Computers, Materials & Continua
DOI:10.32604/cmc.2021.013489
Article

Statistical Inference of Chen Distribution Based on Two Progressive Type-II Censoring Schemes

Hassan M. Aljohani*

Department of Mathematics & Statistics, Faculty of Science, Taif University, Taif, 21944, Saudi Arabia
*Corresponding Author: Hassan M. Aljohani. Email: hmjohani@tu.edu.sa
Received: 08 August 2020; Accepted: 29 October 2020

Abstract: An inverse problem in practical scientific investigations is the process of computing unknown parameters from a set of observations where the observations are only recorded indirectly, such as monitoring and controlling quality in industrial process control. Linear regression can be thought of as linear inverse problems. In other words, the procedure of unknown estimation parameters can be expressed as an inverse problem. However, maximum likelihood provides an unstable solution, and the problem becomes more complicated if unknown parameters are estimated from different samples. Hence, researchers search for better estimates. We study two joint censoring schemes for lifetime products in industrial process monitoring. In practice, this type of data can be collected in fields such as the medical industry and industrial engineering. In this study, statistical inference for the Chen lifetime products is considered and analyzed to estimate underlying parameters. Maximum likelihood and Bayes' rule are both studied for model parameters. The asymptotic distribution of maximum likelihood estimators and the empirical distributions obtained with Markov chain Monte Carlo algorithms are utilized to build the interval estimators. Theoretical results using tables and figures are adopted through simulation studies and verified in an analysis of the lifetime data. We briefly describe the performance of developed methods.

Keywords: Chen distributions; progressive type-II censoring; maximum likelihood; mean posterior; Bayesian estimation; MCMC

1 Introduction

Several types of monitoring data are available. One is the censoring scheme, which is a popular problem in life testing experiments. The oldest censoring projects are the so-called "type-I", and the other is "type-II". In practice, there are usually two random variables, i.e., time and the number of failures of items. This strategy of censoring projects shows how the examiner imagines the experiment based on a predetermined time. A random number of units is accounted for the first type-I of a censoring scheme, which means it may be assumed the exact time of stopping experiment. While the predetermined number of failure units and a random time in the type-II censoring scheme. In these two types of censoring schemes, companies cannot be removed from an experiment until the final stage or the number of units fail. This process allows the detection of some units that are defective after running the experiment. The mixture of these types of censoring schemes is the so-called hybrid censoring system [1]. To remove elements from the test at any stage of the trial is known as a progressive censoring scheme [2]. The topic of progressive censoring has developed in different scientific fields, and has attracted much attention in recent years. Several authors have studied this type of data [3,4]. There are different types of progressive censoring schemes. The idea of the progressive type-I censoring scheme is to test time τ and determine the number m of failure units, and suppose n independent elements are tested under the censoring scheme $r = \{r_1, r_2, \ldots, r_m\}$. The failure unit is removed at $\min(\tau, T_m)$, where T_m is the stopping time of the number of failure units m. After each failure time (T_i, r_i), survival units are removed from the trial, where $i = 1, 2, \ldots, J$ and $J \le m$. In a progressive type-II censoring project, the number m of failure units and $r = \{r_1, r_2, \ldots, r_m\}$ are determined, and we suppose n independent units are examined and the experiment is stopped at T_m. After each failure time (T_i, r_i), survival units are removed from the test, where $i = 1, 2, \ldots, m$. The lifetime products come from different production lines [5,6]. The exact likelihood inference using bootstrap algorithms was studied [7], as was the type-II progressive censoring scheme [8,9] and two censoring schemes [10]. Consider manufactured products that come from two production lines η_1 and η_2 under the same conditions. Assume two independent samples S_1 and S_2 are chosen from these lines for experimental testing. The experiment runs under some consideration of time and cost, and the experimenter reports that it terminates after a predetermined time or number of failures. This is called a joint censoring scheme [11]. The procedure of joint progressive type-II censoring was described previously, where the sample size $S_1 + S_2$ is taken as S_1 from line η_1 and S_2 from line η_2. The integers m and $r = \{r_1, r_2, \ldots, r_m\}$ are determined to satisfy the form $S_1 + S_2 + \sum_{i=1}^{m} r_i$. The element r_1 is removed immediately from the experiment. We observe the first failure unit, say T_1 and has line W_1 from line η_1 or η_2, say (t_1, ω_1). Also, the number r_2 is removed from the test after we examine the second failure unit, say T_2 and has line W_2, say (t_2, ω_2). The experiment continues until (t_m, ω_m) is observed, where w_i takes the value 1 or 0, depending on lines η_1 or η_2. The result of the previous examination $t = \{(t_1, \omega_1), (t_2, \omega_2), \ldots, (t_m, \omega_m)\}$ is called the joint progressive type-II censoring procedure. The concept of a balanced joint progressive type-II censoring scheme was considered by [12] for analytically more straightforward estimators than the other type of progressive censoring procedure. Several authors have discussed statistical inference using different distributions, such as two exponential distributions [12]. The procedure of lifetime using Weibull distributions was investigated [13]. The interpretation of the balanced joint progressive type-II censoring procedure starts with samples of size $S_1 + S_2$, taken from production lines η_1 and η_2, respectively. Integers m and the integers $r = \{r_1, r_2, \ldots, r_m\}$ are determined to satisfy $m + \sum_{i=1}^{m-1} r_i < \min(S_1, S_2)$. The failure times and types are observed, say (t_i, ω_i), $i = 1, 2, \ldots, m$. Fig. 1 shows the main idea of a joint progressive type-II censoring scheme. This study discusses the properties of Chen lifetime estimation procedures under a joint progressive type-II censoring scheme. The Chen lifetime distribution with two parameters was introduced by [14]. This study's objective is to build a balanced joint progressive type-II censoring procedure for the Chen lifetime distribution and parameter estimation with the maximum likelihood estimator (MLE) and Bayes methods. The developed methods are also used to measure the same Chen lifetime products' relative merits under the same conditions. Estimators are evaluated through numerical data analysis and assessed through a simulation study. The remainder of this article is organized as follows. The main principle and model formulation are given in Section 2. Point MLE and interval estimators are introduced in Section 3. Section 4 discusses Bayes point and credible intervals. Estimators under numerical examples and simulation studies are discussed in Section 5. We summarize some comments which are extracted from numerical methods in Section 6.

