{V_{33}}. \tag{23}$$

Based on the likelihood function (11) and the joint prior density (20), the joint posterior of the SS-PALT of the LBWL distribution with parameters θ, λ , and β is given by:

$$\pi(\beta, \lambda, \theta | x, R = r) \propto \frac{\theta^{m+a_1-1} (\theta-1)^m \beta^{m_2+a_3-1}}{\lambda^{2m-a_2+1}} e^{-b_1\theta-b_2\lambda-b_3\beta} \prod_{i=1}^{m_u} (x_i D_i^{-\theta(1+r_i)-1} k_i^{r_i}) \prod_{i=1}^{m_a} [(\tau + \beta(x_i - \tau)) W_i^{-\theta(1+r_i)-1} h_i^{r_i}], \tag{24}$$

Therefore, the Bayesian estimators of the parameters θ, λ , and β under the SELF, which are denoted as $\tilde{\theta}, \tilde{\lambda}$, and $\tilde{\beta}$, can be obtained as the posterior mean, which is expressed as follows:

$$\begin{aligned} \tilde{\theta} &= \int_0^\infty \theta \pi(\beta, \lambda, \theta | x, R = r) d\theta, \quad \tilde{\lambda} = \int_0^\infty \lambda \pi(\beta, \lambda, \theta | x, R = r) d\lambda, \quad \text{and} \\ \tilde{\beta} &= \int_0^\infty \beta \pi(\beta, \lambda, \theta | x, R = r) d\beta. \end{aligned} \tag{25}$$

The integrals presented in (25) are very difficult to solve analytically, so the Markov chain Monte Carlo (MCMC) method is used in this paper for a numerical evaluation. An important sub-class of the MCMC method is the Gibbs sampling and a more general Metropolis within Gibbs samplers. The MCMC method was first introduced in [25,26]. The Metropolis–Hastings (MH) algorithm and the Gibbs sampling are the two most popular variants of the MCMC method. Similar to the acceptance-rejection sampling, the MH method considers that for each iteration of the algorithm, a candidate value can be generated from a proposal distribution. The MH algorithm generates a sequence of draws from this distribution by conducting the following steps:

1. Set the initial value $(\Omega_l^{(0)})$; $\Omega = (\beta, \lambda, \theta)$; $l = 1, 2, 3$ satisfying the condition of $\pi(\Omega_l^{(0)}) > 0$.
2. Using the initial value, sample a candidate point (Ω^*) from proposal $q(\Omega^*)$.
3. Given the candidate point (Ω^*) , calculate the acceptance probability by:

$$A_l = \min \left(1, \frac{L_1(\Omega_l^* | x, R=r) \pi(\Omega_l^*) q(\Omega_l)}{L_1(x | \Omega_l, R=r) \pi(\Omega_l) q(\Omega_l^*)} \right); \quad l = 1, 2, 3 \quad (26)$$

4. Draw a value of u from the uniform $(0, 1)$ distribution; if $u \leq A_l$, accept Ω_l^* as $\Omega_l^{(j)}$.
5. Otherwise, reject Ω_l^* and set $\Omega_l^{(j)} = \Omega_l^{(j-1)}$.
6. Repeat Steps 2–5 $(j+1)$ times until j draws.
7. Obtain the Bayes estimate of Ω_l with respect to the SELF by $\sum_{j=1}^J \frac{(\Omega_l^{(j-1)})_i}{J}$.
8. Repeat Steps (1–7) l times to obtain the Bayesian estimate of Ω_l .

According to [26], the BCIs of the parameters $\Omega = (\beta, \lambda, \theta)$ can be obtained through the following steps:

- (1) Arrange $\Omega_l^{(j)}$; $l = 1, 2, 3$ as $\beta^{[1]}, \beta^{[2]}, \dots, \beta^{[M]}$, $\lambda^{[1]}, \lambda^{[2]}, \dots, \lambda^{[M]}$, and $\theta^{[1]}, \theta^{[2]}, \dots, \theta^{[M]}$, where M denotes the length of the generated simulation.
- (2) The $100(1 - \alpha)\%$ symmetric credible intervals of β, λ , and θ are obtained as:

$$\left(\beta^{[M\frac{\alpha}{2}]}, \beta^{[M(1-\frac{\alpha}{2})]} \right), \left(\lambda^{[M\frac{\alpha}{2}]}, \lambda^{[M(1-\frac{\alpha}{2})]} \right) \text{ and } \left(\theta^{[M\frac{\alpha}{2}]}, \theta^{[M(1-\frac{\alpha}{2})]} \right). \quad (27)$$

5 Bootstrap Confidence Intervals

In this section, different bootstrap CIs of population parameters and acceleration factor based on the PTIC data with binomial removal are proposed for the LBWL distribution.

5.1 Percentile Bootstrap (PB) Confidence Intervals

- (i) Compute estimates of θ, λ , and β by the ML and Bayesian methods.
- (ii) Generate bootstrap samples of θ, λ , and β to obtain the bootstrap estimate of β_k, λ_k , and θ_k denoted as $\hat{\beta}_k^b, \hat{\lambda}_k^b$, and $\hat{\theta}_k^b$, respectively.
- (iii) Repeat Step (ii) B times to obtain $(\beta_k^{b(1)}, \beta_k^{b(2)}, \dots, \beta_k^{b(B)})$, $(\lambda_k^{b(1)}, \lambda_k^{b(2)}, \dots, \lambda_k^{b(B)})$, and $(\theta_k^{b(1)}, \theta_k^{b(2)}, \dots, \theta_k^{b(B)})$.

