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Article

The Bivariate Transmuted Family of Distributions: Theory and Applications

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Abstract: The bivariate distributions are useful in simultaneous modeling of two random variables. These distributions provide a way to model models. The bivariate families of distributions are not much widely explored and in this article a new family of bivariate distributions is proposed. The new family will extend the univariate transmuted family of distributions and will be helpful in modeling complex joint phenomenon. Statistical properties of the new family of distributions are explored which include marginal and conditional distributions, conditional moments, product and ratio moments, bivariate reliability and bivariate hazard rate functions. The maximum likelihood estimation (*MLE*) for parameters of the family is also carried out. The proposed bivariate family of distributions is studied for the Weibull baseline distributions giving rise to bivariate transmuted Weibull (*BTW*) distribution. The new bivariate transmuted Weibull distribution is explored in detail. Statistical properties of the new *BTW* distribution are studied which include the marginal and conditional distributions, product, ratio and conditional momenst. The hazard rate function of the *BTW* distribution is obtained. Parameter estimation of the *BTW* distribution is also done. Finally, real data application of the *BTW* distribution is given. It is observed that the proposed *BTW* distribution is a suitable fit for the data used.

Keywords: Transmuted Distributions; (T−X) family of distributions; bivariate transmuted family of distributions; Weibull distribution; maximum likelihood estimation

1 Introduction

The probability distributions are widely used in several areas of life. Certain situations arise where the standard probability models are not capable of capturing complex behavior of the data and hence some extensions are required. Numerous methods are available in literature to extend a univariate probability distribution by adding some new parameters. These developments provide more flexible statistical distributions by adding new parameters to the baseline distribution. A simple method to extend any baseline distribution is proposed by Shaw et al. [1] and is known as the *transmuted family of distributions*. The cumulative distribution function (cdf) of this family is

$F_X(x) = G(x)[1 + \lambda - \lambda G(x)],$

where $G(x)$ is the *cdf* of the baseline distribution and λ is the transmutation parameter such that $\lambda \in [-1, 1]$.

Various researches have extended the transmuted family of distributions by using the $(T - X)$ family of distributions proposed by Alzaatreh et al. [2]; see for example the transmuted-G family of distributions [3], the Kumaraswamy transmuted-G family of distributions by Afify et al. [4], the beta transmuted-H family by Afify et al. [5] and the T-transmuted X family of distributions by Jayakumar et al. [6].

Different situations arise where the joint modeling of two variables is required and in such cases the bivariate distributions are required. The development of bivariate distributions from univariate marginals has been an area of interest. Several methodologies have been proposed to develop the bivariate distribution from given univariate marginals. A classical method has been proposed by Gumbel [7] which generated a bivariate distribution from given univariate marginals and is known as Gumbel family of distributions. The joint *cdf* of Gumbel bivariate distribution is given as

$F(x, y) = G(x)G(y)[1 + \gamma\{1 - G(x)\}\{1 - G(y)\}],$

where $G(x)$ and $G(y)$ are any marginal *cdf*'s and γ is an association parameter. This family has been studied by various authors. For example, the bivariate Kumaraswamy distribution has been studied by Barreto-Souza et al. [8], the bivariate Pareto distributions by Sankaran et al. [9] among others. Sarabia et al. [10] has extended the $Beta - G$ family of distributions to the bivariate case by using bivariate beta distribution as baseline distribution. This method has opened horizons for development of new bivariate distributions. In this article a bivariate transmuted family of distributions is proposed and some of its properties are studied. The family of distributions is proposed in the following sections.

2 The Bivariate Transmuted Family of Distributions

Alzaatreh et al. [2] have proposed a new method to derive families of distributions using two different distributions. The proposed family is referred to as the *T–X* family of distributions. The *cdf* of this family is

$$F_{T-X}(x) = \int_a^{W[G(x)]} r(t)\mathrm{d}t = R[W\{G(x)\}]\ ,\ x \in \mathrm{R} \tag{1}$$

where $W[G(x)]$ is some function of $G(x)$ such that

$$\left.\begin{array}{l} W[G(x)] \in [a, b] \\ W[G(x)] \text{is differentiable and monotonically non} - \text{decreasing} \\ W[G(x)] \to a \text{ as } x \to -\infty \text{ and } W[G(x)] \to b \text{ as } x \to +\infty. \end{array}\right\}$$

The probability density function (*pdf*) corresponding to (1) is

$$f_{T-X}(x) = \left[\frac{d}{dx} W\{G(x)\}\right] r[W\{G(x)\}].$$

The transmuted family of distributions, proposed by Shaw et al. [1], can be obtained from (1) by using a suitable $r(t)$ with support on $[0, 1]$ and $W[G(x)] = G(x)$. Alizadeh et al. [11] has shown that the transmuted family of distribution can be obtained by using $r(t) = 1 + \lambda - 2\lambda t$ and $W[G(x)] = G(x)$ in (1).

In this article our focus is to extend the transmuted family of distributions to bivariate case. It is, therefore, suitable to discuss the bivariate *T–X* family of distributions and then use it to propose a new family of distribution called *the bivariate transmuted family of distributions*.