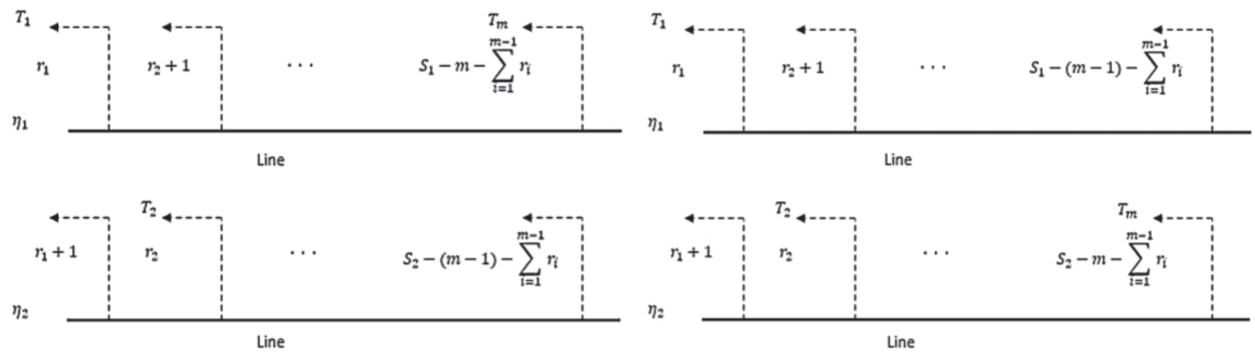

Figure 1: Example of the structure of joint progressive type-II censoring procedures