- (iv) Arrange $(\beta_k^{b(1)}, \beta_k^{b(2)}, \dots, \beta_k^{b(B)})$, $(\lambda_k^{b(1)}, \lambda_k^{b(2)}, \dots, \lambda_k^{b(B)})$, and $(\theta_k^{b(1)}, \theta_k^{b(2)}, \dots, \theta_k^{b(B)})$ in ascending order as $(\beta_k^{b[1]}, \beta_k^{b[2]}, \dots, \beta_k^{b[B]})$, $(\lambda_k^{b[1]}, \lambda_k^{b[2]}, \dots, \lambda_k^{b[B]})$, and $(\theta_k^{b[1]}, \theta_k^{b[2]}, \dots, \theta_k^{b[B]})$.
- (v) A two-sided $100(1 - \alpha)\%$ percentile bootstrap CIs for unknown parameters β, λ , and θ are given by $(\hat{\beta}_k^{[B\frac{\alpha}{2}]}, \hat{\beta}_k^{[B(1-\frac{\alpha}{2})]})$, $(\hat{\lambda}_k^{[B\frac{\alpha}{2}]}, \hat{\lambda}_k^{[B(1-\frac{\alpha}{2})]})$, and $(\hat{\theta}_k^{[B\frac{\alpha}{2}]}, \hat{\theta}_k^{[B(1-\frac{\alpha}{2})]})$, respectively.

5.2 Bootstrap-t (BT) Confidence Interval

Steps 1 and 2 are the same as Steps (i) and (ii) in the previous BP algorithm, respectively.

Step 3. Compute the t -statistic of β, λ , and θ by $T = (\hat{\Omega}_k^b - \hat{\Omega}_k) / (\sqrt{V(\hat{\Omega}_k^b)})$.

Step 4. Repeat Steps 1–3 B times and obtain $T^{(1)}, T^{(2)}, \dots, T^{(B)}$.

Step 5. Arrange $T^{(1)}, T^{(2)}, \dots, T^{(B)}$ in ascending order as $T^{[1]}, T^{[2]}, \dots, T^{[B]}$.

Step 6. A two-sided $100(1 - \alpha)\%$ BT-CIs for the population parameters β, λ , and θ are given by

$$\left(\hat{\beta}_k - T_k^{[B\frac{\alpha}{2}]} \sqrt{V(\hat{\beta}_k)}, \hat{\beta}_k + T_k^{[B(1-\frac{\alpha}{2})]} \sqrt{V(\hat{\beta}_k)} \right), \left(\hat{\lambda}_k - T_k^{[B\frac{\alpha}{2}]} \sqrt{V(\hat{\lambda}_k)}, \hat{\lambda}_k + T_k^{[B(1-\frac{\alpha}{2})]} \sqrt{V(\hat{\lambda}_k)} \right),$$

and $\left(\hat{\theta}_k - T_k^{[B\frac{\alpha}{2}]} \sqrt{V(\hat{\theta}_k)}, \hat{\theta}_k + T_k^{[B(1-\frac{\alpha}{2})]} \sqrt{V(\hat{\theta}_k)} \right)$, respectively.

6 Simulation Study

The performance of the proposed methods for estimation of the LBWL distribution based on the SS-PALT under PTIIC was verified by the Monte-Carlo simulation via R-package. The Bayesian estimators were obtained using the gamma priors under the SELF. The main difficulty in the Bayesian procedure was obtaining the posterior distribution. The MH algorithm and the Gibbs sampling were used to simulate deviates from the posterior density. The simulation steps were as follows:

- Generate 10000 random samples of size $n = 50, 100$ from the LBWL distribution based on the SS-PALT under the PTIIC.
- Use the CDF of the LBWL distribution and the uni-root function in the R-package to generate the random number of the LBWL distribution in the numerical algorithm.
- For different parameters and τ , select two sets of parameters' values as:
 - Set I: $(\beta = 1.5, \lambda = 0.5, \theta = 1.5, \tau = 2), (\beta = 3, \lambda = 0.5, \theta = 1.5, \tau = 2)$,
 - Set II: $(\beta = 1.5, \lambda = 0.5, \theta = 3, \tau = 0.75), (\beta = 3, \lambda = 0.5, \theta = 3, \tau = 0.75)$;
- In the PTIIC, select the sample size (failure items) m as $m = 35, 45$ at $n = 50$, and $m = 70, 80, 90$ at $n = 100$. Set the binomial parameter p as $p = 0.25$ and 0.75 .
- Calculate the ML estimates and associated ACIs, and the Bayesian estimates and associated credible intervals at $\alpha = 0.05$.
- Evaluate the performance of the estimates based on the accuracy measures, including biases, mean square errors (MSEs), and lengths of CIs (L.CIs). The simulation results are presented in [Tabs. 1–4](#).