A simple bivariate extension of $(T - X)$ family of distributions is defined by the joint *cdf*.

$$F_{X,Y}(x,y) = \int_{a_1}^{W_1[G_1(x)]} \int_{a_2}^{W_2[G_2(y)]} r(u_1,u_2)du_1du_2,$$

where $W_1[G_1(x)]$ and $W_2[G_2(y)]$ have usual properties and $r(u_1,u_2)$ is any bivariate distribution with suitable support for random variables U_1 and U_2. If $r(u_1,u_2)$ is a bivariate distribution such that the support of u_1 and u_2 is $[0,1] \times [0,1]$ then a simpler version of bivariate $T-X$ family is given as

$$F_{X,Y}(x,y) = \int_0^{G_1(x)} \int_0^{G_2(y)} r(u_1,u_2)du_1du_2. \tag{2}$$

It can be seen that the Gumbel bivariate family of distributions can be obtained from (2) by using

$r(u_1,u_2) = 1 + \gamma(1-2u_1)(1-2u_2);\ 0 \le u_1,u_2 \le 1.$

The bivariate transmuted family of distributions is obtained by using

$r(u_1,u_2) = 1 + \lambda_1(1-2u_1) + \lambda_2(1-2u_2) + 2\lambda_3(1-u_1-u_2),\ 0 \le u_1,u_2 \le 1,$

in (2) and the joint *cdf* of the proposed bivariate transmuted family is

$$F_{X,Y}(x,y) = G_1(x)G_2(y)[1 + (\lambda_1+\lambda_3)\{1-G_1(x)\} + (\lambda_2+\lambda_3)\{1-G_2(y)\}], \tag{3}$$

where $G_1(x)$ and $G_2(y)$ are any marginal *cdf*'s and $(\lambda_1,\lambda_2,\lambda_3)$ are the transmutation parameters such that $(\lambda_1,\lambda_2,\lambda_3) \in [-1,1]$, $-1 \le \lambda_1+\lambda_3 \le 1$ and $-1 \le \lambda_2+\lambda_3 \le 1$.

The density function corresponding to (3) is

$$f_{X,Y}(x,y) = g_1(x)g_2(y)[1 + (\lambda_1+\lambda_3)\{1-2G_1(x)\} + (\lambda_2+\lambda_3)\{1-2G_2(y)\}]. \tag{4}$$

We will, now, give some properties of the bivariate transmuted family of distributions.

3 Properties of Bivariate Transmuted Family of Distributions

In this section some important properties of the bivariate transmuted family of distributions are studied. These properties include the marginal and conditional distributions, conditional moments, product and ratio moments, bivariate reliability and bivariate hazard rate functions. Maximum likelihood estimation of the parameters of the family is also given when all the parameters of the baseline distribution are known.

3.1 The Marginal and Conditional Distributions

The marginal *cdf*'s of X and Y are readily written from (3) as

$$F_X(x) = G_1(x)[1 + (\lambda_1+\lambda_3)\{1-G_1(x)\}] \tag{5}$$

and

$$F_Y(y) = G_2(y)[1 + (\lambda_2+\lambda_3)\{1-G_2(y)\}], \tag{6}$$

where $G_1(x)$ and $G_2(y)$ are the *cdf*'s of baseline distributions and $(\lambda_1,\lambda_2,\lambda_3) \in [-1,1]$, $-1 \le \lambda_1+\lambda_3 \le 1$ and $-1 \le \lambda_2+\lambda_3 \le 1$. It is easy to see that (5) and (6) are *cdf*'s of transmuted family of distributions. The marginal density functions of X and Y are

$$f_X(x) = g_1(x)[1 + (\lambda_1+\lambda_3) - 2(\lambda_1+\lambda_3)\ G_1(x)] \tag{7}$$

and

$$f_Y(y) = g_2(y)[1 + (\lambda_2 + \lambda_3) - 2(\lambda_2 + \lambda_3)\ G_2(y)], \tag{8}$$

where $g_1(x)$ and $g_2(y)$ are the *pdf*'s of baseline distributions corresponding to $G_1(x)$ and $G_2(y)$, respectively.

The conditional distribution of X given $Y = y$ and Y given $X = x$ for the bivariate transmuted family of distributions are readily written as

$$f_{X|Y}(x|y) = \frac{g_1(x)}{\delta_1(y)}[1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y)\}] \tag{9}$$

and

$$f_{Y|X}(y|x) = \frac{g_2(y)}{\delta_2(x)}[1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y)\}], \tag{10}$$

where $\delta_1(y) = 1 + (\lambda_2 + \lambda_3) - 2(\lambda_2 + \lambda_3)G_2(y)$ and $\delta_2(x) = 1 + (\lambda_1 + \lambda_3) - 2(\lambda_1 + \lambda_3)G_1(x)$. The conditional distributions can be studied for any baseline distribution.

3.2 The Conditional Moments

The rth conditional moment of X given $Y = y$ is obtained by using

$$E(X^r|y) = \int_{-\infty}^{\infty} x^r f(x|y)\mathrm{d}x.$$

Now using the conditional distribution of X given $Y = y$, given in (9), the rth conditional moment of X given $Y = y$ is

$$E(X^r|Y) = \frac{1}{\delta_1(y)}\left[\beta\mu_x^r - (\lambda_1 + \lambda_3)\mu_{x(2:2)}^r - 2(\lambda_2 + \lambda_3)G_2(y)\mu_x^r\right], \tag{11}$$

where $\beta = 1 + \lambda_1 + \lambda_2 + 2\lambda_3$, μ_x^r is rth raw moment of X and $\mu_{x(2:2)}^r$ is rth raw moment of larger observation in a sample of size 2 from $G_1(x)$.