2 Model Formulation

Assume two production lines, and a random sample of size $S_1 + S_2$, where S_1 comes from line η_1 and S_2 from line η_2. The integers m and $r = \{r_1, r_2, \ldots, r_m\}$ are determined to satisfy $m + \sum_{i=1}^{m-1} r_i < \min(S_1, S_2)$. Suppose t_1 is observed from some units that are taken from line η_1, then, r_1 survival component is removed from S_1 and $r_1 + 1$ survival component is removed from S_2 when the second failure t_2 is observed if t_2 is chosen from the line η_2. In that case, $r_2 + 1$ survival component is removed from $S_1 - r_1 - 1$, and r_2 survival component is removed from the sample $S_2 - r_2 - 1$. The test continues in this manner until the mth failure t_m is observed. If the final failure is from line η_1, then the survival components $S_1 - m - \sum_{i=1}^{m-1} r_i$ are removed from η_1, and $S_2 - (m-1) - \sum_{i=1}^{m-1} r_i$ are removed from η_2. If the final failure belongs to line η_2, then the survival units $S_1 - (m-1) - \sum_{i=1}^{m-1} r_i$ are removed from η_1, and $S_2 - m - \sum_{i=1}^{m-1} r_i$ are removed from η_2. Fig. 1 shows the scheme of joint balanced progressive type-II censoring. The observed data $t = \{(t_1, \omega_1), (t_2, \omega_2), \ldots, (t_m, \omega_m)\}$ are called balanced joint progressive type-II censoring samples. Under consideration that S_1 comes from the line η_1, and it has independent and identically distribution of lifetimes $\{X_1, X_2, \ldots, X_{s_1}\}$ and S_2 comes from the line η_2, and ithas independent and identically distribution of lifetimes $\{X_1^*, X_2^*, \ldots, X_{s_2}^*\}$. These samples distributed with populations have probability density (PDFs) and cumulative distribution (CDFs) functions are given, respectively, by the functions $f_j(.)$ and $F_j(.)$, $j = 1, 2$. Then the balanced joint progressive type-II sample $t = \{t_1, t_2, \ldots, t_m\}$ is taken from $\{X_1, X_2, \ldots, X_{m_1}, X_1^*, X_2^*, \ldots, X_{m_2}^*\}$, where $m = m_1 + m_2$, m_1 is the number of failed units from line η_1, and m_2 is the number of failed units from line η_2. The observed balanced joint progressive type-II censoring sample is $t = \{(t_1, \omega_1), (t_2, \omega_2), \ldots, (t_m, \omega_m)\}$, where ω_i takes the value 1 or 0, depends on line η_1 or η_2, $m_1 = \sum_{i=1}^{m} \omega_i$ and $m_2 = \sum_{i=1}^{m} (1 - \omega_i)$.

The joint likelihood rule under two progressive type-II censoring samples $t = \{(t_1, \omega_1), (t_2, \omega_2), \ldots, (t_m, \omega_m)\}$ is

$$L(\boldsymbol{t}) \propto (R_1(t_m))^{I_1} (R_2(t_m))^{I_2} (h_1(t_m))^{m_1} (h_2(t_m))^{1-m_1} \times \prod_{i=1}^{m-1} (h_1(t_i))^{m_i} (h_2(t_i))^{1-m_i} (R_1(t_i) R_2(t_i))^{r_i+1}, \tag{1}$$

where

$$I_j = S_j - (m-1) - \sum_{i=1}^{m-1} r_i, j = 1, 2, \tag{2}$$

and $R_j(.)$ and $h_j(.)$ are reliability and hazard rate functions, respectively. Under the described model, the probability density functions (PDFs) and cumulative distribution functions (CDFs) of the tested unit and chosen from two lines η_1 and η_2 have Chen lifetime distributions with PDFs given by

$$f_j(t) = \alpha_j \lambda_j t^{\alpha_j - 1} \exp\left\{t^{\alpha_j}\right\} \exp\left\{\lambda_j \left(1 - \exp\left\{t^{\alpha_j}\right\}\right)\right\}, \quad t > 0, \ \left(\alpha_j \lambda_j\right) > 0. \tag{3}$$

Reliability and hazard rate functions, respectively, are given by

$$F_j(t) = 1 - \exp\left\{\lambda_j \left(1 - \exp\left\{t^{\alpha_j}\right\}\right)\right\}, \tag{4}$$

$$S_j(t) = \exp\left\{\lambda_j \left(1 - \exp\left\{t^{\alpha_j}\right\}\right)\right\}, \tag{5}$$

and

$$h_j(t) = \alpha_j \lambda_j t^{\alpha_j - 1} \exp\left\{t^{\alpha_j}\right\}, \tag{6}$$

where α_j and λ_j are the respective shape and scale parameters of the Chen distribution. Hence, a bathtub-shaped failure rate is noticed when $\alpha_j \geq 1$, and an exponential form can be obtained when $\alpha_j = 1$ [15]. Fig. 2d plots the properties of the Chen distribution. It is clearly seen that $h(t)$ provides a bathtub-shaped curve when $\alpha = 1$.

