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***Article***

# On Computer Implementation for Comparison of Inverse Numerical Schemes for Non-Linear Equations

**Mudassir Shams[1,*], Naila Rafiq[2], Nazir Ahmad Mir[1,2], Babar Ahmad[3], Saqib Abbasi[1] and Mutee-Ur-Rehman Kayani[1]**

[1]Department of Mathematics and Statistics, Riphah International University, I-14, Islamabad, 44000, Pakistan
[2]Department of Mathematics, NUML, Islamabad, Pakistan
[3]Department of Mathematics, Comsats University, Islamabad, 44000, Pakistan
*Corresponding Author: Mudassir Shams. Email: mudassir4shams@gmail.com
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**Abstract:** In this research article, we interrogate two new modifications in inverse Weierstrass iterative method for estimating all roots of non-linear equation simultaneously. These modifications enables us to accelerate the convergence order of inverse Weierstrass method from 2 to 3. Convergence analysis proves that the orders of convergence of the two newly constructed inverse methods are 3. Using computer algebra system Mathematica, we find the lower bound of the convergence order and verify it theoretically. Dynamical planes of the inverse simultaneous methods and classical iterative methods are generated using MATLAB (R2011b), to present the global convergence properties of inverse simultaneous iterative methods as compared to classical methods. Some non-linear models are taken from Physics, Chemistry and engineering to demonstrate the performance and efficiency of the newly constructed methods. Computational CPU time, and residual graphs of the methods are provided to present the dominance behavior of our newly constructed methods as compared to existing inverse and classical simultaneous iterative methods in the literature.

**Keywords:** Non-linear equation; inverse iterative method; simultaneous method; basins of attraction; lower bound of convergence

## 1 Introduction

A large number of physical and theoretical problems arise in various fields of mathematical, physical and engineering sciences which can be formulated as a non-linear equation:

$$f(r) = 0. \tag{1}$$

The most primitive and popular iterative technique for approximating single root of Eq. (1) is Newton's method [1] (abbreviated as NM) having local quadratic convergence.

$$s^{(t)} = r^{(t)} - \frac{f(r^{(t)})}{f'(r^{(t)})}, \ (t = 0, 1, \ldots). \tag{2}$$

In the year 2016, Nedzhibov et al. [2] presented inverse method (abbreviated as INM) corresponding to method Eq. (2) given as:

$$s^{(t)} = \frac{\left(r^{(t)}\right)^2 f'(r^{(t)})}{r^{(t)} f'(r^{(t)}) + f(r^{(t)})}, \tag{3}$$

In the last few years, lot of work has been done on those numerical iterative methods which approximate single root at a time. Besides these methods in literature, there is another class of derivative free iterative schemes which approximate all roots of Eq. (1) simultaneously. These methods are very popular due to their global convergence and parallel implementation on computer (see, e.g., Weierstrass' [3], Cholakov et al. [4], Ivanov [5], Kyncheva [6], Mir et al. [7], Proinov [8], Shams et al. [9], Farmer [10] and reference cited there in [11–13]).

Among derivative free simultaneous methods, Weierstrass-Dochive method (abbreviated as WDK) is the most attractive method given by:

$$s_i^{(t)} = r_i^{(t)} - w(r_i^{(t)}), \tag{4}$$

where

$$w(r_i^{(t)}) = \frac{f(r_i^{(t)})}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{\overline{n}} (r_i^{(t)} - r_j^{(t)})}, \quad (i, j = 1, 2, 3, \ldots, \overline{n}),$$

is Weierstrass' Correction, Eq. (4) has local quadratic convergence.

G.H Nedzibove presented two new modifications of Eq. (4) namely, inverse WDK and modified inverse WDK as:

First modification (abbreviated as INHB):

$$u_i^{(t)} = \frac{r_i^{(t)}}{1 - \frac{f(r_i^{(t)})}{b_0} \prod\limits_{\substack{j=1 \\ j \neq i}}^{\overline{n}} \left(\frac{r_j^{(t)}}{r_j^{(t)} - r_i^{(t)}}\right)}, \tag{5}$$

where $b_0 = (-1)^n \prod\limits_{j=1}^{n} \zeta_j$ is Vietas formula for monic polynomial.

Second modification (abbreviated as INHH):

$$u_i^{(t)} = \frac{\left(r_i^{(t)}\right)^2 \prod\limits_{\substack{j=1 \\ j \neq i}}^{\overline{n}} (r_i^{(t)} - r_j^{(t)})}{r_i^{(t)} \prod\limits_{\substack{j=1 \\ j \neq i}}^{\overline{n}} (r_i^{(t)} - r_j^{(t)}) + f(r_i^{(t)})}. \tag{6}$$

The main aim of this research article is to accelerate the convergence order of Eqs. (5) and (6) from 2 to 3. The programs written in CAS-Mathematica are presented to find the lower bound of the convergence order of both the new and the existing inverse simultaneous methods and verify the local convergence theoretically. We take some engineering applications as numerical test examples to show the convergence behavior of simultaneous iterative schemes. Computational efficiency, dynamical planes, basins of attraction and residual graphs are presented to demonstrate the dominance performance of our newly constructed methods over existing methods in the literature of same convergence order.