Table 1: ML and Bayesian estimates of LBWL parameters and acceleration factor based on PTIIC data with $p = 0.25$ for Set I

$p = 0.25$		$\beta = 3$						$\beta = 1.5$						
n	m	ML			Bayesian			ML			Bayesian			
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	
50	35	Bias	0.1163	0.2855	-0.0745	0.0635	0.2074	0.5616	0.0964	0.2694	0.0630	0.0480	0.2125	0.3562
		MSE	0.0790	0.2850	1.6826	0.0441	0.1113	1.7911	0.0935	0.2597	0.7398	0.0140	0.1146	0.6540
		L.CI	1.0033	1.7692	5.0790	0.7854	1.0250	4.7642	1.1341	1.7994	3.3602	0.4246	1.0332	2.8476
		BP	0.1401	0.2436	0.6924	0.1429	0.2444	0.7288	0.1617	0.2568	0.4502	0.1278	0.2442	0.4479
		BT	0.1642	0.2643	0.7215	0.1627	0.2712	0.7693	0.1843	0.2803	0.4773	0.1473	0.2707	0.4659
45	45	Bias	0.0834	0.1597	-0.0896	0.0553	0.1343	0.4907	0.0993	0.2218	0.0760	0.0479	0.1708	0.3242
		MSE	0.0468	0.1380	1.2659	0.0237	0.0588	1.4221	0.0625	0.2521	0.7194	0.0128	0.0817	0.5071
		L.CI	0.7829	1.3152	4.3987	0.5722	0.7923	4.2626	0.9045	1.6620	3.3172	0.4029	0.8988	2.4867
		BP	0.1121	0.1819	0.6282	0.1070	0.1854	0.6072	0.1282	0.2318	0.4364	0.1696	0.2644	0.4557
		BT	0.1264	0.1974	0.6527	0.1197	0.2015	0.6112	0.1424	0.2565	0.4674	0.2013	0.2962	0.4901
100	70	Bias	0.0559	0.1506	-0.1022	0.0305	0.1229	0.3555	0.0579	0.1536	-0.0087	0.0277	0.1181	0.2036
		MSE	0.0190	0.0764	0.7218	0.0052	0.0390	0.7188	0.0210	0.0802	0.2958	0.0046	0.0355	0.2417
		L.CI	0.4947	0.9089	3.3078	0.2551	0.6064	3.2410	0.5214	0.9331	2.1326	0.2421	0.5757	1.7552
		BP	0.0535	0.0966	0.3440	0.0513	0.0918	0.3253	0.0519	0.0934	0.2157	0.0519	0.0918	0.2084
		BT	0.0546	0.0965	0.3365	0.0532	0.0950	0.3210	0.0549	0.0984	0.2157	0.0517	0.0909	0.2107
80	80	Bias	0.0387	0.0954	-0.0234	0.0241	0.0830	0.1366	0.0465	0.1116	-0.0021	0.0237	0.0923	0.1982
		MSE	0.0134	0.0506	0.4896	0.0037	0.0223	0.4831	0.0163	0.0563	0.2548	0.0039	0.0254	0.1899
		L.CI	0.4285	0.7986	2.7428	0.2182	0.4870	2.7030	0.4666	0.8211	1.9796	0.2267	0.5098	1.5222
		BP	0.0426	0.0800	0.2800	0.0459	0.0779	0.2862	0.0481	0.0853	0.1949	0.0460	0.0815	0.1995
		BT	0.0433	0.0800	0.2832	0.0463	0.0805	0.2855	0.0476	0.0879	0.2008	0.0476	0.0828	0.2008
90	90	Bias	0.0296	0.0632	-0.0782	0.0229	0.0678	0.1274	0.0385	0.0740	-0.0262	0.0217	0.0688	0.1545
		MSE	0.0102	0.0344	0.4635	0.0034	0.0168	0.4633	0.0128	0.0355	0.2475	0.0031	0.0167	0.1650
		L.CI	0.3782	0.6833	2.6523	0.2108	0.4327	2.5930	0.4176	0.6799	1.9486	0.2001	0.4286	1.4733
		BP	0.0390	0.0692	0.2689	0.0371	0.0662	0.2660	0.0419	0.0679	0.1867	0.0405	0.0715	0.1886
		BT	0.0393	0.0711	0.2756	0.0390	0.0671	0.2712	0.0424	0.0703	0.1943	0.0421	0.0713	0.1940

Based on the obtained simulation results, the following conclusions can be drawn:

- For fixed values of n and p , the biases, MSE values, and L.CIs values of the parameter estimates decreased for both estimation methods with m .
- For fixed values of m and p , the biases, MSE values, and the L.CIs values of the parameter estimates decreased for both estimation methods with n .
- For fixed n and m , the biases, MSE values, and the L.CIs values of the parameter estimates decreased for both estimation methods with p .
- In most sets of parameters, for fixed values of λ, θ , and τ , the biases, MSE values, and the L.CI values of estimates of λ and θ decreased for both estimation methods with β , while those of estimate of β increased with β .
- For fixed n, m, p , and sets of parameters, the biases, MSE values, and the L.CIs values of estimates increased for both estimation methods with τ .

- In most situations, the measures of Bayesian estimates were better than those of the ML estimates.