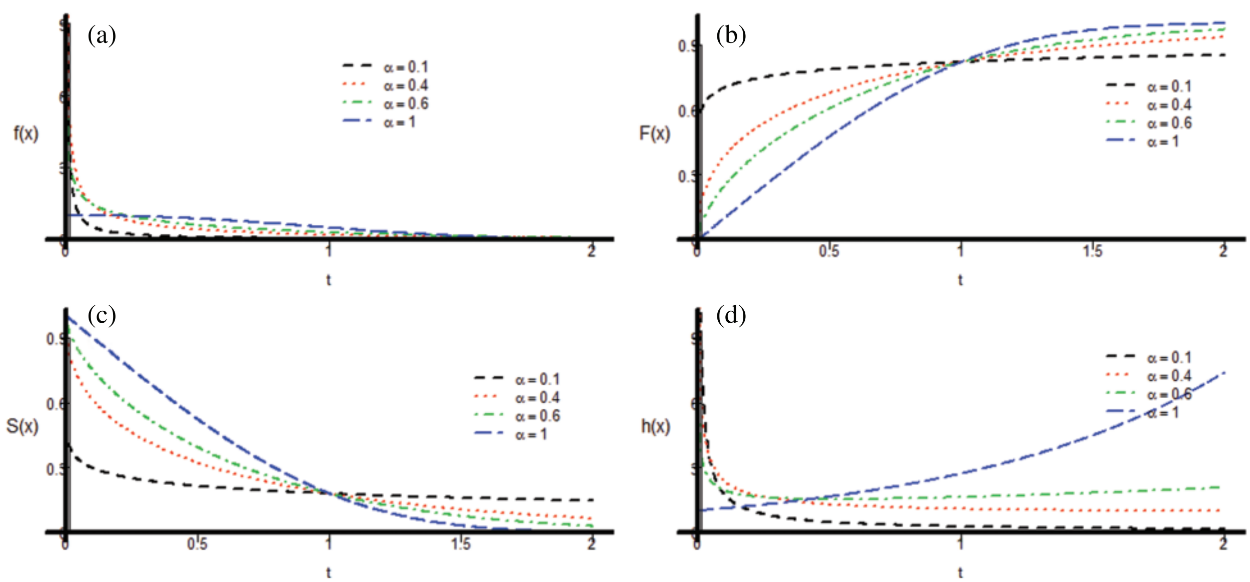

Figure 2: Examples of the scaled Chen distribution for different values of α with $\lambda = 1$: (a) Chen distribution; (b) Cumulative distribution; (c) Reliability function; and (d) Hazard rate function

3 Maximum Likelihood Estimation

The joint likelihood function in Eq. (1) without a normalized constant under a Chen lifetime distribution is defined as

$$L(\alpha_1,\alpha_2,\lambda_1,\lambda_2|\boldsymbol{t})$$
$$\propto (\alpha_1,\lambda_1)^{m_1}(\alpha_2,\lambda_2)^{m_2}\exp\Bigg\{(\alpha_1-1)\sum_{i=1}^{m}\omega_i\log t_i+\sum_{i=1}^{m}\omega_i t_i^{\alpha_1}+\lambda_1\sum_{i=1}^{m-1}(r_i+1)\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)$$
$$+\lambda_1 I_1\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)+(\alpha_2-1)\sum_{i=1}^{m-1}(1-\omega_i)\log t_i+\sum_{i=1}^{m}(1-\omega_i)t_i^{\alpha_2}+\lambda_2\sum_{i=1}^{m-1}(r_i+1)$$
$$\times\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right)+\lambda_2 I_2\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right)\Bigg\}. \tag{7}$$

After taking the logarithms of both sides, the joint likelihood function in Eq. (7) becomes

$$\ell(\alpha_1,\alpha_2,\lambda_1,\lambda_2|\boldsymbol{t})$$
$$\propto m_1(\alpha_1,\lambda_1)+m_2(\alpha_2,\lambda_2)+(\alpha_1-1)\sum_{i=1}^{m}\omega_i\log t_i+\sum_{i=1}^{m}\omega_i t_i^{\alpha_1}+\lambda_1\sum_{i=1}^{m-1}(r_i+1)\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)$$
$$+\lambda_1 I_1\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)+(\alpha_2-1)\sum_{i=1}^{m-1}(1-\omega_i)\log t_i+\sum_{i=1}^{m}(1-\omega_i)t_i^{\alpha_2}$$
$$+\lambda_2\sum_{i=1}^{m-1}(r_i+1)\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right)+\lambda_2 I_2\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right), \tag{8}$$

which is used to represent the point and interval estimators of underlying parameters.