## 2 Construction of Simultaneous Methods

Here, we propose the following methods by replacing $r_j^{(t)}$ by $s_j^{(t)}$ in Eqs. (5) and (6) i.e.,

$$u_i^{(t)} = \frac{r_i^{(t)}}{1 - \frac{f(r_i^{(t)})}{b_0} \prod\limits_{\substack{j=1 \\ j\neq i}}^{\bar{n}} \left(\frac{s_j^{(t)}}{s_j^{(t)} - r_i^{(t)}}\right)}, \tag{7}$$

and

$$u_i^{(t)} = \frac{\left(r_i^{(t)}\right)^2 \prod\limits_{\substack{j=1 \\ j\neq i}}^{\bar{n}} \left(r_i^{(t)} - s_j^{(t)}\right)}{r_i^{(t)} \prod\limits_{\substack{j=1 \\ j\neq i}}^{\bar{n}} \left(r_i^{(t)} - s_j^{(t)}\right) + f\left(r_i^{(t)}\right)}, \tag{8}$$

where $s_j^{(t)} = \dfrac{\left(r_j^{(t)}\right)^2 f'\left(r_j^{(t)}\right)}{r_j^{(t)} f'\left(r_j^{(t)}\right) + f\left(r_j^{(t)}\right)}$. Newly proposed inverse simultaneous methods Eqs. (7) and (8) are abbreviated as IWKM1 and IWKM2 respectively.

### 2.1 Convergence Frame Work

In this section, we prove third order convergence of the methods IWKM1 and IWKM2.

Let $D \in C^n$ be an open convex subset, $\Gamma : D \to C^n$ and u-times differentiable operators $\Gamma^{(u)}(r)$, $\Gamma(r) = (\Gamma_1(r), \ldots, \Gamma_n(r))^T$ be continuous and the sequence $\left(r^{(k)}\right)_{k\in N}$ be defined by $r^{(k+1)} = \Gamma(r^{(k)})$, $r^{(k)} = (r_1^{(k)}, \ldots, r_n^{(k)})$.

$$\Leftrightarrow r_i^{(k+1)} = \Gamma_i(r^{(k)}) \forall i \in \{1, \ldots, n\}, k \in N, \tag{9}$$

where norm in $C^n$ be defined by $norm\|r\| = \max\{|r_1|, \ldots, |r_n|\}$.

**Theorem 1 [2]:** *Let X , Y be normed spaces. Take an open convex subset D of X for a u-times Frēchet differential Operator* $\Gamma$, *i.e.*, $\Gamma : D \to Y$. *Then, for any x,y* $\in D$:

$$\left\| \Gamma(y) - \Gamma(x) - \sum_{j=1}^{u-1} \frac{1}{j!} \Gamma^{(j)}(x) \left( \underbrace{(y-x)\ldots(y-x)}_{j-times} \right) \right\| \leq \frac{\|y-x\|^u}{u!} \sup_{\zeta\in(x,y)} \left\|\Gamma^{(u)}(\zeta)\right\| \tag{10}$$

Using Theorem 1, we have:

**Theorem 2:** *Let* $\beta \in D$, *if*

$(i)$ $\Gamma(\beta) = \beta$

$(ii)$ $\Gamma(\beta) = \Gamma'(\beta) = \Gamma''(\beta) = \ldots = \Gamma^{(u)}(\beta) = 0$,

then there exists, $s > 0$ such that for any $r^{(0)} \in D$, $\left\|r^{(0)} - \beta\right\| < s$, the sequence $r^{(k+1)} = \Gamma(r^{(k)})_{k\in N,}$ converges to $\beta$.

**Proof:** Let $s_0 > 0$ be such that

$v_0 = \{r \in C : \ \|r - \beta\| \leq s_0\} \subset D$

and C $_0 = \max_{z \in v_0} \left\| \Gamma^{(u)}(z_0) \right\|$ then, there exists, $0 < s \leq s_0$, such that

$$\frac{C_0 s^u}{u!} < s \Leftrightarrow \left(\frac{C_0}{u!}\right)^{\frac{1}{u-1}} < s,$$

where $v = \{r \in C^n \ : \ \|r - \beta\| \leq s\}$. If r $\in v$, then (ii) and Theorem 1 implies:

$$\|\Gamma(r) - \beta\| = \left\| \Gamma(r) - \Gamma(\beta) - \sum_{j=1}^{u-1} \frac{1}{j!} \Gamma^{(j)}(\beta) \left( \underbrace{(r-\beta)\ldots(r-\beta)}_{j-times} \right) \right\|$$
$$\leq \frac{1}{u!} \|r - \beta\|^u \sup_{\zeta \in (\beta, r)} \left\| \Gamma^{(u)}(\zeta) \right\|^u \leq \frac{C_0 s^u}{u!} < s$$

Thus, $\Gamma(r) \in v$. Using above relation for $r = r^{(k)}$, we have:

$$\left\| r^{(k+1)} - \beta \right\| = \left\| \Gamma(r^{(k)}) - \beta \right\| \leq \frac{C_0}{u!} \left\| r^{(k)} - \beta \right\|^u. \tag{11}$$

