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Bivariate Beta–Inverse Weibull Distribution: Theory and Applications

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Abstract: Probability distributions have been in use for modeling of random phenomenon in various areas of life. Generalization of probability distributions has been the area of interest of several authors in the recent years. Several situations arise where joint modeling of two random phenomenon is required. In such cases the bivariate distributions are needed. Development of the bivariate distributions necessitates certain conditions, in a field where few work has been performed. This paper deals with a bivariate beta-inverse Weibull distribution. The marginal and conditional distributions from the proposed distribution have been obtained. Expansions for the joint and conditional density functions for the proposed distribution have been obtained. The properties, including product, marginal and conditional moments, joint moment generating function and joint hazard rate function of the proposed bivariate distribution have been studied. Numerical study for the dependence function has been implemented to see the effect of various parameters on the dependence of variables. Estimation of the parameters of the proposed bivariate distribution has been done by using the maximum likelihood method of estimation. Simulation and real data application of the distribution are presented.

Keywords: Bivariate beta distribution; inverse Weibull distribution; conditional moments; maximum likelihood estimation

1 Introduction

The probability distributions are widely used in many areas of life. Standard probability distributions have been extended by various authors to increase the applicability of a given baseline distribution. The exponentiated distributions, proposed by Gupta et al. [1], extend a baseline distribution by exponentiation. The beta generated distributions, proposed by Eugene et al. [2], generalize a baseline distribution by using logit of the beta distribution. The cumulative distribution function (*cdf*) of the *Beta-G* distributions is

$$F_{Beta-G}(x) = \frac{1}{B(a,b)} \int_0^{G(x)} u^{a-1}(1-u)^{b-1} du, \; 0 < G(x) < 1, \; (a,b) > 0, \tag{1}$$

where $G(x)$ is the *cdf* of any baseline distribution and $B(a,b)$ is the complete beta function defined as

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du.$$

The density function of any member of the Beta–G family of distributions is written as

$$f_{Beta-G}(x;a,b) = \frac{1}{B(a,b)} g(x)[G(x)]^{a-1}[1-G(x)]^{b-1}\ ,\ 0 < G(x) < 1. \tag{2}$$

Another method of generating new distributions has been proposed by Cordeiro et al. [3] by using the *cdf* of Kumaraswamy distribution, proposed by Kumaraswamy et al. [4]. The proposed family is referred to as the Kumaraswamy–G (*Kum–G*) family of distributions and its *cdf* is

$$F_{Kum-G}(x) = 1 - [1 - G^a(x)]^b,\ 0 < G(x) < 1,\ a,b > 0. \tag{3}$$

A general method of generating new distributions has been proposed by Alzaatreh et al. [5] by using the *cdf* of any distribution and this method is known as the *T-X* family of distributions. The *cdf* of new distribution by using the *T-X* family of distributions is

$$F_{T-X}(x) = \int_{d_1}^{W[G(x)]} r(t)dt, \tag{4}$$

where $r(t)$ is the density of any random variable defined on $[d_1, d_2]$, where d_1 can be $-\infty$ and d_2 can be $+\infty$, and $W[G(x)]$ is any function of $G(x)$ such that $W(0) = d_1$ and $W(1) = d_2$.

The *Beta–G*, the *Kum–G* and the *T-X* family of distributions provide basis to obtain a new univariate distribution by using the *cdf*, $G(x)$, of any baseline distribution. Different authors have proposed several new distributions by using these families, for example beta-normal by Eugene et al. [2], beta-Weibull by Famoye et al. [6], beta-inverse Weibull by Hanook et al. [7], Kum-inverse Weibull by Shahbaz et al. [8], gamma-normal by Alzaatreh et al. [9], exponentiated-gamma distribution by Nadarajah et al. [10], among others.

Several situations arise where we are interested in the simultaneous modeling of two random variables, for example we may be interested in the simultaneous study of arrival and departure time of the customers at a service station. In such situations some suitable bivariate distribution is required. The bivariate beta distribution, proposed by Olkin et al. [11], is a useful bivariate distribution to model random variables which represent proportion of some events of interest. The density function of the bivariate beta distribution is

$$f_{X,Y}(x,y) = \frac{x^{a-1}y^{b-1}(1-x)^{b+c-1}(1-y)^{a+c-1}}{B(a,b,c)(1-xy)^{a+b+c}}\ ,\ 0 < (x,y) < 1, \tag{5}$$

where $B(a,b,c)$ is extended beta function defined as

$$B(a,b,c) = \int_0^1 \int_0^1 \frac{u^{a-1}v^{b-1}(1-u)^{b+c-1}(1-v)^{a+c-1}}{(1-uv)^{a+b+c}} dudv = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)} \tag{6}$$

and $\Gamma(a)$ is the complete gamma function defined as

$$\Gamma(a) = \int_0^\infty u^{a-1}e^{-u}du.$$

The bivariate beta distribution has been used by Sarabia et al. [12] to propose the bivariate Beta-G_1G_2 (*BB-G_1G_2*) family of distributions. The joint *cdf* of the proposed family is

$$F_{Beta-G_1(x)G_2(y)}(x,y) = F_{Beta-G_1G_2}(x,y)$$
$$= \frac{1}{B(a_1,a_2,b)} \int_0^{G_2(y)} \int_0^{G_1(x)} \frac{u^{a_1-1}v^{a_2-1}(1-u)^{a_2+b-1}(1-v)^{a_1+b-1}}{(1-uv)^{a_1+a_2+b}} dudv, \tag{7}$$

for $(a_1, a_2, b) > 0$. The density function of the *BB-G₁G₂* family of distributions is

$$f_{Beta-G_1G_2}(x,y) = \frac{1}{B(a_1,a_2,b)} g_1(x)g_2(y)[G_1(x)]^{a_1-1}[G_2(y)]^{a_2-1}[1-G_1(x)]^{a_2+b-1}$$
$$\times [1-G_2(y)]^{a_1+b-1}[1-G_1(x)G_2(y)]^{-(a_1+a_2+b)} \ , \ 0 < G_1(x), G_2(y) < 1, \tag{8}$$

and $(a_1, a_2, b) > 0$. It is also shown by Sarabia et al. [12] that the density (8) can be written as

$$f_{Beta-G_1G_2}(x,y) = \frac{\Gamma(a_1+b)\Gamma(a_2+b)}{\Gamma(b)\Gamma(a_1+a_2+b)} \frac{f_{Beta-G_1}(x; a_1, a_2+b) f_{Beta-G_2}(y; a_2, a_1+b)}{[1-G_1(x)G_2(y)]^{(a_1+a_2+b)}}, \tag{9}$$

where $f_{Beta-G}(x; a, b)$ is the density function of the beta-G distribution, given in (2).

The $BB\text{-}G_1G_2$ family of distributions has not be explored and in this paper we will propose a new bivariate beta-inverse Weibull distribution. The structure of the paper is given below.

A new bivariate beta-inverse Weibull distribution is proposed in Section 2 alongside its marginal and conditional distributions. Some properties of the proposed distribution are obtained in Section 3. Estimation of the parameters of the proposed bivariate distribution is given in Section 4. Section 5 contains simulation and real data applications. Conclusions and recommendations are given in Section 6.