Again the sth conditional moment of Y given $X = x$ is defined as

$$E(Y^s|x) = \int_{-\infty}^{\infty} y^s f(y|x)\mathrm{d}y.$$

Using (10), the sth conditional moment of Y given $X = x$ is

$$E(Y^s|x) = \frac{1}{\delta_2(x)}\left[\beta\mu_y^s - (\lambda_2 + \lambda_3)\mu_{y(2:2)}^s - 2(\lambda_1 + \lambda_3)G_1(x)\mu_y^s\right], \tag{12}$$

where μ_y^s is sth raw moment of Y and $\mu_{y(2:2)}^s$ is sth raw moment of larger observation in a sample of size 2 from $G_2(y)$.

3.3 The Product and Ratio Moments

The product and ratio moments are defined as

$$\mu_{X,Y}^{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f_{X,Y}(x,y)dxdy$$

and

$$\mu_{X,Y}^{r,-s} = E\left(\frac{X^r}{Y^s}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^r}{y^s}f_{X,Y}(x,y)dxdy.$$

The product moments for the bivariate transmuted family of distribution are obtained as

$$\mu_{X,Y}^{r,s} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^r y^s g_1(x)g_2(y)[1+(\lambda_1+\lambda_3)\{1-2G_1(x)\}+(\lambda_2+\lambda_3)\{1-2G_2(y)\}]dxdy,$$

which on simplification becomes

$$\mu_{X,Y}^{r,s} = E(X^r Y^s) = \beta\mu_x^r\mu_y^s - (\lambda_1+\lambda_3)\mu_{x(2:2)}^r\mu_y^s - (\lambda_2+\lambda_3)\mu_x^r\mu_{y(2:2)}^s. \tag{13}$$

Similarly, the (r,s)th ratio moments for the bivariate transmuted family of distributions are obtained as

$$\mu_{X,Y}^{r,-s} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^r y^{-s} g_1(x)g_2(y)[1+(\lambda_1+\lambda_3)\{1-2G_1(x)\}+(\lambda_2+\lambda_3)\{1-2G_2(y)\}]dxdy,$$

which on simplification becomes

$$\mu_{X,Y}^{r,-s} = E\left(\frac{X^r}{Y^s}\right) = \beta\mu_x^r\mu_y^{-s} - (\lambda_1+\lambda_3)\mu_{x(2:2)}^r\mu_y^{-s} - (\lambda_2+\lambda_3)\mu_x^r\mu_{y(2:2)}^{-s}, \tag{14}$$

where μ_y^{-s} is sth negative moment of Y and $\mu_{y(2:2)}^{-s}$ is sth negative moment of larger observation in a sample of size 2 from $G_2(y)$.

3.4 The Bivariate Reliability and Hazard Rate Functions

The reliability function indicates the probability that a patient, device or other element of interest has ability to survive after some specific time (for more details see [12,13]). The bivariate reliability function for random variables X and Y is defined as

$$R(x,y) = 1 - \left[F_X(x) + F_Y(y) - F_{X,Y}(x,y)\right].$$

Now using (5), (6) and (3) in above equation, the bivariate reliability function for the bivariate transmuted family of distributions is

$$R(x,y) = [1-G_1(x)][1-G_2(y)][1-(\lambda_1+\lambda_3)G_1(x)-(\lambda_2+\lambda_3)G_2(y)], \tag{15}$$

which can be obtained for different baseline *cdf*'s $G_1(x)$ and $G_2(y)$.

The hazard rate function is important in reliability studies. It describes the instantaneous rate of failure at any given time. The bivariate hazard rate function (see for example [14]) is defined as

$$h_{X,Y}(x,y) = \frac{f_{X,Y}(x,y)}{R_{X,Y}(x,y)}.$$

The bivariate hazard rate function for the bivariate transmuted family of distributions is obtained by using (4) and (15) in above equation and is

$$h_{X,Y}(x,y) = \frac{g_1(x)g_2(y)[1+(\lambda_1+\lambda_3)\{1-2G_1(x)\}+(\lambda_2+\lambda_3)\{1-2G_2(y)\}]}{[1-G_1(x)][1-G_2(y)][1-(\lambda_1+\lambda_3)G_1(x)-(\lambda_2+\lambda_3)G_2(y)]}. \tag{16}$$

The bivariate hazard rate function can be computed for $G_1(x)$ and $G_2(y)$.

3.5 Dependence

The dependence between two random variables is an important measure to study the relationship between two variables. In this section two improtant dependence measures for the bivariate transmuted family of distributions are obtained. These dependence measures include Kendall's tau and Spearman's rho. The Kendall's tau coefficient for two continuous random variables is computed by using

$$\tau = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{X,Y}(x,y)f_{X,Y}(x,y)dxdy - 1,$$

which for the bivariate transmuted family of distributions is

$$\tau = -\frac{2}{9}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3). \tag{17}$$

Again, the Spearman's rho for two continuous random variables is obtained as

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F_{X,Y}(x,y) - F_X(x)F_Y(y)\right]f_X(x)f_Y(y)dxdy,$$

which for the bivariate transmuted family of distributions is given as

$$\rho = -\frac{1}{3}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3). \tag{18}$$

We can readily see that the Kendall's tau will always be larger than the Spearman's rho for the bivariate transmuted family of distributions.