Using Eq. (11), recursively, we have:

$$\left\| r^{(k)} - \beta \right\| \leq \frac{C_0}{u!} \left\| r^{(k)} - \beta \right\|^u \leq \frac{C_0}{u!} \left( \frac{C_0}{u!} \left\| r^{(k)} - \beta \right\|^u \right)^u$$
$$\leq \ldots \leq \left( \frac{C_0}{u!} \right)^{1+u+\ldots+u^k} \left\| r^{(0)} - \beta \right\|^{u^k}$$
$$\leq \left( \left( \frac{C_0}{u!} \right)^{\frac{1}{u-1}} s \right)^{u^k} \rightarrow 0 \text{ for } k \rightarrow 0.$$

Thus, from last inequality, the convergence order of $(r)_{k \in \mathrm{N}}^{(k)}$ is at least $u$. Now, consider IWKM1 as a vector function, i.e., $\Gamma(r) = (\Gamma_1(r), \ldots, \Gamma_n(r))$, where

$$\Gamma(r) = \frac{r_i^{(t)}}{1 - \frac{f(r_i^{(t)})}{b_0} \prod_{\substack{j=1 \\ j \neq i}}^{\overline{n}} \left( \frac{s_j^{(t)}}{s_j^{(t)} - r_i^{(t)}} \right)}. \tag{12}$$

For a fixed point $\beta = (\beta_1, \ldots, \beta_n)$ , it is not difficult to prove $\frac{\partial \Gamma_i(\zeta)}{\partial r_i} = \frac{\partial^2 \Gamma_i(\zeta)}{\partial r_i \partial r_j} = 0$ and higher order partial derivative is not equal to zero. Thus, IWKM1 has at least third order convergence.

**Theorem 3:** *Let $\zeta_1, \ldots, \zeta_n$ be simple roots of Eq. (1) and for sufficiently close initial distinct estimations $r_1^{(0)}, \ldots, r_n^{(0)}$ of the roots respectively, IWKM2 then has convergence order 3.*

**Proof:** Let $\varepsilon_i = r_i^{(t)} - \zeta_i$, $\varepsilon_i' = u_i^{(t)} - \zeta_i$ be the errors in $r_i$ and $u_i$ respectively. From the first-step of *IWKM2*, we have:

$$u_i - \zeta_i = r_i - \zeta_i - \frac{\frac{r_i f(r_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{\bar{n}} (r_i - s_j)}}{r_i + \frac{f(r_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{\bar{n}} (r_i - s_j)}}.$$

Thus, we get:

$$\varepsilon_i' = \varepsilon_i \left[ 1 - \frac{\prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)}}{1 + \frac{f(r_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{\bar{n}} (r_i - s_j)}} \right] = \varepsilon_i \left[ \frac{1 - \prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)} + \frac{f(r_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{\bar{n}} (r_i - s_j)}}{1 + \frac{f(r_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{\bar{n}} (r_i - s_j)}} \right]. \tag{13}$$

Using the expression $\prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)} - 1 = \sum\limits_{k\neq i}^{n} \frac{\varepsilon_k^2}{r_i - s_k} \prod\limits_{j\neq i}^{k-1} \frac{(r_i - \zeta_k)}{(r_i - s_j)}$ [2] in Eq. (13), we have:

$$\varepsilon_i' = \varepsilon_i \left[ \frac{\frac{\varepsilon_i^2}{r_i} \prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)} - \sum\limits_{k\neq i}^{n} \frac{\varepsilon_k^2}{r_i - s_k} \prod\limits_{j\neq i}^{k-1} \frac{(r_i - \zeta_k)}{(r_i - s_j)}}{1 + \frac{\varepsilon_k^2}{r_i} \prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)}} \right]. \tag{14}$$

If we assume all error are of the same order, i.e., $|\varepsilon_i| = |\varepsilon_k| = O(|\varepsilon|)$, then

$$\varepsilon_i' = |\varepsilon|^3 \left[ \frac{\frac{1}{r_i} \prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta j)}{(r_i - s_j)} - \sum\limits_{k\neq i}^{n} \frac{1}{r_i - s_k} \prod\limits_{j\neq i}^{k-1} \frac{(r_i - \zeta k)}{(r_i - s_j)}}{1 + \frac{\varepsilon_k^2}{r_i} \prod\limits_{\substack{j\neq i\\j=1}}^{n} \frac{(r_i - \zeta_j)}{(r_i - s_j)}} \right] = O\left(|\varepsilon|^3\right). \tag{15}$$

Hence, from Eq. (15), third order convergence is proved.

### *2.2 Using CAS for Verification of Convergence Order*

Consider

$$f(r) = (r - \theta)(r - \phi)(r - \varphi), \tag{16}$$

and the first components of $\Gamma_1(\mathbf{r})$ iterative scheme to find roots of Eq. (16), $r^{(k+1)} = \Gamma(r^{(k)})$ simultaneously. In order to verify conditions of Theorem 2, we have to express the differential of an operator $\Gamma(r)$ in terms of their partial derivate of its component as $\Gamma_i(r)$ .