2 The Bivariate Beta-Inverse Weibull Distribution

The inverse Weibull distribution, also known as the Fréchet [13] distribution, is a useful lifetime distribution. The density and distribution functions of the inverse Weibull distribution are

$$g(x) = \frac{\beta\alpha}{x^{\alpha+1}} \exp(-\beta x^{-\alpha}) \ , \ x, \beta, \alpha > 0, \tag{10}$$

and

$$G(x) = \exp(-\beta x^{-\alpha}) \ , \ x, \beta, \alpha > 0, \tag{11}$$

where β is the rate and α is the shape parameter. A random variable X having the inverse Weibull distribution is written as $IW(\alpha, \beta)$. Now, suppose that the random variable X has $IW(\alpha_1, \beta_1)$ and the random variable Y has $IW(\alpha_2, \beta_2)$ distribution with *cdf*'s

$$G_1(x) = \exp(-\beta_1 x^{-\alpha_1}) \quad \text{and} \quad G_2(y) = \exp(-\beta_2 y^{-\alpha_2}).$$

Using the distribution functions of X and Y in (7), the distribution function of the bivariate beta-inverse Weibull (*BBIW* for short) distribution is

$$F_{X,Y}(x,y) = \frac{1}{B(a_1,a_2,b)} \int_0^{\exp(-\beta_2 y^{-\alpha_2})} \int_0^{\exp(-\beta_1 y^{-\alpha_1})} \frac{u^{a_1-1}v^{a_2-1}(1-u)^{a_2+b-1}(1-v)^{a_1+b-1}}{(1-uv)^{a_1+a_2+b}} \ dudv,$$

or

$$F_{X,Y}(x,y) = I_{\exp(-\beta_1 x^{\alpha_1}),\,\exp(-\beta_2 y^{\alpha_2})}(x,y) = \frac{B_{\exp(-\beta_1 x^{\alpha_1}),\exp(-\beta_2 y^{\alpha_2})}(a_1,a_2,b)}{B(a_1,a_2,b)}, \tag{12}$$

where

$$B_{x,y}(a_1,a_2,b) = \int_0^x \int_0^y \frac{u^{a_1-1}v^{a_2-1}(1-u)^{a_2+b-1}(1-v)^{a_1+b-1}}{(1-uv)^{a_1+a_2+b}}\, dudv \tag{13}$$

is the extended incomplete beta function and

$$I_{x,y}(a_1,a_2,b) = \frac{B_{x,y}(a_1,a_2,b)}{B(a_1,a_2,b)} \tag{14}$$

is the extended incomplete beta function ratio. The density function of the *BBIW* distribution is written from (8) as

$$\begin{aligned} f_{X,Y}(x,y) =& \frac{1}{B(a_1,a_2,b)} \frac{\alpha_1\alpha_2\beta_1\beta_2}{x^{\alpha_1+1}y^{\alpha_2+1}} \exp(-a_1\beta_1 x^{-\alpha_1}) \exp(-a_2\beta_2 y^{-\alpha_2})[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} \\ & \times [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1+b-1} \times [1-\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})]^{-(a_1+a_2+b)}, \ (x,y)>0, \end{aligned} \tag{15}$$

and $(\beta_1,\alpha_1,\beta_2,\alpha_2) > 0$. The random variables X and Y having joint density (15) is written as $BBIW(\beta_1,\beta_2,\alpha_1,\alpha_2,a_1,a_2,b)$. The plots of the density function for $\beta_1=\beta_2=1$ and for different choices of the other parameters are shown in Fig. 1 below.

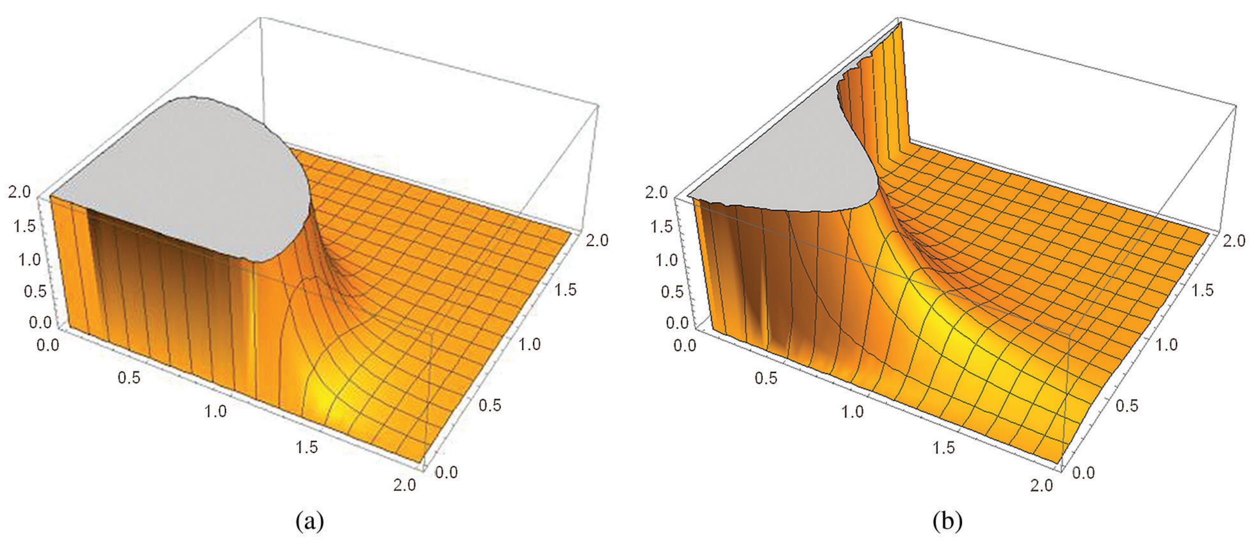

Figure 1: Plot of density function of bivariate beta inverse Weibull distributions for various choices of parameters. (a) $a_1=2.5, a_2=1.75, b=0.75, \alpha_1=1.5, \alpha_2=2.5$ (b) $a_1=2.5, a_2=1.75, b=0.75, \alpha_1=1.5, \alpha_2=2.5$

The plot of joint distribution function for $\beta_1=\beta_2=1$, $\alpha_1=2.5$, $\alpha_2=1.5$, $a_1=1.5$, $a_2=2.5$ and $b=2.0$ is given in Fig. 2.