Table 2: ML and Bayesian estimates of LBWL parameters and acceleration factor based on PTIIC data with $p = 0.75$ for Set I

$p = 0.75$		$\beta = 3$						$\beta = 1.5$						
n	m	ML			Bayesian			ML			Bayesian			
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	
50	35	Bias	0.1045	0.2351	-0.0962	0.0851	0.1866	0.5419	0.1061	0.2383	0.0879	0.0560	0.1870	0.3544
		MSE	0.0585	0.2149	1.5883	0.6514	0.1003	1.7584	0.0687	0.2131	0.7568	0.0149	0.0912	0.5757
		L.CI	0.8556	1.5672	4.9284	3.1477	1.0034	4.7467	0.9399	1.5505	3.3945	0.4246	0.9303	2.6313
		BP	0.1248	0.2241	0.7027	0.1156	0.2154	0.7226	0.1274	0.2180	0.4610	0.1321	0.2132	0.4626
		BT	0.1378	0.2568	0.7506	0.1318	0.2300	0.7165	0.1373	0.2383	0.4843	0.1462	0.2392	0.4966
45	45	Bias	0.0724	0.1374	-0.1044	0.0486	0.1295	0.4656	0.0746	0.1368	0.0337	0.0439	0.1291	0.3008
		MSE	0.0379	0.1035	1.3028	0.0228	0.0549	1.4474	0.0409	0.1133	0.5681	0.0110	0.0593	0.5028
		L.CI	0.7090	1.1406	4.4577	0.5609	0.7655	4.3507	0.7372	1.2065	2.9532	0.3728	0.8094	2.5183
		BP	0.0985	0.1589	0.6030	0.0998	0.1571	0.6115	0.1025	0.1635	0.4318	0.1068	0.1664	0.4127
		BT	0.1060	0.1735	0.6538	0.1111	0.1708	0.6213	0.1107	0.1844	0.4287	0.1132	0.1778	0.4320
100	70	Bias	0.0554	0.1271	-0.1011	0.0325	0.1068	0.3239	0.0492	0.1168	0.0153	0.0271	0.0989	0.2042
		MSE	0.0239	0.0798	0.7203	0.0057	0.0341	0.8516	0.0192	0.0654	0.3318	0.0048	0.0292	0.2323
		L.CI	0.5664	0.9897	3.3048	0.2662	0.5908	3.3891	0.5073	0.8919	2.2584	0.2487	0.5469	1.7122
		BP	0.0558	0.1002	0.3229	0.0543	0.0979	0.3176	0.0516	0.0908	0.2248	0.0512	0.0920	0.2229
		BT	0.0598	0.1066	0.3316	0.0594	0.1022	0.3242	0.0528	0.0919	0.2292	0.0543	0.0963	0.2280
80	80	Bias	0.0415	0.0878	-0.0895	0.0280	0.0830	0.3100	0.0416	0.0816	-0.0205	0.0238	0.0748	0.1725
		MSE	0.0149	0.0481	0.5109	0.0043	0.0236	0.7193	0.0160	0.0440	0.2872	0.0042	0.0199	0.1962
		L.CI	0.4510	0.7879	2.7813	0.2331	0.5069	3.0960	0.4678	0.7580	2.1002	0.2350	0.4694	1.6001
		BP	0.0455	0.0821	0.2757	0.0456	0.0763	0.2881	0.0454	0.0761	0.2074	0.0453	0.0750	0.1983
		BT	0.0485	0.0810	0.2788	0.0476	0.0789	0.2874	0.0471	0.0759	0.2049	0.0474	0.0790	0.2032
90	90	Bias	0.0337	0.0654	-0.0493	0.0230	0.0642	0.3268	0.0405	0.0725	0.0076	0.0243	0.0658	0.1825
		MSE	0.0123	0.0363	0.4597	0.0032	0.0162	0.7179	0.0131	0.0361	0.2438	0.0034	0.0152	0.1957
		L.CI	0.4153	0.7021	2.6521	0.2043	0.4317	3.0660	0.4194	0.6893	1.9361	0.2074	0.4091	1.5806
		BP	0.0401	0.0723	0.2608	0.0402	0.0686	0.2610	0.0399	0.0690	0.1979	0.0406	0.0665	0.1903
		BT	0.0422	0.0749	0.2710	0.0428	0.0718	0.2639	0.0418	0.0698	0.1969	0.0421	0.0684	0.1961

7 Application to Real Data

In order to further demonstrate the performance of the proposed method, a real data set was used. The R-statistical programming language was used for computation. The dataset was an uncensored dataset consisting of the remission times (in months) of a random observation of 128 bladder cancer patients reported in [27]. The LBWL distribution was fitted to real data using the Kolmogorov-Smirnov goodness of the fit test. The estimated values of parameters were: $\hat{\theta} = 4.325$, $\hat{\lambda} = 10.974$, and $p = 0.9983$. The empirical CDF, the histogram of the PDF, PP plots, and QQ plots are displayed in Fig. 1.