3.1 MLEs

The likelihood rule is obtained from Eq. (8) by taking partial derivatives with respect to the parameter vectors $(\alpha_1,\alpha_2,\lambda_1,\lambda_2)$ and equating to zero.

The equation $\dfrac{\partial\ell(\alpha_1,\alpha_2,\lambda_1,\lambda_2|\boldsymbol{t})}{\partial\lambda_1}=0$ is reduced to

$$\lambda_1=\frac{m_1}{\sum_{i=1}^{m-1}(r_i+1)\left(\exp\left\{t_i^{\alpha_1}\right\}-1\right)+I_1\left(\exp\left\{t_m^{\alpha_1}\right\}-1\right)}. \tag{9}$$

The equation $\dfrac{\partial\ell(\alpha_1,\alpha_2,\lambda_1,\lambda_2|\boldsymbol{t})}{\partial\lambda_2}=0$ is reduced to

$$\lambda_2=\frac{m_2}{\sum_{i=1}^{m-1}(r_i+1)\left(\exp\left\{t_i^{\alpha_2}\right\}-1\right)+I_2\left(\exp\left\{t_m^{\alpha_2}\right\}-1\right)}. \tag{10}$$

The equation $\dfrac{\partial \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_1} = 0$ is reduced to

$$\frac{m_1}{\alpha_1} + \sum_{i=1}^{m} \omega_i (1 + t_i^{\alpha_1}) \log t_i - \lambda_1 \sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_1} \log t_i \exp\left\{t_i^{\alpha_1}\right\} - \lambda_1 I_1 t_m^{\alpha_1} \log t_m \exp\left\{t_m^{\alpha_1}\right\} = 0. \tag{11}$$

The equation $\dfrac{\partial \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_2} = 0$ is reduced to

$$\frac{m_2}{\alpha_2} + \sum_{i=1}^{m} (1 - \omega_i)(1 + t_i^{\alpha_2}) \log t_i - \lambda_2 \sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_2} \log t_i \exp\left\{t_i^{\alpha_2}\right\} - \lambda_2 I_2 t_m^{\alpha_2} \log t_m \exp\left\{t_m^{\alpha_2}\right\} = 0. \tag{12}$$

After replacing λ_1 in (9)–(11) and λ_2 in (10)–(12), we obtain

$$\frac{m_1}{\alpha_1} + \sum_{i=1}^{m} \omega_i (1 + t_i^{\alpha_1}) \log t_i - m_1 \frac{\sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_1} \log t_i \exp\left\{t_i^{\alpha_1}\right\} - I_1 t_m^{\alpha_1} \log t_m \exp\left\{t_m^{\alpha_1}\right\}}{\sum_{i=1}^{m-1} (r_i + 1)(\exp\left\{t_i^{\alpha_1}\right\} - 1) - I_1 (\exp\left\{t_m^{\alpha_1}\right\} - 1)} = 0, \tag{13}$$

and

$$\frac{m_2}{\alpha_2} + \sum_{i=1}^{m} (1 - \omega_i)(1 + t_i^{\alpha_2}) \log t_i - m_2 \frac{\sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_2} \log t_i \exp\left\{t_i^{\alpha_2}\right\} - I_2 t_m^{\alpha_2} \log t_m \exp\left\{t_m^{\alpha_2}\right\}}{\sum_{i=1}^{m-1} (r_i + 1)(\exp\left\{t_i^{\alpha_2}\right\} - 1) - I_2 (\exp\left\{t_m^{\alpha_2}\right\} - 1)} = 0. \tag{14}$$

Nonlinear Eqs. (13) and (14) with only one parameter can be solved using any iteration method such as Newton-Raphson or fixed point iteration. The parameter estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are obtained, and parameter estimates $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are obtained from Eqs. (9) and (10) after replacing α_1 and α_2 by $\hat{\alpha}_1$ and $\hat{\alpha}_2$. If $m_1 = 0$ or $m_2 = 0$, then the parameter values α_1 and λ_1 or α_2 and λ_2 cannot be obtained [16].