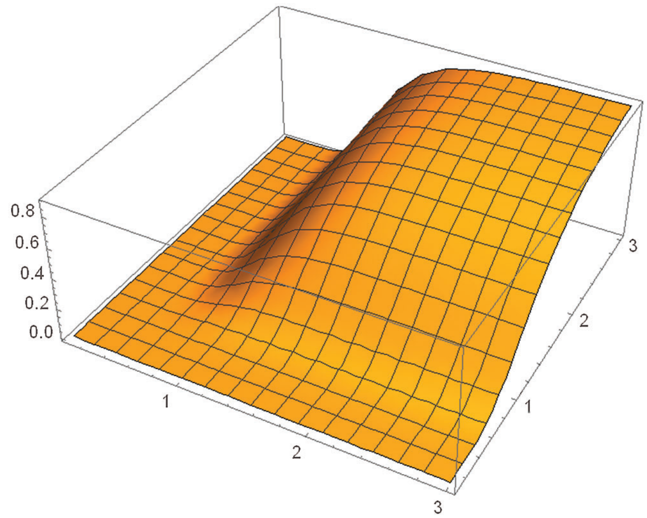

Figure 2: The distribution function of bivariate beta-inverse Weibull distribution

The *BBIW* distribution provides various other distributions as special case. For example, for $\alpha_1 = \alpha_2 = 1$ we have the bivariate beta inverse exponential (*BBIE*) distribution and for $\alpha_1 = \alpha_2 = 2$ the distribution reduces to the bivariate beta inverse Rayleigh (*BBIR*) distribution.

The joint hazard rate function of the distribution is obtained by using

$$\lambda(x,y) = \frac{f_{X,Y}(x,y)}{1 - F_{X,Y}(x,y)}$$

and is given as

$$\begin{aligned}\lambda(x,y) = \Bigg\{ & \frac{\alpha_1\alpha_2\beta_1\beta_2}{x^{\alpha_1+1}y^{\alpha_2+1}} \exp(-a_1\beta_1 x^{-\alpha_1})\exp(-a_2\beta_2 y^{-\alpha_2})[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} \\ & \times[1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1+b-1}[1-\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})]^{-(a_1+a_2+b)}\Bigg\} \\ & \div \left[B(a_1,a_2,b) - B_{\exp(-\beta_1 x^{\alpha_1}),\exp(-\beta_2 y^{\alpha_2})}(a_1,a_2,b)\right].\end{aligned} \tag{16}$$

The hazard rate function can be computed for different values of the parameters.

The density function of the *BBIW* distribution can also be written in the form of beta-inverse Weibull density functions as

$$f_{X,Y}(x,y) = \frac{\Gamma(a_1+b)\Gamma(a_2+b)}{\Gamma(b)\Gamma(a_1+a_2+b)} \frac{f_{Beta-IW}(x;a_1,a_2+b,\beta_1,\alpha_1) f_{Beta-IW}(y;a_2,a_1+b,\beta_2,\alpha_1)}{[1-\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})]^{(a_1+a_2+b)}}, \ (x,y) > 0,$$

where $f_{Beta-IW}(x;\boldsymbol{\theta})$ is the density function of the beta-inverse Weibull random variable with parameter vector $\boldsymbol{\theta}$.

The marginal distributions of X and Y are obtained, from (15), below.

Now

$$f_X(x) = \frac{1}{B(a_1,a_2,b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp(-a_1\beta_1x^{-\alpha_1})[1-\exp(-\beta_1x^{-\alpha_1})]^{a_2+b-1}\int_0^\infty \frac{\alpha_2\beta_2}{y^{\alpha_2+1}}\exp(-a_2\beta_2y^{-\alpha_2})$$
$$\times[1-\exp(-\beta_2y^{-\alpha_2})]^{a_1+b-1}[1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})]^{-(a_1+a_2+b)}dy$$

Making the transformation $\exp(-\beta_2y^{-\alpha_2}) = v$, we have

$$f_X(x) = \frac{1}{B(a_1,a_2,b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp(-a_1\beta_1x^{-\alpha_1})[1-\exp(-\beta_1x^{-\alpha_1})]^{a_2+b-1}\int_0^1 \frac{v^{a_2}(1-v)^{a_1+b-1}}{[1-\exp(-\beta_1x^{-\alpha_1})v]^{a_1+a_2+b}}dv$$

or

$$f_X(x) = \frac{1}{B(a_1,b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp(-a_1\beta_1x^{-\alpha_1})[1-\exp(-\beta_1x^{-\alpha_1})]^{b-1}, \; x>0, \tag{17}$$

which is density function of $BIW(a_1, b, \beta_1, \alpha_1)$. Similarly, it can be shown that the marginal distribution of Y is $BIW(a_2, b, \beta_2, \alpha_2)$.

The conditional distribution of Y given X is readily obtained from (15) and (17) and is

$$\begin{aligned} f_{Y|X}(y|x) = {} & \frac{1}{B(a_1,a_2,b)}\frac{\alpha_2\beta_2}{y^{\alpha_2+1}}\exp(-a_2\beta_2y^{-\alpha_2})[1-\exp(-\beta_2y^{-\alpha_2})]^{a_1+b-1}[1-\exp(-\beta_1x^{-\alpha_1})]^{a_2} \\ & \times[1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})]^{-(a_1+a_2-b)}, \; (x,y)>0. \end{aligned} \tag{18}$$

Similarly, the conditional distribution of X given Y is

$$\begin{aligned} f_{X|Y}(x|y) = {} & \frac{1}{B(a_1,a_2,b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp(-a_1\beta_1x^{-\alpha_1})[1-\exp(-\beta_1x^{-\alpha_1})]^{a_2+b-1}[1-\exp(-\beta_2y^{-\alpha_2})]^{a_1} \\ & \times[1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})]^{-(a_1+a_2-b)}, \; (x,y)>0. \end{aligned} \tag{19}$$

The conditional distributions are useful in computing conditional moments.

In the following we will obtain some useful expansions for the joint density function of *BBIW* distribution. For this, we will use following expansions, for any real a,

$$(1-z)^{-a} = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{j!\Gamma(a)}z^j \quad \text{and} \quad (1-z)^{a} = \sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(a+1)}{j!\Gamma(a-j+1)}z^j.$$

Now, the density function of the *BBIW* distribution is

$$f_{X,Y}(x,y) = \frac{1}{B(a_1,a_2,b)}\frac{\alpha_1\alpha_2\beta_1\beta_2}{x^{\alpha_1+1}y^{\alpha_2+1}}\exp(-a_1\beta_1x^{-\alpha_1})\exp(-a_2\beta_2y^{-\alpha_2})[1-\exp(-\beta_1x^{-\alpha_1})]^{a_2+b-1}$$
$$\times[1-\exp(-\beta_2y^{-\alpha_2})]^{a_1+b-1}[1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})]^{-(a_1+a_2+b)}.$$

Using following series expansion

$$[1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})]^{-(a_1+a_2+b)} = \sum_{j=0}^{\infty}\frac{\Gamma(a_1+a_2+b+j)}{j!\Gamma(a_1+a_2+b)}\exp(-j\beta_1x^{-\alpha_1})\exp(-j\beta_2y^{-\alpha_2})$$

we have

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \frac{1}{B(a_1,a_2,b)} \frac{\alpha_1\alpha_2\beta_1\beta_2}{x^{\alpha_1+1}y^{\alpha_2+1}} \exp(-a_1\beta_1 x^{-\alpha_1}) \exp(-a_2\beta_2 y^{-\alpha_2})[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} \\
& \times [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1+b-1} \sum_{j=0}^{\infty} \frac{\Gamma(a_1+a_2+b+j)}{j!\Gamma(a_1+a_2+b)} \exp(-j\beta_1 x^{-\alpha_1}) \exp(-j\beta_2 y^{-\alpha_2}).
\end{aligned}
$$