Table 3: ML and Bayesian estimates of LBWL parameters and acceleration factor based on PTIIC data with $p = 0.25$ for Set II

$p = 0.25$		$\beta = 3$						$\beta = 1.5$							
		ML			Bayesian			ML			Bayesian				
n	m	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$		
50	35	Bias	1.2179	0.4533	0.0846	0.7050	0.2937	0.2897	1.4356	0.5339	-0.1055	0.7714	0.3238	0.5260	
		MSE	5.6031	0.7057	0.6773	1.5621	0.2039	0.4696	6.3702	0.8539	1.6049	1.8459	0.2482	1.6412	
		L.CI	7.9605	2.7739	3.2106	4.0475	1.3455	2.4356	8.1415	2.9579	4.9513	4.3864	1.4848	4.5813	
		BP	1.0753	0.3777	0.4688	1.1813	0.4138	0.4666	1.1691	0.4313	0.6963	1.1846	0.4264	0.7239	
		BT	1.2008	0.4141	0.4898	1.2680	0.4446	0.4587	1.2989	0.4809	0.7300	1.2855	0.4579	0.7698	
	45	Bias	0.7448	0.2492	0.0743	0.7058	0.2448	0.2233	0.8013	0.2662	-0.0751	0.5727	0.2162	0.3990	
		MSE	2.8757	0.3102	0.5028	1.7947	0.1939	0.6439	3.3029	0.3520	1.1263	1.0225	0.1278	1.3917	
		L.CI	5.9748	1.9535	2.7657	4.4656	1.4353	3.0227	6.3976	2.0793	4.1518	3.2684	1.1165	4.3541	
		BP	0.8531	0.2843	0.4046	0.8129	0.2680	0.4024	0.8832	0.2942	0.5621	0.8853	0.2928	0.5765	
		BT	0.9565	0.3190	0.4268	0.9027	0.2991	0.4181	1.0657	0.3558	0.5737	1.0282	0.3291	0.5940	
	100	70	Bias	0.6470	0.2326	-0.0179	0.4846	0.1994	0.1737	0.5900	0.2089	-0.0791	0.4906	0.1943	0.2991
			MSE	1.6544	0.1912	0.3188	0.7836	0.1077	0.2602	1.4936	0.1715	0.8438	0.8360	0.1078	0.7666
			L.CI	4.3598	1.4522	2.2135	2.9054	1.0221	1.8811	4.1977	1.4027	3.5892	3.0262	1.0380	3.2273
			BP	0.4084	0.1387	0.2146	0.4362	0.1462	0.2181	0.4045	0.1356	0.3698	0.4234	0.1425	0.3670
			BT	0.4297	0.1527	0.2205	0.4614	0.1551	0.2217	0.4186	0.1438	0.3664	0.4439	0.1487	0.3664
80		Bias	0.4714	0.1582	-0.0047	0.4066	0.1553	0.1321	0.5382	0.1847	-0.0593	0.4416	0.1705	0.2739	
		MSE	1.1551	0.1141	0.2768	0.5583	0.0689	0.1606	1.3707	0.1467	0.7824	0.6872	0.0872	0.6455	
		L.CI	3.7880	1.1702	2.0635	2.4586	0.8299	1.4839	4.0778	1.3156	3.4613	2.7515	0.9454	2.9623	
		BP	0.4120	0.1248	0.2115	0.3610	0.1111	0.2012	0.4050	0.1284	0.3450	0.3951	0.1221	0.3428	
		BT	0.4214	0.1270	0.2169	0.3768	0.1161	0.2088	0.4261	0.1326	0.3546	0.4107	0.1313	0.3524	
90		Bias	0.3966	0.1270	-0.0065	0.3582	0.1321	0.1175	0.3510	0.1172	-0.0282	0.3329	0.1232	0.2536	
		MSE	1.0418	0.0960	0.2337	0.5229	0.0600	0.1375	0.8271	0.0886	0.5825	0.4740	0.0563	0.5787	
		L.CI	3.6885	1.1083	1.8958	2.4636	0.8092	1.3795	3.2906	1.0729	2.9913	2.3635	0.7948	2.8129	
		BP	0.3579	0.1128	0.1867	0.3353	0.1026	0.1743	0.3382	0.1108	0.3055	0.3263	0.1077	0.2854	
		BT	0.3806	0.1157	0.1900	0.3650	0.1058	0.1759	0.3584	0.1185	0.3040	0.3520	0.1114	0.2935	

Based on the real data, the SS-PALT of the LBWL distribution under the PTIIC with binomial removals was considered. The ML and Bayesian estimates of parameters and accelerated factor were calculated. In addition, the ACIs for parameters and accelerated factor of the LBWL distribution at a different significant level for $n = 70$, $\beta = 8$, and $p = 0.75$ were also calculated.

Tab. 5 presents the ML and Bayesian estimates of θ , λ , and β and their standard errors (SEs) at $\tau = 9, \beta = 8$ and $\tau = 9, \beta = 15$.

The history plots, approximate marginal posterior density, and MCMC convergence of θ , λ , and β are represented in Fig. 2.

The ACIs for parameters of the LBWL distribution based on the SS-PALT under the PTIIC data with binomial removals at a different level of significance for $n = 70$, $\beta = 8$, and $p = 0.75$ are presented in Fig. 3.