3.2 Asymptotic Confidence Interval

To obtain interval estimates of unknown parameters requires the computation of the Fisher information matrix, which is defined by the negative expectation of the partial second derivative of the log-likelihood rule using (8),

$$\sum = -\mathrm{E}\left(\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \theta_i \partial \theta_j}\right), \tag{15}$$

where $\theta = (\alpha_1, \alpha_2, \lambda_1, \lambda_2)$. In practice, the Fisher information matrix with a large sample can be approximated using the approximate information matrix,

$$\widehat{\sum}_0 = -\left(\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \theta_i \partial \theta_j}\right)\Bigg|_{\hat{\theta} = \left(\hat{\alpha}_1, \hat{\lambda}_1, \hat{\alpha}_2, \hat{\lambda}_2\right)}. \tag{16}$$

Therefore, under the rule of asymptotic normality distribution of computing $\hat{\theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}_1, \hat{\lambda}_2)$ with mean $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ and variance covariance matrix $\widehat{\sum}_0$. The approximate confidence intervals for model parameters are defined as

$\hat{\alpha}_1 \mp \mathrm{Z}_{\gamma} \sqrt{e_{11}}$,

$\hat{\alpha}_2 \mp \mathrm{Z}_{\gamma} \sqrt{e_{22}}$,

$$\hat{\lambda}_1 \mp Z_\gamma \sqrt{e_{33}},$$
$$\widehat{\lambda_2} \mp Z_\gamma \sqrt{e_{44}}, \tag{17}$$

where the diagonal of the approximate variance-covariance matrix $\widehat{\sum}_0$ represents the values e_{11}, e_{22}, e_{33}, and e_{44}, and Z_γ has a standard normal distribution with right-tail probability γ. The other variances are obtained using the partial derivative of the log-likelihood rule in Eq. (8),

$$\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_j^2} = -\frac{m_j}{\lambda_j^2}, \quad j = 1, 2, \tag{18}$$

$$\begin{aligned}\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_1^2} &= -\frac{m_1}{\alpha_1^2} + \sum_{i=1}^{m} \omega_i t_i^{\alpha_1} \log^2 t_i - \lambda_1 \sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_1} \left(1 + t_i^{\alpha_1}\right) \log^2 t_i \exp\left\{t_i^{\alpha_1}\right\} \\ &\quad - \lambda_1 I_1 t_m^{\alpha_1} \left(1 + t_i^{\alpha_1}\right) \log^2 t_m \exp\left\{t_m^{\alpha_1}\right\},\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_2^2} &= -\frac{m_1}{\alpha_2^2} + \sum_{i=1}^{m} \left(1 - \omega_i\right) t_i^{\alpha_2} \log^2 t_i - \lambda_2 \sum_{i=1}^{m-1} (r_i + 1) t_i^{\alpha_2} \left(1 + t_i^{\alpha_2}\right) \log^2 t_i \exp\left\{t_i^{\alpha_2}\right\} \\ &\quad - \lambda_2 I_2 t_m^{\alpha_2} \left(1 + t_i^{\alpha_2}\right) \log^2 t_m \exp\left\{t_m^{\alpha_2}\right\},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_1 \partial \lambda_1} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_1 \partial \alpha_1} &= -\sum_{i=1}^{m-1} (r_i + 1)\, t_i^{\alpha_1} \log t_i \exp\left\{t_i^{\alpha_1}\right\} \\ &\quad - I_1 t_m^{\alpha_1} \log t_m \exp\left\{t_m^{\alpha_1}\right\},\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_2 \partial \lambda_2} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_2 \partial \alpha_2} &= -\sum_{i=1}^{m-1} (r_i + 1)\, t_i^{\alpha_2} \log t_i \exp\left\{t_i^{\alpha_2}\right\} \\ &\quad - I_2 t_m^{\alpha_2} \log t_m \exp\left\{t_m^{\alpha_2}\right\},\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_1 \partial \lambda_2} &= \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_2 \partial \alpha_1} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_2 \partial \lambda_1} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_1 \partial \alpha_2} \\ &= \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \lambda_2 \partial \lambda_1} = \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_1 \partial \alpha_2} \\ &= \frac{\partial^2 \ell\left(\alpha_1, \alpha_2, \lambda_1, \lambda_2 | \boldsymbol{t}\right)}{\partial \alpha_2 \partial \alpha_1}.\end{aligned} \tag{23}$$