$$\frac{\partial \Gamma_1(\mathbf{r})}{\partial r_1} \qquad \frac{\partial \Gamma_1(\mathbf{r})}{\partial r_2} \qquad \frac{\partial \Gamma_1(\mathbf{r})}{\partial r_3}$$

$$\frac{\partial^2\Gamma_1(\mathbf{r})}{\partial r_1^2}\ \frac{\partial^2\Gamma_1(\mathbf{r})}{\partial r_1\partial r_2}\ \frac{\partial^2\Gamma_1(\mathbf{r})}{\partial r_2^2}\ \frac{\partial^2\Gamma_1(\mathbf{r})}{\partial r_2\partial r_3}$$

$$\frac{\partial^3\Gamma_1(\mathbf{r})}{\partial r_1^3}\ \frac{\partial^3\Gamma_1(\mathbf{r})}{\partial r_1^2\partial r_2}\ \frac{\partial^3\Gamma_1(\mathbf{r})}{\partial r_1\partial r_2^2}\ \frac{\partial^3\Gamma_1(\mathbf{r})}{\partial r_2^3}\ \frac{\partial^3\Gamma_1(\mathbf{r})}{\partial r_2^2\partial r_3}$$

$$\vdots\vdots\vdots\vdots\vdots\ldots$$

and so on.

The lower bound of the convergence is obtained until the first non-zero element of row is found. The Mathematica codes are given for each of the considered methods as:

**Weierstrass-Dochive Method WDK**

$$\Gamma_1(r_1, r_2, r_3) := \mathbf{r} - \frac{f(\mathbf{r})}{\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}(\mathbf{r}_i - \mathbf{r}_j)}, (i, j = 1, .., \bar{n})$$

In[1] := D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_1$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}

Out[1] := 0

In[2] := D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_2$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}

Out[2] := 0

In[2] := D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_2$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}

Out[2] := 0

In[3] := Simplify[D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_1$, r$_2$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}]

Out[3] := $\frac{1}{-\theta + \phi}$

**Modified Inverse Weierstrass Method-INHH**

$$\Gamma_1(r_1, r_2, r_3) := \frac{(\mathbf{r})^2\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}(\mathbf{r}_i - \mathbf{r}_j)}{\mathbf{r}\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}(\mathbf{r}_i - \mathbf{r}_j) + f(\mathbf{r})},$$

In[1] := D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_2$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}

Out[1] := 0

In[2] := D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_3$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}

Out[2] := 0

In[3] := Simplify[D[$\Gamma_1$[r$_1$,r$_2$,r$_3$],r$_1$, r$_1$]/.{r$_1 \to \theta$, r$_2 \to \phi$, r$_3 \to \varphi$}]

Out[3] := $\frac{2\theta(\theta-\phi)(\theta-\varphi)}{\theta\phi\varphi}$

**Inverse Weierstrass Method - INHB**

$$\Gamma_1(r_1, r_2, r_3) := \frac{\mathbf{r}}{1 - \frac{f(\mathbf{r})}{\theta\phi\varphi}\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}\left(\frac{\mathbf{r}_j}{\mathbf{r}_j - \mathbf{r}_i}\right)}$$

In[1] := D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_2$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}

Out[1] := 0

⋮

In[3] := Simplify[D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_1$, $r_2$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}]

Out[3] := $\frac{1}{-\theta + \phi}$

**IWKM1 Method**

$$\Gamma_1(r_1, r_2, r_3) := \frac{\mathbf{r}}{1 - \frac{f(\mathbf{r})}{\theta\phi\varphi}\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}\left(\frac{\mathbf{s}_j}{\mathbf{s}_j - \mathbf{r}_i}\right)}, \text{ where } \mathbf{s}_j = \frac{(\mathbf{r}_j)^2 f'(\mathbf{r}_j)}{\mathbf{r}_j f'(\mathbf{r}_j) + f(\mathbf{r}_j)}.$$

In[1] := D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_2$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}

Out[1] := 0

⋮

In[14] := Simplify[D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_1$, $r_1$, $r_1$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}]

Out[14] := $\frac{6(-1 + \varphi)}{\theta^2\varphi^3}$

**IWKM2 Method**

$$\Gamma_1(r_1, r_2, r_3) := \frac{(\mathbf{r})^2 \prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}(\mathbf{r}_i - \mathbf{s}_j)}{\mathbf{r}\prod\limits_{\substack{j=1\\ j\neq i}}^{\bar{n}}(\mathbf{r}_i - \mathbf{s}_j) + f(\mathbf{r})}, \text{ where } \mathbf{s}_j = \frac{(\mathbf{r}_j)^2 f'(\mathbf{r}_j)}{\mathbf{r}_j f'(\mathbf{r}_j) + f(\mathbf{r}_j)}.$$

In[1] := D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_1$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}

Out[1] := 0

⋮

In[13] := Simplify[D[$\Gamma_1$[$r_1$,$r_2$,$r_3$],$r_1$, $r_1$, $r_1$]/.{$r_1 \to \theta$, $r_2 \to \phi$, $r_3 \to \varphi$}]

Out[13] := $-\frac{-12}{\theta^2}$

## 3 Basins of Attraction

To provoke the basins of attraction of iterative schemes NM, INM, WDK, INHB, INHH, IWKM1, IWKM2 for the roots of non-linear equation, we execute the real and imaginary parts of the starting approximations represented as two axes over a mesh of $250 \times 250$ in complex plane. We use

$\left|r^{(t+1)} - r^{(t)}\right| < 10^{-3}$ as a stopping criteria or maximum number of iterations as 5 due to wider convergence region of simultaneous methods. We allow different colors to mark to which root the iterative schemes converge and black in other cases. Color brightness in basins shows less number of iterations. For the generation of basins of attraction, we consider a non-linear polynomial equation $f_1(r) = r^3 + r + 40$.