Table 4: ML and Bayesian estimates of LBWL parameters and acceleration factor based on PTIIC data with $p = 0.75$ for Set II

$p = 0.75$		$\beta = 3$						$\beta = 1.5$							
n	m	ML			Bayesian			ML			Bayesian				
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$		
50	35	Bias	1.0128	0.3503	0.0577	0.6533	0.2534	0.2685	1.1666	0.4031	-0.0919	0.6373	0.2523	1.0128	
		MSE	4.3978	0.5125	0.6634	1.3490	0.1713	0.4901	5.3027	0.6070	1.6960	1.2438	0.1579	4.3978	
		L.CI	7.2019	2.4488	3.1864	3.7663	1.2834	2.5357	7.7865	2.6147	5.0949	3.5897	1.2040	7.2019	
		BP	0.9604	0.3267	0.4338	1.0201	0.3403	0.4283	1.0927	0.3657	0.7062	1.1202	0.3828	0.9604	
	45	BT	1.0822	0.3700	0.4649	1.0934	0.3875	0.4630	1.2372	0.4215	0.7171	1.3084	0.4241	1.0822	
		Bias	0.6903	0.2277	0.0882	0.6393	0.2194	0.2590	0.8294	0.2734	-0.0967	0.6338	0.2270	0.6903	
		MSE	2.7967	0.3019	0.5431	1.6704	0.1725	0.7200	3.2640	0.3657	1.1745	1.4751	0.1595	2.7967	
		L.CI	5.9741	1.9613	2.8696	4.4053	1.3830	3.1690	6.2948	2.1155	4.2335	4.0634	1.2884	5.9741	
	100	70	BP	0.8230	0.2714	0.4074	0.8494	0.2699	0.3903	0.8798	0.2942	0.6169	0.8707	0.2881	0.8230
			BT	0.9137	0.3074	0.4189	0.9569	0.3132	0.4233	0.9962	0.3506	0.6370	0.9763	0.3226	0.9137
			Bias	0.5706	0.1912	-0.0066	0.4841	0.1830	0.1736	0.5619	0.1885	-0.1175	0.4911	0.1850	0.5706
			MSE	1.5067	0.1583	0.3454	0.8220	0.0994	0.2654	1.6568	0.1778	0.7943	0.7772	0.0975	1.5067
		80	L.CI	4.2624	1.3687	2.3050	3.0066	1.0067	1.9025	4.5418	1.4793	3.4649	2.8716	0.9870	4.2624
			BP	0.4258	0.1383	0.2312	0.4354	0.1412	0.2364	0.4200	0.1436	0.3446	0.4762	0.1570	0.4258
			BT	0.4726	0.1513	0.2319	0.4631	0.1470	0.2440	0.4807	0.1561	0.3449	0.5112	0.1646	0.4726
			Bias	0.4964	0.1621	-0.0086	0.4005	0.1483	0.1423	0.4234	0.1434	-0.0314	0.4171	0.1549	0.4964
90		MSE	1.2415	0.1291	0.2751	0.5713	0.0684	0.1589	1.0769	0.1202	0.6752	0.5825	0.0711	1.2415	
		L.CI	3.9123	1.2575	2.0567	2.5139	0.8450	1.4602	3.7157	1.2377	3.2204	2.5066	0.8509	3.9123	
		BP	0.3966	0.1256	0.2079	0.3765	0.1233	0.2090	0.3612	0.1181	0.3158	0.3910	0.1279	0.3966	
		BT	0.4095	0.1299	0.2113	0.3981	0.1293	0.2092	0.3902	0.1281	0.3211	0.4113	0.1334	0.4095	
90		Bias	0.4219	0.1304	-0.0107	0.3645	0.1266	0.1284	0.3574	0.1176	-0.0630	0.3378	0.1244	0.4219	
		MSE	1.0500	0.1046	0.2377	0.5126	0.0577	0.1604	0.8826	0.0934	0.6297	0.4146	0.0502	1.0500	
		L.CI	3.6623	1.1610	1.9119	2.4169	0.8003	1.4880	3.4075	1.1062	3.1024	2.1500	0.7308	3.6623	
		BP	0.3730	0.1186	0.1935	0.3590	0.1111	0.1916	0.3468	0.1142	0.3107	0.3419	0.1088	0.3730	
90	BT	0.3915	0.1283	0.2020	0.3766	0.1157	0.1954	0.3495	0.1181	0.3123	0.3649	0.1151	0.3915		