Re-arranging, the above density can be written as

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \sum_{j=0}^{\infty} dA(j) \frac{1}{B(a_1+j,a_2+b)} \frac{\alpha_1\beta_1}{x^{\alpha_1+1}} \exp[-(a_1+j)\beta_1 x^{-\alpha_1}][1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} \\
& \times \frac{1}{B(a_2+j,a_1+b)} \frac{\alpha_2\beta_2}{y^{\alpha_2+1}} \exp[-(a_2+j)\beta_2 y^{-\alpha_2}][1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1+b-1}
\end{aligned}
$$

or

$$
f_{X,Y}(x,y) = \sum_{j=0}^{\infty} dA(j)\, f_{Beta-IW}(x;a_1+j,a_2+b)\, f_{Beta-IW}(y;a_2+j,a_1+b), \tag{20}
$$

where $f_{Beta-IW}(x;a,b)$ is the density function of beta-inverse Weibull distribution. Also

$$
d = \frac{\Gamma(a_1+b)\Gamma(a_2+b)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b)}
$$

and

$$
A(j) = \frac{\Gamma(a_1+j)\Gamma(a_2+j)}{\Gamma(a_1+a_2+b+j)} \cdot \frac{1}{j!}
$$

From (20), it is easy to see that the joint density function of the *BBIW* distribution is the weighted sum of product of marginal density functions of the beta-inverse Weibull distributions.

The expansion of the density function, given in (20), can also be written as the weighted sum of density functions of the inverse Weibull distributions. For this, we use following expansions

$$
[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} = \sum_{h=0}^{\infty} (-1)^h \frac{\Gamma(a_2+b)}{h!\Gamma(a_2+b-h)} \exp(-h\beta_1 x^{-\alpha_1})
$$

and

$$
[1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1+b-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(a_1+b)}{k!\Gamma(a_1+b-j)} \exp(-k\beta_2 y^{-\alpha_2}).
$$

Using these expansions in (20), the density function of the *BBIW* distribution is written as

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k) \frac{\alpha_1(a_1+j+h)\beta_1}{x^{\alpha_1+1}} \exp[-(a_1+j+h)\beta_1 x^{-\alpha_1}] \\
& \times \frac{\alpha_2(a_2+j+k)\beta_2}{y^{\alpha_2+1}} \exp[-(a_2+j+k)\beta_2 y^{-\alpha_2}],
\end{aligned} \tag{21}
$$

or

$$f_{X,Y}(x,y) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)\, f_{IW}(x;(a_1+j+h)\beta_1,\alpha_1) f_{IW}(y;(a_2+j+k)\beta_2,\alpha_2),$$

where $f_{IW}(x;\beta,\alpha)$ is the density function of the inverse Weibull random variable with parameters (β,α) and

$$A(j,h,k) = A(j)\cdot\frac{(-1)^{h+k}}{h!k!}\cdot\frac{\Gamma(a_1+b)}{\Gamma(a_1+b-h)}\cdot\frac{\Gamma(a_2+b)}{\Gamma(a_2+b-k)}\cdot\frac{1}{(a_1+j+h)(a_2+j+k)}.$$

From (21), we can see that the density function of the *BBIW* distribution is the weighted sum of product of the inverse Weibull density functions. The expression (21) is useful in computing moments of the distribution.

The expansion for the conditional density function of X given Y is readily obtained by using

$$[1-\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})]^{-(a_1+a_2+b)} = \sum_{j=0}^{\infty}\frac{\Gamma(a_1+a_2+b+j)}{j!\Gamma(a_1+a_2+b)}\exp(-j\beta_1 x^{-\alpha_1})\exp(-j\beta_2 y^{-\alpha_2})$$

in (19) and is

$$f_{X|Y}(x|y) = \frac{1}{B(a_1,a_2,b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp(-a_1\beta_1 x^{-\alpha_1})[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1}[1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(a_1+a_2+b+j)}{j!\Gamma(a_1+a_2+b)}\exp(-j\beta_1 x^{-\alpha_1})\exp(-j\beta_2 y^{-\alpha_2}),$$

or

$$f_{X|Y}(x|y) = \frac{\Gamma(a_2+b)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a_1+j)}{j!}\frac{1}{B(a_1+j,a_2+b)}\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp[-(a_1+j)\beta_1 x^{-\alpha_1}]$$
$$\times[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1}\exp(-j\beta_2 y^{-\alpha_2})[1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1},$$

or

$$f_{X|Y}(x|y) = [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}\sum_{j=0}^{\infty} d_1 A_1(j)\, f_{Beta-IW}(x;a_1+j,a_2+b)\exp(-j\beta_2 y^{-\alpha_2}), \tag{22}$$

where $d_1 = \dfrac{\Gamma(a_2+b)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b)}$ and $A_1(j) = \dfrac{\Gamma(a_1+j)}{j!}$.

The expansion of the conditional density function of Y given X is similarly written as

$$f_{Y|X}(y|x) = [1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2}\sum_{j=0}^{\infty} d_2 A_2(j) f_{Beta-IW}(y;a_2+j,a_1+b)\exp(-j\beta_1 x^{-\alpha_1}), \tag{23}$$

where $d_2 = \dfrac{\Gamma(a_1+b)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b)}$ and $A_2(j) = \dfrac{\Gamma(a_2+j)}{j!}$.

The expansions (22) and (23) are helpful in computing the conditional moments of the distribution.

3 Properties of the Distribution

In this section some properties of the *BBIW* distribution are discussed. These properties include joint, marginal and conditional moments and are discussed in the following sub-sections.

3.1 Product and Ratio Moments of the Distribution

The product and ratio moments of the *BBIW* distribution are obtained by using the joint density function, given in (15), or the expansion of the density function, given in (21). It is easier to obtain the product and ratio moments from (21) and are obtained below.

The (r,s)th product moment for two random variables is defined as

$$\mu'_{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f_{X,Y}(x,y)dxdy.$$

Using the joint density function of the *BBIW* distribution, the expression for the product moment is

$$\mu'_{r,s} = E(X^r Y^s) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)\int_0^{\infty}\int_0^{\infty} x^r y^s \frac{\alpha_1(a_1+j+h)\beta_1}{x^{\alpha_1+1}}\exp[-(a_1+j+h)\beta_1 x^{-\alpha_1}]$$
$$\times \frac{\alpha_2(a_2+j+k)\beta_2}{y^{\alpha_2+1}}\exp[-(a_2+j+k)\beta_2 y^{-\alpha_2}]dxdy.$$

Making the transformations $u = (a_1+j+h)\beta_1 x^{-\alpha_1}$ and $v = (a_2+j+k)\beta_2 y^{-\alpha_2}$, the expression for the product moment for the *BBIW* distribution is

$$\mu'_{r,s} = E(X^r Y^s) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)[(a_1+j+h)\beta_1]^{r/\alpha_1}[(a_2+j+k)\beta_2]^{s/\alpha_2}\Gamma\left(1-\frac{r}{\alpha_1}\right)\Gamma\left(1-\frac{s}{\alpha_2}\right), \quad (24)$$

which exists for $r < \alpha_1$ and $s < \alpha_2$.