3.6 Estimation of the Parameters

In this section the maximum likelihood estimation of the parameters is done for the bivariate transmuted family of distributions under the assumption that all the parameters of baseline distributions $G_1(x)$ and $G_2(y)$ are known. For this, suppose $X_1, X_2, \cdots, X_n$ is a random sample of size n from the bivariate transmuted family of distributions. The likelihood function is

$$L = \prod_{i=1}^{n} f_{X,Y}(x_i, y_i) = \prod_{i=1}^{n} g_1(x_i)g_2(y_i)[1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x_i)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y_i)\}]$$

and the log-likelihood function is

$$\ell = \sum_{i=1}^{n} \ln g_1(x_i) + \sum_{i=1}^{n} \ln g_2\left(x_y\right) + \sum_{i=1}^{n} \ln[1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x_i)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y_i)\}]. \tag{19}$$

The derivatives of log-likelihood function with respect to λ_1, λ_2 and λ_3 are

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2G_1(x_i)}{1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x_i)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y_i)\}}, \tag{20}$$

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{1 - 2G_2(y_i)}{1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x_i)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y_i)\}}, \tag{21}$$

and

$$\frac{\partial \ell}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{2\{1 - G_1(x_i) - G_2(y_i)\}}{1 + (\lambda_1 + \lambda_3)\{1 - 2G_1(x_i)\} + (\lambda_2 + \lambda_3)\{1 - 2G_2(y_i)\}}. \tag{22}$$

The maximum likelihood estimators of λ_1, λ_2 and λ_3 are obtained by equating (20), (21) and (22) to zero and numerically solving the resulting equations.

The bivariate transmuted family of distributions can be explored for any baseline distribution. In the following section we will study a member of the bivariate transmuted family of distributions for baseline Weibull distribution. The resulting distribtuion is named as *bivariate transmuted Weibull distribution*.

4 The Bivariate Transmuted Weibull Distribution

The Weibull distribution is a widely used distribution in statistical analysis, especially in the field of reliability. The Weibull distribution, introduced by Weibull [15], has been explored by several authors. Various extensions of the Weibull distribution have been proposed from time-to-time. The transmuted Weibull (*TW* for short) distribution has proposed by Aryal and Tsokos [16].

In the following we will propose the bivariate transmuted Weibull distribution by using *cdf* of Weibull distribution in the bivariate transmuted family of distributions. Distributional properties of the proposed distribution will be studied. Parameter estimation of the proposed distribution will also be discussed alongside real data application.

The bivariate transmuted Weibull distribution is obtained by using following *cdf* of Weibull distribution for X and Y

$$G_1(x) = 1 - e^{-(x/\theta_1)^{\alpha_1}}, \quad x, \theta_1, \alpha_1 \geq 0 \text{ and } G_2(y) = 1 - e^{-(y/\theta_2)^{\alpha_2}}, \quad y, \theta_2, \alpha_2 \geq 0 \tag{23}$$

in (3). The *cdf* of bivariate transmuted Weibull (*BTW* for short) distribution is

$$F_{X,Y}(x,y) = \left[1 - e^{-(x/\theta_1)^{\alpha_1}}\right]\left[1 - e^{-(y/\theta_2)^{\alpha_2}}\right]\left[1 + (\lambda_1 + \lambda_3)e^{-(x/\theta_1)^{\alpha_1}} + (\lambda_2 + \lambda_3)e^{-(y/\theta_2)^{\alpha_2}}\right]. \tag{24}$$

The density function of the distribution is

$$\begin{aligned} f_{X,Y}(x,y) = {} & \frac{\alpha_1 \alpha_2}{\theta_1^{\alpha_1}\theta_2^{\alpha_2}} x^{\alpha_1 - 1} y^{\alpha_2 - 1} e^{-(x/\theta_1)^{\alpha_1}} e^{-(y/\theta_2)^{\alpha_2}} \left[1 + (\lambda_1 + \lambda_3)\left\{2e^{-(x/\theta_1)^{\alpha_1}} - 1\right\}\right. \\ & \left. + (\lambda_2 + \lambda_3)\left\{2e^{-(y/\theta_2)^{\alpha_2}} - 1\right\}\right], \end{aligned} \tag{25}$$

where $(\theta_1, \theta_2) > 0$ are scale parameters, $(\alpha_1, \alpha_2) > 0$ are shape parameters and $(\lambda_1, \lambda_2, \lambda_3)$ are the transmutation parameters such that all λ's $\in [-1, 1]$.

5 Properties of Bivariate Transmuted Weibull Distribution

In this section some statistical properties of the *BTW* distribution are studied.