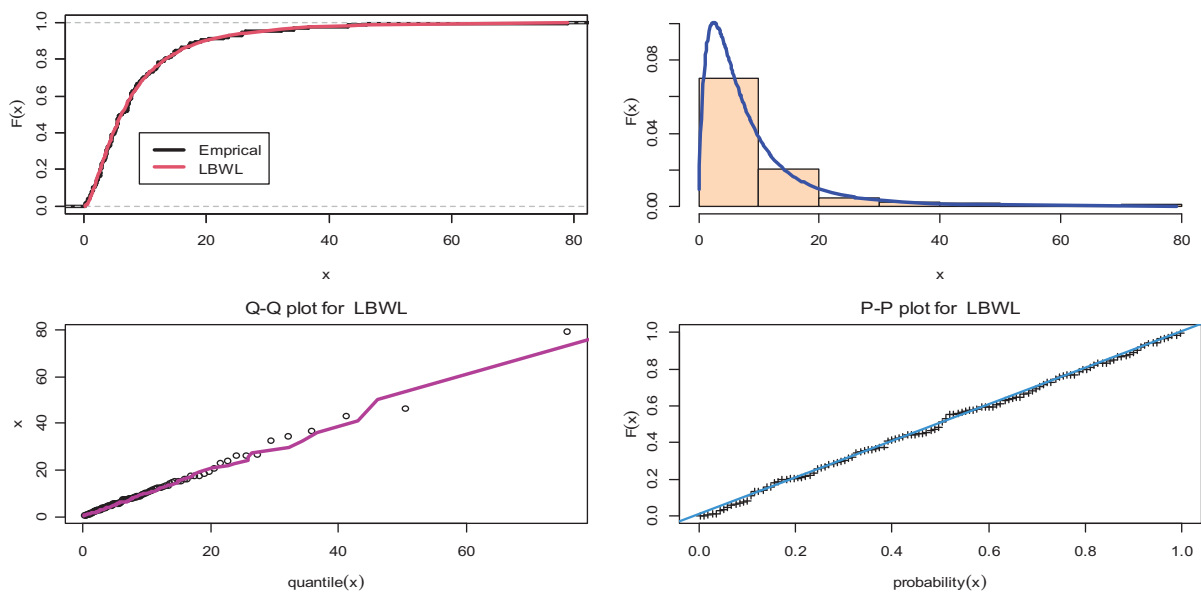


Figure 1: Plots of empirical cdf, histogram, PP-plots and QQ-plots for the LBWL distribution

Table 5: ML, Bayesian estimates and their SEs based on SS-PALT under PTIC data with binomial removals for real data

m	p	Methods		$\tau = 9, \beta = 8$			$\tau = 9, \beta = 15$			
				$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\beta}$	
70	0.25	ML	Estimates	7.6616	24.0890	7.9564	3.4136	7.1470	15.6050	
			SE	6.3438	24.3121	3.2046	1.1863	4.1016	7.2071	
		Bayesian	Estimates	11.6077	39.4794	8.0091	3.9091	9.1546	17.1497	
			SE	4.7009	17.9081	3.1449	1.2401	4.5333	8.2739	
		0.5	ML	Estimates	3.0285	6.3043	12.2879	3.0321	6.3173	22.9839
				SE	0.8693	3.2489	5.0929	0.8708	3.2558	9.5087
	Bayesian		Estimates	3.0573	6.5321	13.7553	3.2284	7.2684	25.0118	
			SE	0.5837	2.1964	5.0300	0.6934	2.7585	8.9943	
	0.75		ML	Estimates	3.5528	7.7159	8.5280	3.5571	7.7294	15.9533
				SE	1.2798	4.4997	4.0122	1.2818	4.5065	7.4926
		Bayesian	Estimates	3.8300	9.0410	10.3091	3.6750	8.4149	19.1855	
			SE	0.9736	3.6979	4.0303	0.9067	3.4011	7.4354	
100		0.25	ML	Estimates	3.7854	9.0819	9.9502	3.7825	9.0720	18.6744
				SE	1.2215	4.5667	3.5475	1.2194	4.5592	6.6596
	Bayesian		Estimates	4.1864	10.9759	10.8710	4.2122	10.8904	18.7347	
			SE	1.0984	4.1604	3.5064	0.9592	3.7099	5.4908	
	0.5		ML	Estimates	4.0116	9.6960	8.5389	4.0172	9.7184	15.9979
				SE	1.3900	5.1062	3.0495	1.3972	5.1340	5.7210
		Bayesian	Estimates	4.8383	13.1819	8.5757	4.4484	11.5132	16.4234	
			SE	1.5162	5.0121	2.4670	1.2147	4.8149	4.5385	
		0.75	ML	Estimates	4.1199	10.0591	7.0695	4.1182	10.0534	13.2684
				SE	1.5367	5.6271	2.6016	1.5377	5.6304	4.8911
	Bayesian		Estimates	4.3458	11.1512	7.9142	4.9837	13.5235	13.1067	
			SE	1.2083	4.7402	2.4686	1.3139	5.0768	4.2847	