The (r,s)th ratio moment for two random variables is defined as

$$\mu'_{r,-s} = E(X^r Y^{-s}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^{-s} f_{X,Y}(x,y)dxdy.$$

Using the joint density function of the *BBIW* distribution, the ratio moment is

$$\mu'_{r,-s} = E(X^r Y^{-s}) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)\int_0^{\infty}\int_0^{\infty} x^r y^{-s} \frac{\alpha_1(a_1+j+h)\beta_1}{x^{\alpha_1+1}}\exp[-(a_1+j+h)\beta_1 x^{-\alpha_1}]$$
$$\times \frac{\alpha_2(a_2+j+k)\beta_2}{y^{\alpha_2+1}}\exp[-(a_2+j+k)\beta_2 y^{-\alpha_2}]dxdy$$

Again making the transformations $u = (a_1+j+h)\beta_1 x^{-\alpha_1}$ and $v = (a_2+j+k)\beta_2 y^{-\alpha_2}$, the expression of the ratio moment for the *BBIW* distribution is

$$\mu'_{r,-s} = E\left(\frac{X^r}{Y^s}\right) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)\frac{[(a_1+j+h)\beta_1]^{r/\alpha_1}}{[(a_2+j+k)\beta_2]^{s/\alpha_2}}\Gamma\left(1-\frac{r}{\alpha_1}\right)\Gamma\left(1+\frac{s}{\alpha_2}\right). \quad (25)$$

Similarly, the expression for the ratio moment, $\mu'_{-r,s}$, for the *BBIW* distribution is

$$\mu'_{-r,s} = E\left(\frac{Y^s}{X^r}\right) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} dA(j,h,k)\frac{[(a_2+j+k)\beta_2]^{s/\alpha_2}}{[(a_1+j+h)\beta_1]^{r/\alpha_1}}\Gamma\left(1+\frac{r}{\alpha_1}\right)\Gamma\left(1-\frac{s}{\alpha_2}\right). \quad (26)$$

It is to be noted that the parameters a_1, a_2 and b has no effect on the means and variances of the random variables X and Y as these are controlled by parameters $\alpha_1, \alpha_2, \beta_1$ and β_2.

3.2 Conditional Moments of the Distribution

The conditional moment of a distribution is useful in studying the behavior of one variable under the conditions on the other variable. In case of a bivariate distribution we can compute conditional moment of X given $Y = y$ and of Y given $X = x$. In the following, the conditional moments for the *BBIW* distribution are obtained.

The conditional moment of X given $Y = y$ is computed as

$$E(X^r|y) = \int_0^{\infty} x^r f_{X|Y}(x|y)dx.$$

Now, using the conditional distribution of X given $Y = y$, from (22), we have

$$E(X^r|y) = [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}\sum_{j=0}^{\infty} d_1 A_1(j)\exp(-j\beta_2 y^{-\alpha_2})\int_0^{\infty} x^r f_{Beta-IW}(x; a_1+j, a_2+b)dx.$$

Using the density function of the beta-inverse Weibull distribution, we have

$$E(X^r|y) = [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}\sum_{j=0}^{\infty} d_1 A_1(j)\exp(-j\beta_2 y^{-\alpha_2})\int_0^{\infty} x^r \frac{1}{B(a_1+j, a_2+b)}$$
$$\times\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp[-(a_1+j)\beta_1 x^{-\alpha_1}][1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1}dx.$$

Now, using the expansion

$$[1-\exp(-\beta_1 x^{-\alpha_1})]^{a_2+b-1} = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(a_2+b)}{k!\Gamma(a_2+b-k)}\exp(-k\beta_1 x^{-\alpha_1})$$

in above equation we have

$$E(X^r|y) = [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_1 A_1(j)\exp(-j\beta_2 y^{-\alpha_2})\frac{(-1)^k\Gamma(a_1+a_2+j+b)}{j!\Gamma(a_2+b-k)\Gamma(a_1+j)}$$
$$\times\int_0^{\infty} x^r\frac{\alpha_1\beta_1}{x^{\alpha_1+1}}\exp[-(a_1+j+k)\beta_1 x^{-\alpha_1}]dx,$$

or

$$E(X^r|y) = [1-\exp(-\beta_2 y^{-\alpha_2})]^{a_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_1 A_{11}(j,k)\exp(-j\beta_2 y^{-\alpha_2})[(a_1+j+k)\beta_1]^{r/\alpha_1}\Gamma\left(1-\frac{r}{\alpha_1}\right), \quad (27)$$

where

$$A_{11}(j,k) = \frac{(-1)^k A_1(j)\Gamma(a_1 + a_2 + j + b)}{(a_1 + j + k)k!\Gamma(a_2 + b - k)\Gamma(a_1 + j)}.$$

Similarly, the expression for the conditional moment of Y given $X = x$ is

$$E(Y^s|x) = [1 - \exp(-\beta_1 x^{-\alpha_1})]^{a_1} \sum_{j=0}^{\infty}\sum_{h=0}^{\infty} d_2 A_{21}(j,h)\exp(-j\beta_1 x^{-\alpha_1})[(a_2 + j + h)\beta_2]^{s/\alpha_2}\Gamma\left(1 - \frac{s}{\alpha_2}\right), \quad (28)$$

where

$$A_{21}(j,h) = \frac{(-1)^h A_2(j)\Gamma(a_1 + a_2 + j + b)}{(a_2 + j + h)h!\Gamma(a_1 + b - h)\Gamma(a_2 + j)}.$$

The conditional means and variances can be obtained from (27) and (28).

3.3 The Dependence Function and Correlation Coefficient

The dependence function for a bivariate distribution is, given by Holland et al. [14],

$$\gamma(x,y) = \frac{\partial^2}{\partial x \partial y}\ln f_{X,Y}(x,y)$$

and Sarabia et al. [12] has shown that the dependence function for the *BB–G_1G_2* family is

$$\gamma(x,y) = \frac{(a_1 + a_2 + b)g_1(x)g_2(y)}{[1 - G_1(x)G_2(y)]^2}. \quad (29)$$

Using the density and distribution functions of the inverse Weibull distribution in (29), the dependence function for the *BBIW* distribution is

$$\gamma(x,y) = \frac{(a_1 + a_2 + b)\alpha_1\beta_1\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})}{x^{\alpha_1+1}y^{\alpha_2+1}[1 - \exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})]^2}, \quad (30)$$

which is always positive. It is to be noted that the dependence function is different from the linear correlation coefficient which is to be computed from the product moment. The values of correlation coefficient for different

Table 1: Correlation coefficient between X and Y for the *BBIW* distribution

a_1	a_2	b	$\alpha_1 = 3$			$\alpha_1 = 4$			$\alpha_1 = 5$		
			$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$	$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$	$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$
2	2	2	0.603	0.596	0.590	0.461	0.457	0.454	0.372	0.369	0.367
		3	0.596	0.590	0.586	0.457	0.453	0.451	0.369	0.366	0.364
		4	0.590	0.586	0.583	0.454	0.451	0.448	0.367	0.364	0.362
	3	2	0.649	0.642	0.637	0.511	0.507	0.504	0.420	0.417	0.416
		3	0.641	0.637	0.633	0.506	0.503	0.499	0.417	0.414	0.413
		4	0.635	0.633	0.630	0.503	0.500	0.498	0.414	0.412	0.411
	4	2	0.675	0.669	0.664	0.542	0.538	0.535	0.452	0.449	0.447
		3	0.667	0.664	0.660	0.537	0.534	0.533	0.448	0.446	0.444
		4	0.662	0.660	0.657	0.534	0.531	0.529	0.446	0.444	0.442