The basins of attraction of single root finding iterative schemes NM and INM are shown in Figs. 1 and 2. The basins of attraction of simultaneous iterative schemes WDK, INHB, INHH, IWKM1 and IWKM2 are presented in Figs. 3–6 and Fig. 7 respectively. The elapsed time from Tab. 1 and brightness in color in Figs. 6 and 7 show the dominance behavior of IWKM1 and IWKM2 as compared to NM, INM, WDK, INHB and INHH respectively.

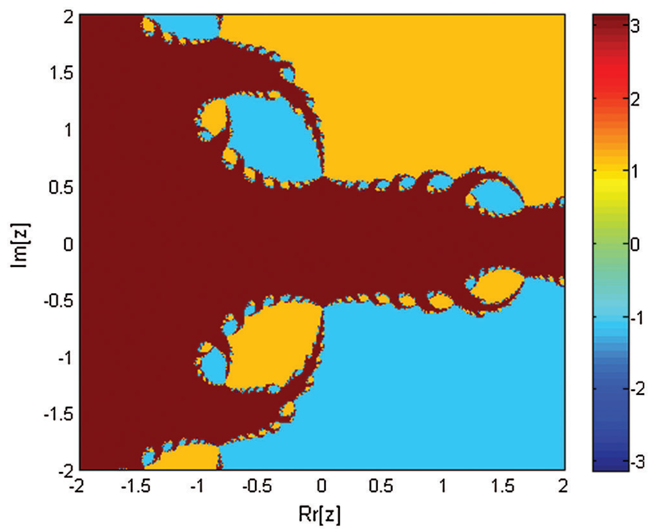


**Figure 1:** Basin of attraction of iterative method NM for polynomial equation $f_1(r) = r^3 + r + 40$

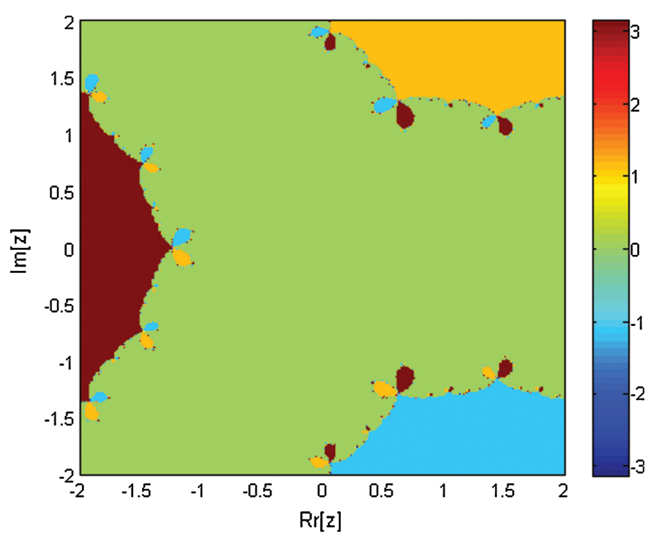


**Figure 2:** Basin of attraction of iterative method INM for polynomial equation $f_1(r) = r^3 + r + 40$

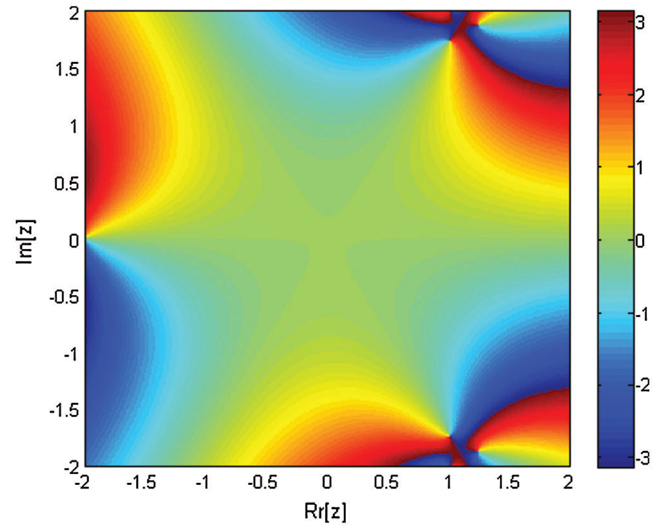


**Figure 3:** Basin of attraction of iterative method WDK for polynomial equation $f_1(r) = r^3 + r + 40$

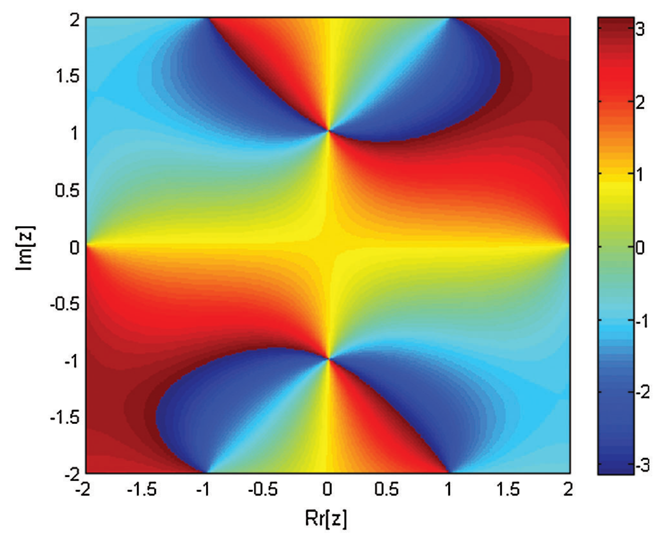


**Figure 4:** Basin of attraction of iterative method INHB for polynomial equation $f_1(r) = r^3 + r + 40$

## 4 Numerical Results

Some non-linear models from engineering and applied sciences are considered to illustrate the performance and efficiency of WDK, INHB, INHH, IWKM1 and IWKM2. All calculations are done with 64 digits floating point arithmetic. The following stopping criteria are used to terminate the computer program:

$$e_i = \left\| r_i^{(t+1)} - r_i^{(t)} \right\|_2 < \in,$$

where $e_i$ represents the absolute error and $\in = 10^{-30}$. In Tabs. 2–4, CO represents convergence order of iterative simultaneous schemes.

**Example 1** [14]**: Fractional Conversion**

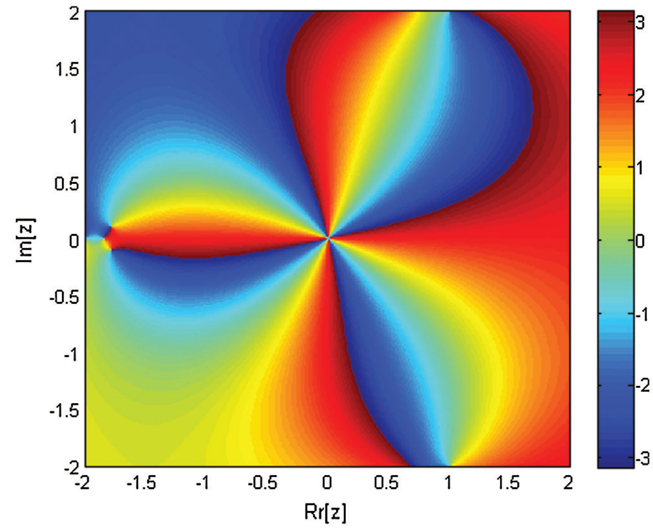


**Figure 5:** Basin of attraction of iterative method INHH for polynomial equation $f_1(r) = r^3 + r + 40$

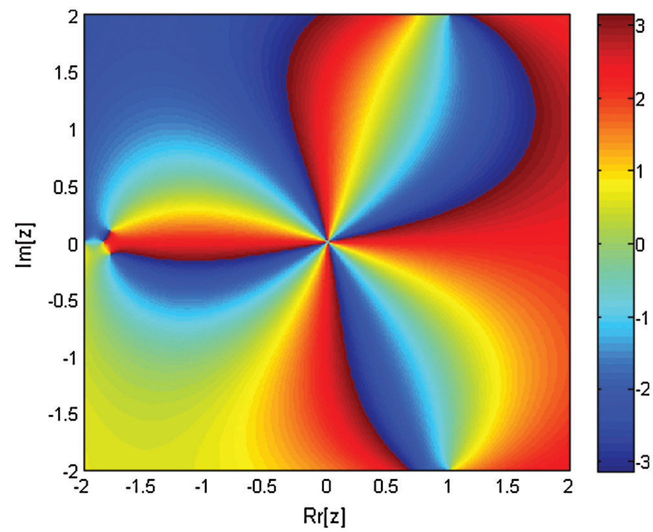


**Figure 6:** Basin of attraction of iterative method IWKM1 for polynomial equation $f_1(r) = r^3 + r + 40$

As described in [14,15],

$$f_2(r) = r^4 - 7.79075r^3 + 14.7445r^2 + 2.511r - 1.674, \tag{17}$$

is the fractional conversion of nitrogen, hydrogen feed at 250 atm and 227k.

The exact roots of Eq. (17) are:

$\zeta_1 = 3.9485 + 0.3161i$, $\zeta_2 = 3.9485 - 0.3161i$, $\zeta_3 = -0.3841$, $\zeta_4 = 0.2778$.

The initial calculated values of Eq. (17) have been taken as:

$\overset{(0)}{r_1} = 3.5 + 0.3i$, $\overset{(0)}{r_2} = 3.5 - 0.3i$, $\overset{(0)}{r_3} = -0.3 + 0.01i$, $\overset{(0)}{r_4} = 1.8 + 0.01i$.

**Example 2** [16]**: Beam Designing Model**

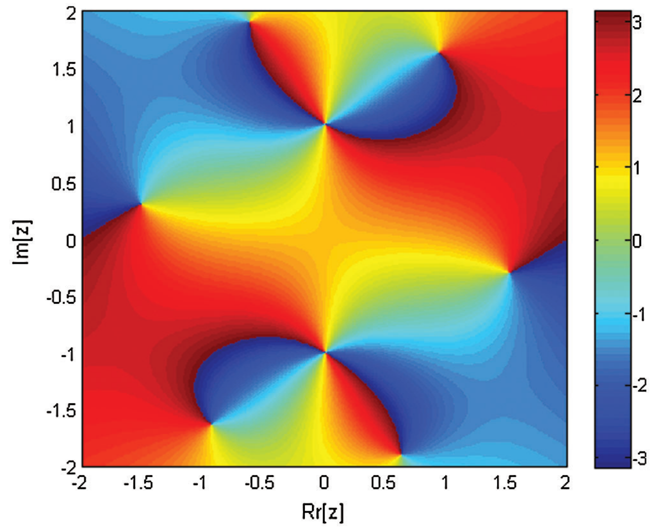


**Figure 7:** Basin of attraction of iterative method IWKM2 for polynomial equation $f_1(r) = r^3 + r + 40$

**Table 1:** Elapsed time in seconds

| NM | INM | WDK | INHB | INHH | IWKM1 | IWKM2 |
|---|---|---|---|---|---|---|
| 1.00205 | 0.08826 | 0.12388 | 0.09124 | 0.12212 | 0.04352 | 0.03521 |