5.1 The Marginal and Conditional Distributions

The marginal *cdf*'s of X and Y are immediately written from (24) as

$$F_{TW}(x) = \left[1 - e^{-(x/\theta_1)^{\alpha_1}}\right]\left[1 + (\lambda_1 + \lambda_3)e^{-(x/\theta_1)^{\alpha_1}}\right], \tag{26}$$

and

$$F_{TW}(y) = \left[1 - e^{-(y/\theta_2)^{\alpha_2}}\right]\left[1 + (\lambda_2 + \lambda_3)e^{-(y/\theta_2)^{\alpha_2}}\right]. \tag{27}$$

It can be easily seen that both of the marginal *cdf*'s are transmuted Weibull distributions. The marginal *pdf*'s of X and Y are obtained from (25) as

$$f_X(x) = \frac{\alpha_1}{\theta_1^{\alpha_1}} x^{\alpha_1 - 1} e^{-(x/\theta_1)^{\alpha_1}} \left[1 + (\lambda_1 + \lambda_3) - 2(\lambda_1 + \lambda_3)\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\}\right] \tag{28}$$

and

$$f_Y(y) = \frac{\alpha_2}{\theta_2^{\alpha_2}} y^{\alpha_2 - 1} e^{-(y/\theta_2)^{\alpha_2}} \left[1 + (\lambda_2 + \lambda_3) - 2(\lambda_2 + \lambda_3)\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}\right]. \tag{29}$$

The conditional *pdf* of X given $Y = y$ for the *BTW* distribution is obtained by using the density and distribution function of Weibull random variable in (9) and is

$$f_{BTW}(x|y) = \frac{\alpha_1 x^{\alpha_1 - 1} e^{-(x/\theta_1)^{\alpha_1}} \left[1 + b\left\{2e^{-(x/\theta_1)^{\alpha_1}} - 1\right\} + c\left\{2e^{-(y/\theta_2)^{\alpha_2}} - 1\right\}\right]}{\theta_1^{\alpha_1}\left[1 + c - 2c\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}\right]}. \tag{30}$$

Similarly, the conditional *pdf* of Y given $X = x$ for the *BTW* distribution is obtained by using density and distribution function of Weibull distribution in (10) and is

$$f_{BTW}(y|x) = \frac{\alpha_2 y^{\alpha_2 - 1} e^{-(y/\theta_2)^{\alpha_2}} \left[1 + b\left\{2e^{-(x/\theta_1)^{\alpha_1}} - 1\right\} + c\left\{2e^{-(y/\theta_2)^{\alpha_2}} - 1\right\}\right]}{\theta_2^{\alpha_2}\left[1 + b - 2b\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\}\right]}, \tag{31}$$

where $b = \lambda_1 + \lambda_3$ and $c = \lambda_2 + \lambda_3$. The conditional distributions are useful in obtaining the conditional moments of the distribution which will be obtained in the following subsection.

5.2 The Conditional Moments

We have seen in (11) and (12) that the conditional moments for the bivariate transmuted family of distributions involve raw moments and moments of order statistics for the baseline distribution. We know that the rth raw moment and rth moment of larger observation in a sample of size 2 from Weibull distribution are, respectively, given as

$$E(W^r) = \theta^r \Gamma\left(1 + \frac{r}{\alpha}\right) \tag{32}$$

and

$$E\left(W_{2:2}^r\right) = 2\theta^r \Gamma\left(1 + \frac{r}{\alpha}\right)\left(1 - 2^{-(r/\alpha + 1)}\right). \tag{33}$$

Now using (32) and (33) in (11), the rth conditional moment of X given $Y = y$ for the *BTW* distribution is

$$E(X^r|y) = \frac{\theta_1^r}{\Delta_1(y)} \Gamma\left(\frac{r}{\alpha_1} + 1\right)\left[\beta - 2b\left\{1 - 2^{-(r/\alpha_1 + 1)}\right\} - 2c\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}\right]. \tag{34}$$

Similarly, the sth conditional moment of Y given $X = x$ is

$$E(Y^s|x) = \frac{\theta_2^s}{\Delta_2(x)}\Gamma\left(\frac{s}{\alpha_2}+1\right)\left[\beta - 2c\left\{1 - 2^{-(s/\alpha+1)}\right\} - 2b\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\}\right], \tag{35}$$

where $\Delta_1(y) = 1 + c - 2c\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}$ and $\Delta_2(x) = 1 + b - 2b\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\}$. The conditional moments are useful to obtain the conditional mean and conditional variance of the distribution.

5.3 The Product and Ratio Moments

The (r, s)th product moment for the *BTW* distribution are obtained by using simple moments and moments of maximum in a sample of size 2 from the Weibull distribution in (13). Now, using these moments, the expression for (r, s)th product moment for the *BTW* distribution is

$$\mu_{r,s} = \theta_1^r\theta_2^s\Gamma\left(1+\frac{r}{\alpha_1}\right)\Gamma\left(1+\frac{s}{\alpha_2}\right)\left[\beta - 2b\left\{1 - 2^{-(r/\alpha_1+1)}\right\} - 2c\left\{1 - 2^{-(s/\alpha_2+1)}\right\}\right]. \tag{36}$$

Again using the raw moments and moments of order statistics from Weibull distribution in (14), the (r, s)th ratio moment for the *BTW* distribution is obtained as

$$\mu_{r,-s} = \theta_1^r\theta_2^{-s}\Gamma\left(1+\frac{r}{\alpha_1}\right)\Gamma\left(1-\frac{s}{\alpha_2}\right)\left[\beta - 2b\left\{1 - 2^{-(r/\alpha_1+1)}\right\} - 2c\left\{1 - 2^{(s/\alpha_2-1)}\right\}\right], \tag{37}$$

where β, b and c are defined earlier.