(Continued)

Table 1 (continued).

a_1	a_2	b	$\alpha_1 = 3$			$\alpha_1 = 4$			$\alpha_1 = 5$		
			$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$	$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$	$\alpha_2 = 3$	$\alpha_2 = 4$	$\alpha_2 = 5$
3	2	2	0.649	0.641	0.635	0.511	0.506	0.503	0.420	0.417	0.414
		3	0.642	0.637	0.633	0.507	0.503	0.500	0.417	0.414	0.412
		4	0.637	0.633	0.630	0.504	0.499	0.498	0.416	0.413	0.411
	3	2	0.698	0.691	0.686	0.566	0.562	0.559	0.475	0.472	0.470
		3	0.691	0.687	0.683	0.562	0.558	0.556	0.472	0.469	0.467
		4	0.686	0.683	0.680	0.559	0.556	0.554	0.470	0.467	0.466
	4	2	0.726	0.720	0.715	0.600	0.596	0.593	0.510	0.507	0.505
		3	0.719	0.716	0.713	0.596	0.593	0.592	0.507	0.505	0.503
		4	0.714	0.713	0.710	0.593	0.590	0.589	0.505	0.503	0.501
4	2	2	0.675	0.667	0.662	0.542	0.537	0.534	0.452	0.448	0.446
		3	0.669	0.664	0.660	0.538	0.534	0.531	0.449	0.446	0.444
		4	0.664	0.660	0.657	0.535	0.533	0.529	0.447	0.444	0.442
	3	2	0.726	0.719	0.714	0.600	0.596	0.593	0.510	0.507	0.505
		3	0.720	0.716	0.713	0.596	0.593	0.590	0.507	0.505	0.503
		4	0.715	0.713	0.710	0.593	0.592	0.589	0.505	0.503	0.501
	4	2	0.756	0.750	0.745	0.636	0.632	0.630	0.548	0.545	0.543
		3	0.750	0.747	0.743	0.632	0.630	0.627	0.545	0.543	0.541
		4	0.745	0.743	0.741	0.630	0.627	0.625	0.543	0.541	0.540

combinations of the parameters are given in Tab. 1 below. The values of correlation coefficient indicate that the effect of a_1 and a_2 on the correlation coefficient is positive, that is increase in the values of a_1 and a_2 will increase the linear correlation between X and Y for the *BBIW* distribution. The effect of other parameters on the correlation coefficient is negative, that is increase in b, α_1 and α_2 will decrease the correlation coefficient.

3.4 The Shannon Entropy

The Shannon entropy, Shannon [15], in a bivariate distribution is defined as

$$I_{Sh}\left[f_{X,Y}(x,y)\right] = E\left[-\ln\left\{f_{X,Y}(x,y)\right\}\right] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\ln f_{X,Y}(x,y)\right]f_{X,Y}(x,y)dxdy.$$

Now, for the *BBIW* distribution we have

$$\begin{aligned}\ln f_{X,Y}(x,y) = & -\ln B(a_1,a_2,b) + \ln(\alpha_1\alpha_2\beta_1\beta_2) - (\alpha_1+1)\ln x - (\alpha_2+1)\ln y - a_1\beta_1 x^{-\alpha_1} - a_2\beta_2 y^{-\alpha_2} \\ & + (a_2+b-1)\ln[1-\exp(-\beta_1 x^{-\alpha_1})] + (a_1+b-1)\ln[1-\exp(-\beta_2 y^{-\alpha_2})] \\ & - (a_1+a_2+b)\ln[1-\exp(-\beta_1 x^{-\alpha_1})\exp(-\beta_2 y^{-\alpha_2})].\end{aligned}$$

Using this in above equation, the Shannon entropy for the *BBIW* distribution is

$$I_{Sh}\left[f_{X,Y}(x,y)\right]=\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}dA(j,h,k)\int_0^{\infty}\int_0^{\infty}[-\ln B(a_1,a_2,b)+\ln(\alpha_1\alpha_2\beta_1\beta_2)-(\alpha_1+1)\ln x-(\alpha_2+1)\ln y$$
$$-a_1\beta_1x^{-\alpha_1}-a_2\beta_2y^{-\alpha_2}+(a_2+b-1)\ln\{1-\exp(-\beta_1x^{-\alpha_1})\}+(a_1+b-1)\ln\{1-\exp(-\beta_2y^{-\alpha_2})\}$$
$$-(a_1+a_2+b)\ln\{1-\exp(-\beta_1x^{-\alpha_1})\exp(-\beta_2y^{-\alpha_2})\}]\frac{\alpha_1(a_1+j+h)\beta_1}{x^{\alpha_1+1}}$$
$$\times\exp[-(a_1+j+h)\beta_1x^{-\alpha_1}]\frac{\alpha_2(a_2+j+k)\beta_2}{y^{\alpha_2+1}}\exp[-(a_2+j+k)\beta_2y^{-\alpha_2}]dxdy,$$

or

$$\begin{aligned}I_{Sh}\left[f_{X,Y}(x,y)\right] = {} & \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}dA(j,h,k)\Bigg[-\ln B(a_1,a_2,b)+\ln(\alpha_1\alpha_2\beta_1\beta_2)-\frac{(\alpha_1+1)}{\alpha_1}\\ & \times\{\gamma+\ln\beta_1+\ln(a_1+j+h)\}-\frac{(\alpha_2+1)}{\alpha_2}\{\gamma+\ln\beta_2+\ln(a_2+j+k)\}-\frac{a_1\beta_1}{\beta_1(a_1+j+h)}\\ & -\frac{a_2\beta_2}{\beta_2(a_2+j+k)}-\frac{(a_2+b-1)}{(a_1+j+h+1)}H_{(a_1+j+h+1)}-\frac{(a_1+b-1)}{(a_2+j+k+1)}H_{(a_2+j+k+1)}\\ & +\frac{(a_1+a_2+b)}{(a_1+j+h+1)(a_2+j+k+1)^2}\left\{(a_2+j+k+1)H_{(a_1+j+h+1)}+1\right\}\Bigg],\end{aligned} \tag{31}$$

where γ is the Euler's gamma and H_k is the harmonic number of order k.

3.5 Random Number Generation

The random sample from the *BBIW* distribution can be generated by using the conditional distribution method. In this method the random sample from a bivariate distribution is generated using following two steps

- Generate a random variate from the marginal distribution of X.
- Generate a random variate from the conditional distribution of Y given $X = x$.

Using this method, the random sample from the *BBIW* distribution can be generated by using following algorithm

- Generate a random variate X from the beta-inverse Weibull distribution with given parameters $(a_1, b, \beta_1, \alpha_1)$ and denote it by x.
- Generate a random variate Y from the conditional distribution of Y given $X = x$, given in (18).
- The pair (x,y) is a single random observation from the *BBIW* distribution.
- Repeat the process until sample of size n is obtained.