**Table 2:** Simultaneous determination of all roots of $f_2(r)$

| Method | CO | CPU | n | $\breve{e}_1$ | $\breve{e}_2$ | $\breve{e}_3$ | $\breve{e}_4$ |
|---|---|---|---|---|---|---|---|
| WDK | 2 | 0.188 | 8 | 2.5e-13 | 2.1e-13 | 5.1e-9 | 1.5e-9 |
| INHB | 2 | 0.172 | 8 | 0.009 | 0.009 | 7.2e-17 | 3.3e-18 |
| INHH | 2 | 0.140 | 8 | 4.0e-12 | 3.9e-12 | 3.4e-14 | 6.8e-12 |
| IWKM1 | 3 | 0.141 | 8 | 0.0 | 0.0 | 5.8e-40 | 6.8e-13 |
| IWKM2 | 3 | 0.125 | 8 | 8.2e-25 | 6.1e-25 | 8.2e-23 | 6.7e-13 |

**Table 3:** Simultaneous determination of all roots of $f_3(r)$

| Method | CO | CPU | n | $\breve{e}_1$ | $\breve{e}_2$ | $\breve{e}_3$ | $\breve{e}_4$ |
|---|---|---|---|---|---|---|---|
| WDK | 2 | 0.031 | 7 | 0.002 | 0.002 | 0.0 | 5.8e-26 |
| INHB | 2 | 0.031 | 7 | 0.002 | 0.002 | 4.9e-26 | 0.0 |
| INHH | 2 | 0.047 | 7 | 0.003 | 0.003 | 4.3e-24 | 3.3e-21 |
| IWKM1 | 3 | 0.016 | 7 | 2.5e-4 | 2.5e-4 | 3.8e-25 | 1.0e-28 |
| IWKM2 | 3 | 0.015 | 7 | 5.5e-4 | 5.6e-4 | 0.0 | 4.9e-25 |

**Table 4:** Simultaneous determination of all roots of $f_4(r)$

| Method | CO | CPU | n | $\breve{e}_1$ | $\breve{e}_2$ | $\breve{e}_3$ |
|---|---|---|---|---|---|---|
| WDK | 2 | 0.017 | 4 | 7.2 | 7.4 | 9.8 |
| INHB | 2 | 0.013 | 4 | 5.2 | 5.2 | 0.05 |
| INHH | 2 | 0.016 | 4 | 6.4e-18 | 7.8e-18 | 1.3e-12 |
| IWKM1 | 3 | 0.011 | 4 | 1.5e-18 | 5.0e-17 | 0.8e-20 |
| IWKM2 | 3 | 0.010 | 4 | 8.0e-20 | 8.0e-20 | 3.2e-65 |

Problem of beam positioning [16], results a non-linear function as:

$$f_3(r) = r^4 + 4r^3 - 24r^2 + 16r + 16. \tag{18}$$

The exact roots of Eq. (18) are:

$$\zeta_{1,2} = 2,\ \zeta_3 = -4 - 2\sqrt{3},\ \zeta_3 = 2,\ \zeta_4 = -4 + 2\sqrt{3}.$$

The initial calculated values of Eq. (18) have been taken as:

$$\overset{(0)}{r_{1,}} = 1.9,\ \overset{(0)}{r_2} = 1.6, \overset{(0)}{r_3} = -7.4641,\ \overset{(0)}{r_4} = -0.5359.$$

**Example 3** [17]**: Predator-Prey Model**

In Predator-Prey model, predation rate is denoted by

$$P(r) = \frac{kr^3}{a^3 + r^3},\ a,\ k > 0 \tag{19}$$

where r is number of aphids as preys [17] and lady bugs as a predator. Obeying the Malthusian Model, the growth rate of aphids is defined as $G(r) = r_1 r,\ r_1 > 0$ . To find the solution of the problem, we take the aphid density for which $P(r) = G(r)$ implies

$$r_1 r^3 - kr^2 + r_1 a^3 = 0. \tag{20}$$

Taking k = 30 (aphids eaten rate), a = 20 (number of aphids) and $r_1 = 2^{-\frac{1}{3}}$ (rate per hour) in Eq. (20), we get:

$$f_4(r) = 0.7937005260r^3 - 30r^2 + 6349.604208. \tag{21}$$

The exact roots of Eq. (21) are:

$$\zeta_1 = 25.198,\ \zeta_2 = 25.198,\ \zeta_3 = 12.84.$$

The initial estimates for $f_2(r)$ are taken as:

$$\overset{(0)}{r_1} = 1.8 + 8.7i,\ \overset{(0)}{r_2} = 1.8 - 8.7i,\ \overset{(0)}{r_3} = 0.1 + 0.1i$$