5.4 The Bivariate Reliability and Hazard Rate Functions

The bivariate reliability function of X and Y can be obtained by using (15). The expression for bivariate reliability function for the *BTW* distribution is

$$R(x,y) = e^{-(x/\theta_1)^{\alpha_1}}e^{-(y/\theta_2)^{\alpha_2}}\left[1 - b\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\} - c\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}\right]. \tag{38}$$

Now using the bivariate density function (25) and bivariate reliability function (38) in (16), the bivariate hazard rate function for the *BTW* distribution is obtained as

$$h_{BTW}(x,y) = \frac{\alpha_1\alpha_2x^{\alpha_1-1}y^{\alpha_2-1}\left[1 + b\left\{2e^{-(x/\theta_1)^{\alpha_1}} - 1\right\} + c\left\{2e^{-(y/\theta_2)^{\alpha_2}} - 1\right\}\right]}{\theta_1^{\alpha_1}\theta_2^{\alpha_2}\left[1 - b\left\{1 - e^{-(x/\theta_1)^{\alpha_1}}\right\} - c\left\{1 - e^{-(y/\theta_2)^{\alpha_2}}\right\}\right]}, \tag{39}$$

where b and c are defined earlier. The plots of bivariate hazard rate function for $\theta_1 = 1, \theta_2 = 1, \lambda_1 = 0.25, \lambda_2 = -0.25$ and $\lambda_3 = 0.5$ at varios combinations of α_1 and α_2 are given in Fig. 1 below.

We can see that the distribution has decreasing hazard rate for $(\alpha_1, \alpha_2) < 1$ and has increasing hazard rate for $(\alpha_1, \alpha_2) > 1$. The bivariate reliability and bivariate hazard rate functions can be computed and plotted for different other combinations of the parameters.

6 Parameter Estimation for Bivariate Transmuted Weibull Distribution

In this section the maximum likelihood estimation of the parameters of *BTW* distribution is given. For this, let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the *BTW* distribution. The likelihood function is