The random variates from the beta-inverse Weibull distribution can be easily obtained by using any of the *R* package; for example Newdistns, by Nadarajah et al. [16] or MPS by Teimouri [17].

4 Statistical Inference

In this section, parameter estimation of the *BBIW* distribution is presented. For this, we first see that the likelihood function for a sample of size n from the distribution is

$$L(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=\frac{1}{[B(a_1,a_2,b)]^n}\frac{\alpha_1^n\alpha_2^n\beta_1^n\beta_2^n}{x^{\alpha_1+1}y^{\alpha_2+1}}\exp\left(-a_1\beta_1\sum\nolimits_{i=1}^{n}x_i^{-\alpha_1}-a_2\beta_2\sum\nolimits_{i=1}^{n}y_i^{-\alpha_2}\right)\prod_{i=1}^{n}[1-\exp(-\beta_1x_i^{-\alpha_1})]^{a_2+b-1}$$
$$\times\prod_{i=1}^{n}[1-\exp(-\beta_2y_i^{-\alpha_2})]^{a_1+b-1}\prod_{i=1}^{n}[1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})]^{-(a_1+a_2+b)},$$

where $\boldsymbol{\theta}=(a_1,a_2,b,\alpha_1,\alpha_2,\beta_1,\beta_2)$. The log-likelihood function is

$$\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=-n\ln B(a_1,a_2,b)+n\ln\alpha_1+n\ln\alpha_2+n\ln\beta_1+n\ln\beta_2-(\alpha_1+1)\sum_{i=1}^{n}x_i-(\alpha_2+1)\sum_{i=1}^{n}y_i$$
$$-a_1\beta_1\sum_{i=1}^{n}x_i^{-\alpha_1}-a_2\beta_2\sum_{i=1}^{n}y_i^{-\alpha_2}+(a_2+b-1)\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})]+(a_1+b-1)\tag{32}$$
$$\times\sum_{i=1}^{n}\ln[1-\exp(-\beta_2y_i^{-\alpha_2})]-(a_1+a_2+b)\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})].$$

The derivatives of the log-likelihood function with respect to the unknown parameters are

$$\frac{\partial}{\partial a_1}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=n[\psi(a_1+a_2+b)-\psi(a_1)]-\beta_1\sum_{i=1}^{n}x_i^{-\alpha_1}+\sum_{i=1}^{n}\ln[1-\exp(-\beta_2y_i^{-\alpha_2})]$$
$$-\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})],\tag{33}$$

$$\frac{\partial}{\partial a_2}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=n[\psi(a_1+a_2+b)-\psi(a_2)]-\beta_2\sum_{i=1}^{n}y_i^{-\alpha_2}+\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})]$$
$$-\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})],\tag{34}$$

$$\frac{\partial}{\partial b}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=n[\psi(a_1+a_2+b)-\psi(b)]+\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})]$$
$$+\sum_{i=1}^{n}\ln[1-\exp(-\beta_2y_i^{-\alpha_2})]-\sum_{i=1}^{n}\ln[1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})],\tag{35}$$

$$\frac{\partial}{\partial\alpha_1}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=\frac{n}{\alpha_1}-\sum_{i=1}^{n}\ln x_i+a_1\beta_1\sum_{i=1}^{n}x_i^{-\alpha_1}\ln x_i-(a_2+b-1)\sum_{i=1}^{n}\frac{\beta_1x_i^{-\alpha_1}\ln x_i\exp(-\beta_1x_i^{-\alpha_1})}{1-\exp(-\beta_1x_i^{-\alpha_1})}$$
$$+(a_1+a_2+b)\sum_{i=1}^{n}\frac{\beta_1x_i^{-\alpha_1}\ln x_i\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})}{1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})},\tag{36}$$

$$\frac{\partial}{\partial\alpha_2}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta})=\frac{n}{\alpha_2}-\sum_{i=1}^{n}\ln y_i+a_2\beta_2\sum_{i=1}^{n}y_i^{-\alpha_2}\ln y_i-(a_1+b-1)\sum_{i=1}^{n}\frac{\beta_2y_i^{-\alpha_2}\ln y_i\exp(-\beta_2y_i^{-\alpha_2})}{1-\exp(-\beta_2y_i^{-\alpha_2})}$$
$$+(a_1+a_2+b)\sum_{i=1}^{n}\frac{\beta_2y_i^{-\alpha_2}\ln y_i\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})}{1-\exp(-\beta_1x_i^{-\alpha_1})\exp(-\beta_2y_i^{-\alpha_2})},\tag{37}$$

$$\frac{\partial}{\partial\beta_1}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta}) = \frac{n}{\beta_1} - a_1\sum_{i=1}^{n}x_i^{-\alpha_1} + (a_2+b-1)\sum_{i=1}^{n}\frac{x_i^{-\alpha_1}\exp(-\beta_1 x_i^{-\alpha_1})}{1-\exp(-\beta_1 x_i^{-\alpha_1})}$$
$$-(a_1+a_2+b)\sum_{i=1}^{n}\frac{x_i^{-\alpha_1}\exp(-\beta_1 x_i^{-\alpha_1})\exp(-\beta_2 y_i^{-\alpha_2})}{1-\exp(-\beta_1 x_i^{-\alpha_1})\exp(-\beta_2 y_i^{-\alpha_2})} \quad (38)$$

and

$$\frac{\partial}{\partial\beta_2}\ell(\mathbf{x},\mathbf{y};\boldsymbol{\theta}) = \frac{n}{\beta_2} - a_2\sum_{i=1}^{n}y_i^{-\alpha_2} + (a_1+b-1)\sum_{i=1}^{n}\frac{y_i^{-\alpha_2}\exp(-\beta_2 y_i^{-\alpha_2})}{1-\exp(-\beta_2 y_i^{-\alpha_2})}$$
$$-(a_1+a_2+b)\sum_{i=1}^{n}\frac{y_i^{-\alpha_2}\exp(-\beta_1 x_i^{-\alpha_1})\exp(-\beta_2 y_i^{-\alpha_2})}{1-\exp(-\beta_1 x_i^{-\alpha_1})\exp(-\beta_2 y_i^{-\alpha_2})}. \quad (39)$$

The maximum likelihood estimates are obtained by equating the derivatives, given in Eqs. (33) and (39), to zero and simultaneously solving the resulting equations.

5 Applications

In this section some numerical applications of the proposed *BBIW* distribution are given. We will first give a simulation study to see the performance of the maximum likelihood estimates and then we will apply the proposed *BBIW* distribution on a real data set. These applications are discussed in the following subsections.

5.1 Simulation Study

In the following, simulation results for the proposed *BBIW* distribution are given. The algorithm for the simulation study is given below:

1. Draw random sample of a specific size from the *BBIW* distribution, for specific values of the parameters, using the procedure given in Section 3.4.
2. Obtain maximum likelihood estimates of the parameters for the generated sample.
3. Repeat Steps 1 and 2 for the specific number of simulations.
4. Obtain average value of the estimates and the standard errors.