## 5 Conclusion

Here, we have developed two new inverse simultaneous methods of order three for determining all the roots of non-linear equations simultaneously. It must be pointed out that so far there exists an inverse

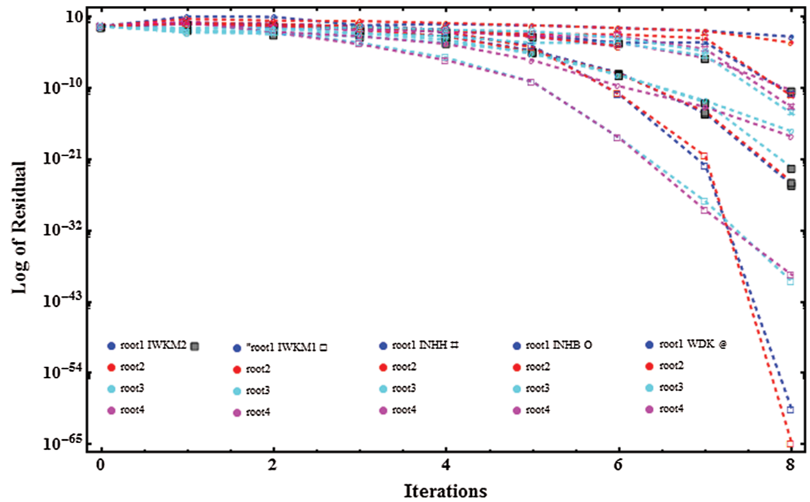


Figure 8: Residual graphs of WDK, INHB, INHH, IWKM1 and IWKM2 for non-linear function $f_2(r)$

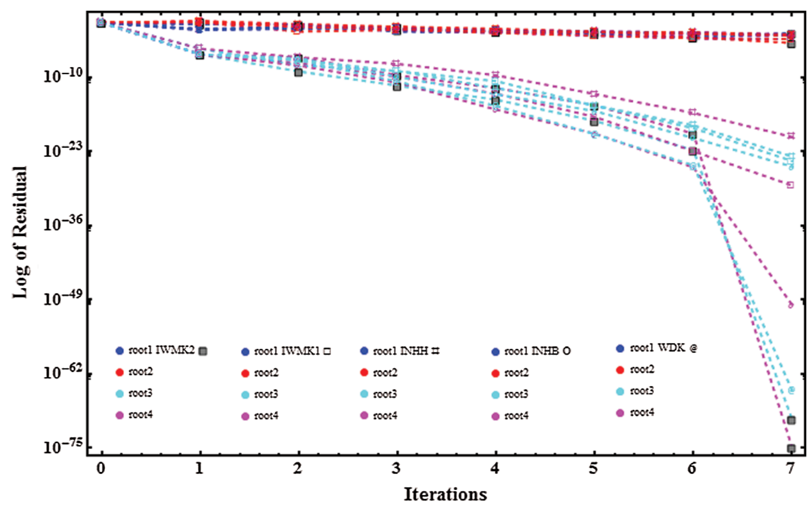


Figure 9: Residual graphs of WDK, INHB, INHH, IWKM1 and IWKM2 for non-linear function $f_3(r)$

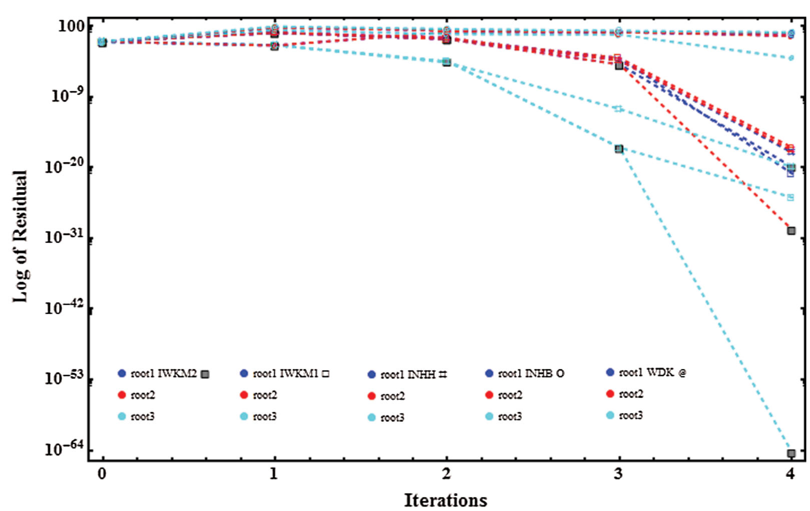


Figure 10: Residual graphs of WDK, INHB, INHH, IWKM1 and IWKM2 for non-linear function $f_4(r)$

simultaneous iterative scheme of order two only in the literature. We have made here comparison with the methods INHB, INHH and with classical Weierstrass-Dochive method WDK all of order two. The dynamical behavior/basins of attractions of iterative methods IWKM1, IWKM2 are also discussed here to show the global convergence behavior. Single root finding methods may have divergence region. From Tabs. 1–4 and Figs. 1–10, we observe that our numerical results are much better in terms of absolute error, number of iterations, CPU time and lapsed time of dynamical planes.

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