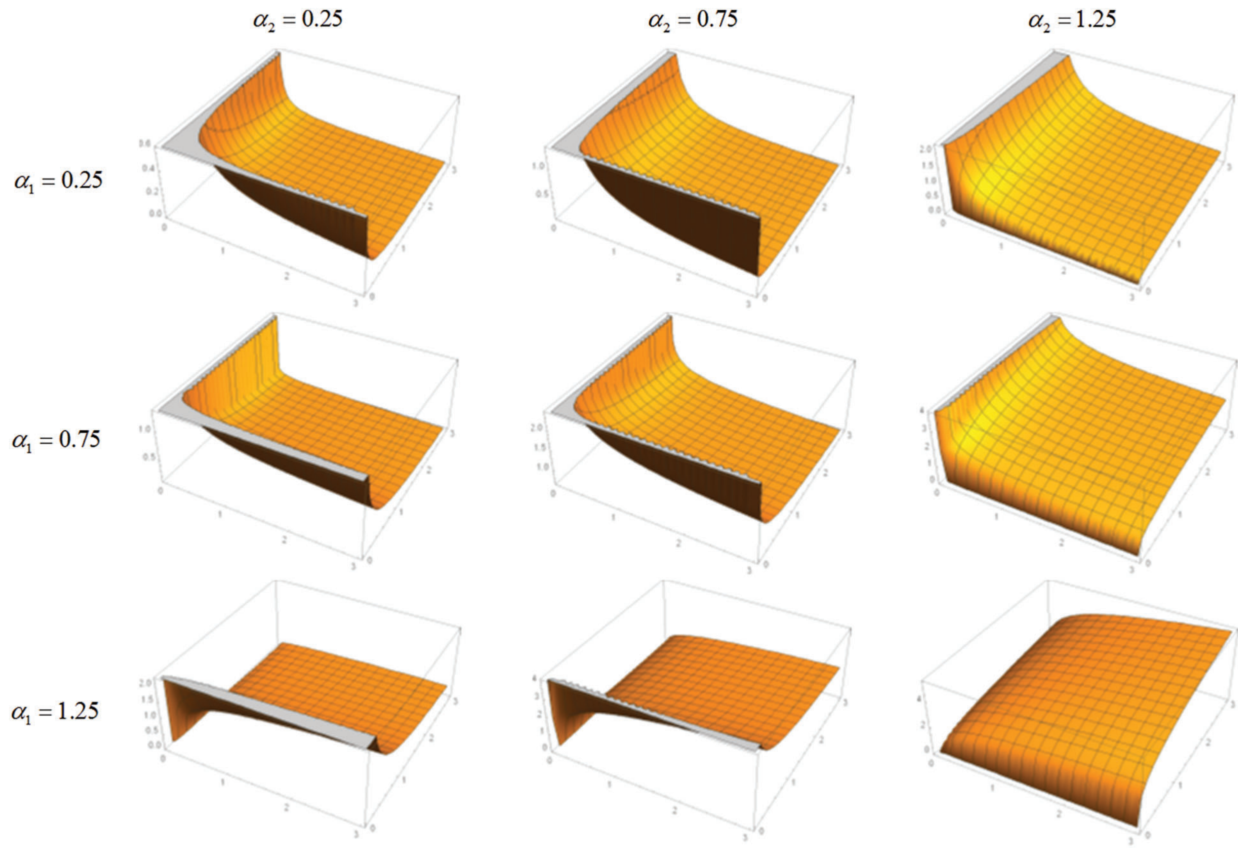

Figure 1: Hazard rate function of bivariate transmuted Weibull distribution

$$L = \frac{\alpha_1^n \alpha_2^n}{\theta_1^{n\alpha_1}\theta_2^{n\alpha_2}}\left(\prod_{i=1}^{n} x_i^{\alpha_1-1}\right)\mathrm{e}^{-\sum_{i=1}^{n}(x_i/\theta_1)^{\alpha_1}}\left(\prod_{i=1}^{n} y_i^{\alpha_2-1}\right)\mathrm{e}^{-\sum_{i=1}^{n}(y_i/\theta_2)^{\alpha_2}}$$
$$\times\prod_{i=1}^{n}\left[1+(\lambda_1+\lambda_3)\left\{2\mathrm{e}^{-(x/\theta_1)^{\alpha_1}}-1\right\}+(\lambda_2+\lambda_3)\left\{2\mathrm{e}^{-(y/\theta_2)^{\alpha_2}}-1\right\}\right].$$

The log-likelihood function is

$$\ell = n\ln\alpha_1 + n\ln\alpha_2 - n\alpha_1\ln\theta_1 - n\alpha_2\ln\theta_2 + (\alpha_1-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta_1}\right)^{\alpha_1} + (\alpha_2-1)\sum_{i=1}^{n}\ln y_i$$
$$-\sum_{i=1}^{n}\left(\frac{y_i}{\theta_2}\right)^{\alpha_2} + \sum_{i=1}^{n}\ln\left[1+(\lambda_1+\lambda_3)\left\{2\mathrm{e}^{-(x_i/\theta_1)^{\alpha_1}}-1\right\}+(\lambda_2+\lambda_3)\left\{2\mathrm{e}^{-(y_i/\theta_2)^{\alpha_2}}-1\right\}\right]. \quad (40)$$

The *MLEs* of the parameters in $\Theta = (\theta_1, \theta_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda_3)$ are obtained by maximizing the log-likelihood function (40). The derivatives with respect to unknown parameters are

$$\frac{\partial\ell}{\partial\theta_1} = -\frac{n\alpha_1}{\theta_1} + \frac{\alpha_1}{\theta_1^2}\sum_{i=1}^{n}\left\{x_i\left(\frac{x_i}{\theta_1}\right)^{\alpha_1-1}\right\} + \frac{2\alpha_1(\lambda_1+\lambda_3)}{\theta_1^2}\sum_{i=1}^{n}\frac{x_i(x_i/\theta_1)^{\alpha_1-1}\mathrm{e}^{-(x_i/\theta_1)^{\alpha_1}}}{\left[1+b\left\{2\mathrm{e}^{-(x_i/\theta_1)^{\alpha_1}}-1\right\}+c\left\{2\mathrm{e}^{-(y_i/\theta_2)^{\alpha_2}}-1\right\}\right]},$$

$$\frac{\partial \ell}{\partial \theta_2} = -\frac{n\alpha_2}{\theta_2} + \frac{\alpha_2}{\theta_2^2}\sum_{i=1}^{n}\left\{y_i\left(\frac{y_i}{\theta_2}\right)^{\alpha_2-1}\right\} + \frac{2\alpha_2(\lambda_2+\lambda_3)}{\theta_2^2}\sum_{i=1}^{n}\frac{y_i\left(y_i/\theta_2\right)^{\alpha_2-1}e^{-\left(y_i/\theta_2\right)^{\alpha_2}}}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]},$$

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n}{\alpha_1} - n\ln\theta_1 + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta_1}\right)^{\alpha_1}\ln\left(\frac{x_i}{\theta_1}\right) - \sum_{i=1}^{n}\frac{2(\lambda_1+\lambda_3)(x_i/\theta_1)^{\alpha_1}\ln(x_i/\theta_1)e^{-\left(x_i/\theta_1\right)^{\alpha_1}}}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]},$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{n}{\alpha_2} - n\ln\theta_2 + \sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{n}\left(\frac{y_i}{\theta_2}\right)^{\alpha_2}\ln\left(\frac{y_i}{\theta_2}\right) - \sum_{i=1}^{n}\frac{2(\lambda_2+\lambda_3)(y_i/\theta_2)^{\alpha_2}\ln(y_i/\theta_2)e^{-\left(y_i/\theta_2\right)^{\alpha_2}}}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]},$$

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n}\frac{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]},$$

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n}\frac{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]},$$

and

$$\frac{\partial \ell}{\partial \lambda_3} = \sum_{i=1}^{n}\frac{2\left[e^{-\left(x_i/\theta_1\right)^{\alpha_1}}+e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right]}{\left[1+b\left\{2e^{-\left(x_i/\theta_1\right)^{\alpha_1}}-1\right\}+c\left\{2e^{-\left(y_i/\theta_2\right)^{\alpha_2}}-1\right\}\right]}.$$

The maximum likelihood estimators of components of parameter vector $\Theta = (\theta_1, \theta_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda_3)$ are obtained by equating above derivatives to zero and numerically solving the resulting equations.