In our simulation study, random samples of sizes 50, 100, 200 and 500 are drawn and the results are simulated for 50000 times. The results of simulation study are given in Tab. 2 below (standard error of the estimate is in the parenthesis).

The simulation results indicate that the estimate converges to the true parameter value with increase in the sample size. We can also see that the standard error of the estimate decreases with increase in the sample size.

5.2 Real Data Application

In this section, a real data application of the proposed *BBIW* distribution is given. We have used data on GNI *per capita* of all the countries of the world for 2016 (as *X*) and for 2017 (as *Y*). The data is obtained from *UNDP* site http://hdr.undp.org/en/data [18]. For the analysis, the data is transformed by dividing actual values with 10000. The bivariate histogram of the data is shown in Fig. 3 below.

We have fitted three bivariate distributions to the data including the bivariate beta inverse Weibull (*BBIW*) distribution, the bivariate Weibull (*BW*) distribution, proposed by Shahbaz et al. [19], and the bivariate inverse Weibull (BIW) distribution obtained by using F-G-M family with density.

Table 2: Simulation results for the *BBIW* distribution

n	$a_1 = 2.0$	$a_2 = 3.0$	$b = 1.5$	$\alpha_1 = 2.5$	$\alpha_2 = 2.0$	$\beta_1 = 1.5$	$\beta_2 = 0.5$
50	2.014	3.111	1.541	2.363	2.082	1.488	0.500
	(0.010)	(0.048)	(0.113)	(0.214)	(0.175)	(0.142)	(0.076)
100	1.927	3.029	1.487	2.501	2.094	1.368	0.361
	(0.012)	(0.019)	(0.108)	(0.185)	(0.181)	(0.144)	(0.081)
200	2.089	3.107	1.602	2.365	1.887	1.398	0.399
	(0.012)	(0.019)	(0.085)	(0.174)	(0.152)	(0.127)	(0.074)
500	1.852	3.080	1.524	2.630	1.991	1.485	0.418
	(0.009)	(0.018)	(0.074)	(0.142)	(0.137)	(0.129)	(0.068)
$\boldsymbol{n}$	$a_1 = 3.0$	$a_2 = 2.0$	$b = 2.0$	$\alpha_1 = 1.5$	$\alpha_2 = 2.5$	$\beta_1 = 2.0$	$\beta_2 = 2.5$
50	3.089	2.148	2.084	1.350	2.413	2.142	2.502
	(0.465)	(0.421)	(0.406)	(0.461)	(0.421)	(0.463)	(0.462)
100	3.087	1.863	2.133	1.453	2.463	2.116	2.475
	(0.421)	(0.392)	(0.406)	(0.353)	(0.444)	(0.421)	(0.373)
200	2.884	2.036	1.995	1.420	2.574	1.934	2.446
	(0.318)	(0.328)	(0.325)	(0.272)	(0.291)	(0.289)	(0.306)
500	2.850	2.058	2.078	1.511	2.416	2.053	2.470
	(0.261)	(0.293)	(0.255)	(0.267)	(0.257)	(0.269)	(0.293)

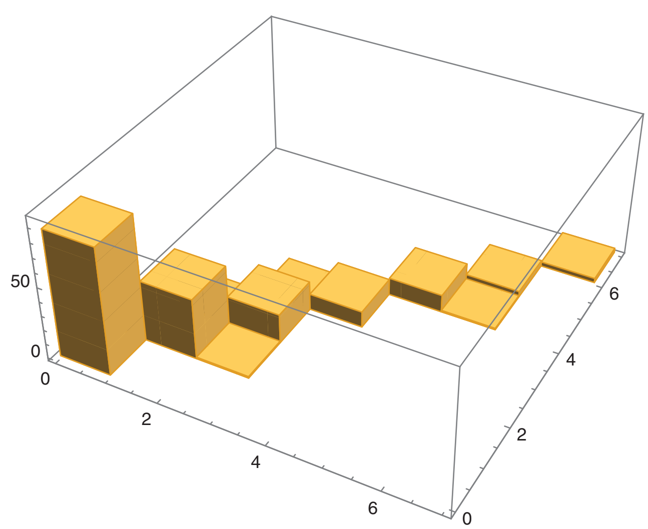

Figure 3: Bivariate histogram of the GNI data

$$f_{X,Y}(x,y) = \frac{\beta_1\beta_2\alpha_1\alpha_2}{x^{\alpha_1+1}y^{\alpha_2+1}}\exp\left[-\left(\frac{\beta_1}{x^{\alpha_1}}+\frac{\beta_2}{y^{\alpha_2}}\right)\right]\left[1+b\left\{2\exp\left(-\frac{\beta_1}{x^{\alpha_1}}\right)-1\right\}\left\{2\exp\left(-\frac{\beta_1}{x^{\alpha_1}}\right)-1\right\}\right]. \quad (40)$$

Results of the fitted models are given in Tab. 3 below.

Table 3: Results of fitted distributions to GNI data

	BW	BIW	BBIW
a_1		–	0.975
a_2		–	1.172
b	0.449	−0.494	0.00017
α_1	0.423	1.028	0.971
α_2	0.543	1.025	1.088
β_1		0.835	0.373
β_2		0.177	0.013
AIC	879.288	378.64	177.87

From the table, we can see that the *BBIW* distribution is the best fit to the data as it has smallest *AIC*. The fitted *BBIW* distribution is shown in Fig. 4 below.

The graph of the fitted *BBIW* distribution is reasonably close to the bivariate histogram of the data.

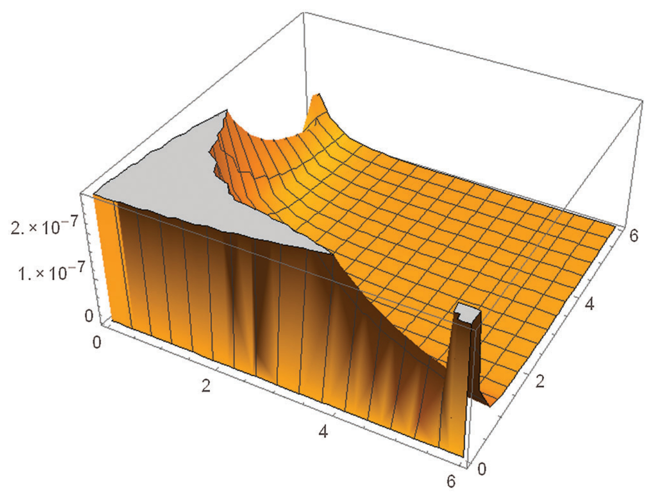

Figure 4: Fitted *BBIW* distribution for GNI data

6 Conclusions

In this paper a new bivariate beta inverse Weibull distribution is proposed by using the logit of bivariate beta distribution. The distribution is very flexible and is useful in modeling of the complex data. The properties of the distribution have been studied and it is found that the correlation coefficient is controlled by combination of three parameters α_1, α_2 and b. The proposed bivariate beta inverse Weibull distribution has been fitted to the world *GNI* data of two years and it is found that the proposed *BBIW* is better fit as compared with the other models involved in the study

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