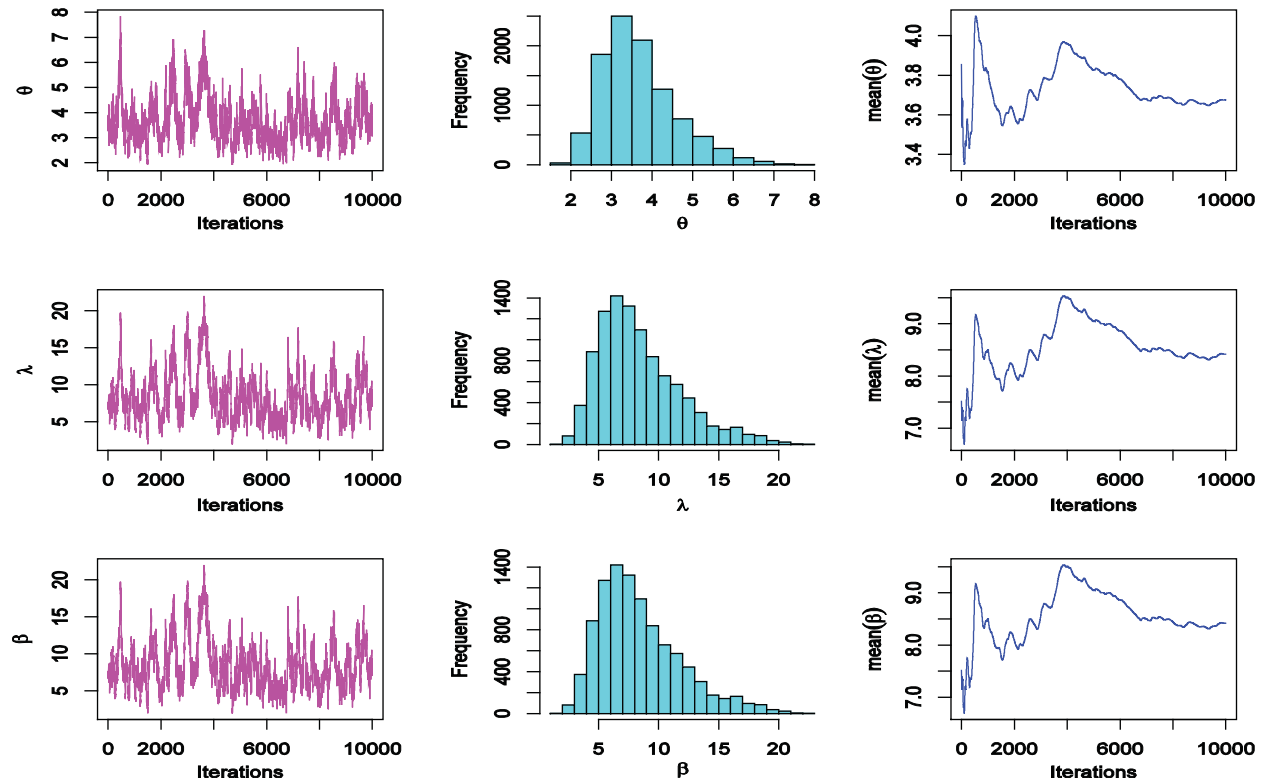


Figure 2: The MCMC plots for data based on SS-PALT under PTIC data with binomial removals

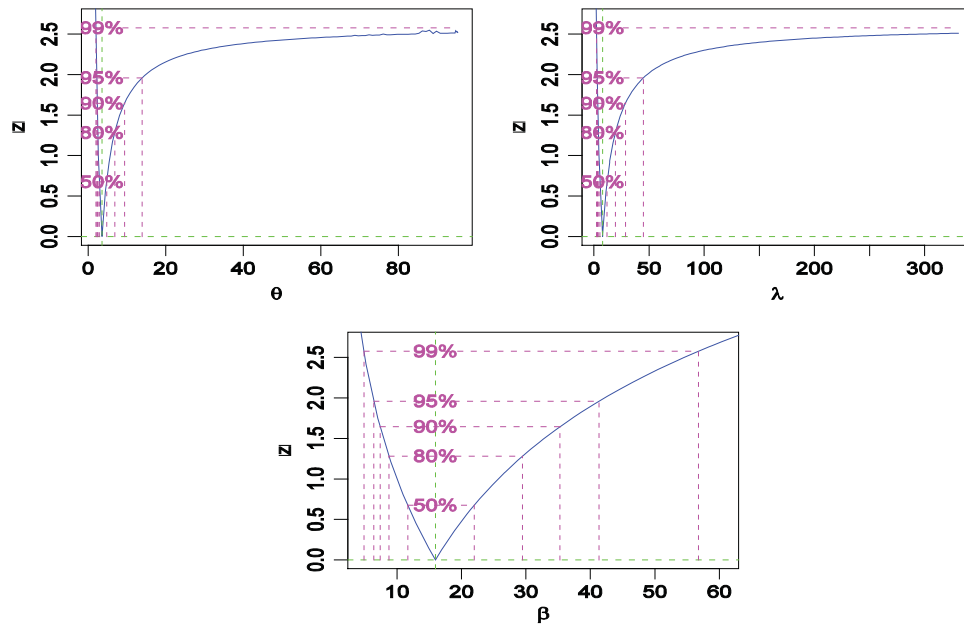


Figure 3: Plots of ACIs for SS-PALT under PTIIC data with binomial removals of real data

8 Conclusion

In this paper, the Bayesian and maximum likelihood estimation methods for the LBWL distribution are discussed based on SS-PALT using the PTIIC data with binomial removals. The approximate confidence intervals of the ML estimators of the model parameters are assessed based on the Fisher information matrix. In addition, the percentile bootstrap and bootstrap-t confidence intervals are determined. Moreover, the effects of sample size n , failure size m , and removal probability p on the accuracy of estimates are studied by simulation studies. The application of the proposed methods to real data is given for illustration purposes.

The simulation results show that, for fixed values of m and p , the performances of both estimation methods improve with n , and for fixed n and p , their performances improve with m , also for fixed n and m , their performances improve with p . Generally, the Bayesian estimates are better than the ML estimates in most situations. The lengths of the percentile bootstrap and bootstrap-t are smaller than those of the corresponding ACIs. For small sample sizes, Bootstrap-t CIs are better than the Bootstrap-p CIs in the sense of having smaller widths. However, the differences between the lengths of CIs using both methods decrease when sample sizes increase. In fact, the removal probability p is an important factor on the accuracy of the parameter estimates. When p is large, $n-m$ of the n test units would be dropped out at the early stage of the life test.

As a future work, this study can be extended to explore the situation under type-I progressive censoring. Evaluation of the coverage probabilities can also be computed rather than the lengths of the CIs.

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