7 Real Data Application

In this section two real data applications of the bivariate transmuted Weibull distribution are given. We have modeled two data sets by using the *BTW* distribution alongside three other bivariate distributions. The distributions that we have used, for comparison, are bivariate Pseudo exponential propose by Mohsin et al. [17], bivariate Weibull propose by Hanif Shahbaz et al. [18] and Gumbel bivariate Weibull distribution. The distributions are fitted by obtaining the maximum likelihood estimates of the parameters. The maximum likelihood estimates of the model parameters are obtained by using the R-package “maxLik” [19]. In order to assess the performance of the *BTW* with the competing distributions the Akaike’s information criterion (*AIC*) and Bayesian information criterion (*BIC*) are computed.

7.1 Optical Data

The first data set is related to abortion index. The random variable *X* in the data set represents the *Time* and random variable *Y* represents the *Absorption Index*. Tab. 1 contains the summary statistics for the data set.

The results of maximum likelihood estimates are given in Tab. 2. The computed values of *AIC* and *BIC* are given in Tab. 3. From Tab. 3, we can see that the bivariate transmuted Weibull distribution has smallest values of *AIC* and *BIC* and hence is considered as the best fit for this data.

Table 1: Summary statistics for optical data

Min	Q_1	Median	Mean	Q_3	Max
X 1.4189	1.7275	2.2573	3.0286	3.4346	9.7145
Y 4.9000e-12	4.6357e-04	7.4168e-04	0.5202	1.6549e-03	1.2344e+01

Table 2: *MLEs* and *SEs* for the selected models

Distribution	Parameter	Estimate	SE
	θ_1	0.0032	0.0019
	θ_2	0.0042	0.0008
	α_1	0.4014	0.0309
	α_2	0.3766	0.0217
Bivariate Transmuted Weibull	λ_1	0.5107	0.0205
	λ_2	0.1387	0.0943
	λ_3	0.8438	0.0323
Bivariate Pseudo Exponential	α	0.3302	0.0660
Bivariate Weibull	β_1	0.7204	0.0776
	β_2	0.2237	0.0326
	α_1	1.7595	0.2631
	α_2	0.2617	0.0372
Gumbel Bivariate Weibull	β_1	9.5022	4.6777
	β_2	0.2855	0.0694
	b	0.9999	1.7169

Table 3: Selection criteria for the selected models

Distribution	LogLik	AIC	BIC
Bivariate Transmuted Weibull	647.0981	–1280.1962	–1271.6641
Bivariate Pseudo Exponential	–48.7663	99.5326	100.7515
Bivariate Weibull	32.56451	–61.1290	–58.6913
Gumbel Bivariate Weibull	61.2213	–112.4425	–106.3482

7.2 Atom Arrays Data

In the second data set X represents the *Light Emissions* and Y represents the *Dipole Blockade* of atom arrays. Tab. 4 gives some descriptive statistics of the data. The results of maximum likelihood estimates are given in Tab. 5 and computed values of *AIC* and *BIC* are listed in Tab. 6. According to Tab. 6, it is obvious that the bivariate transmuted Weibull distribution is the best fit for the second data as it has the smallest values of *AIC* and *BIC*.

Table 4: Summary statistics for atom array data

Min.	Q_1	Median	Mean	Q_3	Max.
X 0.0095	0.0132	0.0152	0.1296	0.0415	1.3413
Y 0.0094	0.0131	0.0153	0.1309	0.0454	1.3559

Table 5: *MLEs* and *SEs* for the selected models

Distribution	Parameter	Estimate	*SE*
	θ_1	1.2451	0.0158
	θ_2	3.1611	0.0635
	α_1	0.9730	0.0652
	α_2	2.5970	0.1162
Bivariate Transmuted Weibull	λ_1	0.7003	0.0887
	λ_2	–0.0709	0.0031
	λ_3	–0.7237	0.0687
Bivariate Pseudo Exponential	α	7.7150	1.6450
Bivariate Weibull	β_1	0.2004	0.0374
	β_2	0.3272	0.0645
	α_1	0.5996	0.0897
	α_2	0.5980	0.0895
Gumbel Bivariate Weibull	β_1	0.2144	0.0572
	β_2	0.2164	0.0576
	b	0.9999	1.4543

Table 6: Selection criteria for the selected models

Distribution	LogLik	AIC	BIC
Bivariate Transmuted Weibull	325.0031	–636.0062	–628.3689
Bivariate Pseudo Exponential	–56.5817	115.1634	116.2545
Bivariate Weibull	21.5436	–39.0873	–36.9052
Gumbel Bivariate Weibull	69.5337	–129.0675	–123.6122

8 Conclusions

In this paper we have introduced a new bivariate family of transmuted distributions. The proposed family has been studied for Weibull baseline distribution giving rise to bivariate transmuted Weibull distribution. Various properties of the proposed family and bivariate transmuted Weibull distribution have been studied. We have also applied the bivariate transmuted Weibull distribution on two real data sets. We have seen that the proposed bivariate Weibull distribution turned out to be the best fit for modeling of the data used. The proposed bivariate transmuted family of distributions can be further explored for different other baseline distributions which can be useful in modeling of complex bivariate data.

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