4 Bayes with MCMC Methods

We need to use Bayes approaches with the MCMC method because of the dimensionality of the model. Bayes estimation requires prior information about the model parameters, which are considered in this study to be independent gamma priors. Then, the available prior information is modeled as

$$\theta_i \xrightarrow{distributed\ \ as} \text{gamma}\,(a_i, b_i)\,, (a_i, b_i > 0)\,, \quad i = 1, 2, 3, 4, \tag{24}$$

where $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \lambda_1, \lambda_2)$. The joint distribution of prior densities is formed by

$$P^*\,(\theta_i) = \prod_{i=1}^{4} \frac{b_i^{a_i}}{\Gamma(a_i)} \theta_i^{a_i-1} \exp\{-b_i\theta_i\}. \tag{25}$$

Following this, the information about the model parameters is obtained from the prior information and the data, which provides the posterior distribution as

$$P\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t}) = \frac{P^*\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t})\, L\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t})}{\int\limits_{\alpha_1}\int\limits_{\alpha_2}\int\limits_{\lambda_1}\int\limits_{\lambda_2} P^*\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t})\, L\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t})\, d\lambda_2 d\lambda_1 d\alpha_2 d\alpha_1}, \tag{26}$$

where the denominator of the fraction can be removed since it contains no information about $\boldsymbol{\theta}$. The proportional form from posterior distribution (26) with prior distribution (25) and likelihood rule (7) is defined as

$$\begin{aligned} P\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t}) \propto\, & \alpha_1^{m_1+a_1-1} \alpha_2^{m_2+a_2-1} \lambda_1^{m_1+a_3-1} \lambda_2^{m_2+a_4-1} \\ & \times \exp\Bigg\{ -b_1 a_1 - b_3\lambda_1 + (\alpha_1 - 1) \sum_{i=1}^{m} \omega_i \log t_i + \sum_{i=1}^{m} \omega_i t_i^{\alpha_1} \\ & + \lambda_1 \sum_{i=1}^{m-1} (r_i + 1) \left(1 - \exp\left\{t_i^{\alpha_1}\right\}\right) + \lambda_1 I_1 \left(1 - \exp\left\{t_i^{\alpha_1}\right\}\right) \\ & - b_2 a_2 - b_4\lambda_2 + (\alpha_2 - 1) \sum_{i=1}^{m} (1 - \omega_i) \log t_i + \sum_{i=1}^{m} (1 - \omega_i)\, t_i^{\alpha_2} \\ & + \lambda_2 \sum_{i=1}^{m-1} (r_i + 1) \left(1 - \exp\left\{t_i^{\alpha_2}\right\}\right) + \lambda_2 I_2 \left(1 - \exp\left\{t_i^{\alpha_2}\right\}\right) \Bigg\}. \end{aligned} \tag{27}$$

The Bayes estimators are computed with respect to the loss rule; then the Bayes method of any function $\pi\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ under the rule of the squared-error loss (SEL) function is presented by

$$\hat{\pi}\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \mathrm{E}_P\,(\pi\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2)) = \int_{\alpha_1}\int_{\alpha_2}\int_{\lambda_1}\int_{\lambda_2} P\,(\alpha_1, \alpha_2, \lambda_1, \lambda_2|\boldsymbol{t})\, d\lambda_2 d\lambda_1 d\alpha_2 d\alpha_1. \tag{28}$$

The integrals in Eqs. (26) and (28) generally cannot be obtained in explicit form, but can be solved by approximation, such as numerical integration or Lindley approximation. One of the most frequently applied methods is the MCMC method, which is used to compute point and interval estimates as follows. The full conditional distributions can be described as

$$P_1(\alpha_1|\boldsymbol{t},\lambda_1) \propto \alpha_1^{m_1+a_1-1}\exp\left\{-b_1a_1+\alpha_1\sum_{i=1}^{m}\omega_i\log t_i+\sum_{i=1}^{m}\omega_i t_i^{\alpha_1}\right.$$
$$\left.-\lambda_1\sum_{i=1}^{m-1}(r_i+1)\exp\left\{t_i^{\alpha_1}\right\}-\lambda_1 I_1\exp\left\{t_i^{\alpha_1}\right\}\right\}, \tag{29}$$

$$P_2(\alpha_2|\boldsymbol{t},\lambda_2) \propto \alpha_2^{m_2+a_2-1}\exp\left\{-b_2a_2+\alpha_2\sum_{i=1}^{m}\omega_i\log t_i+\sum_{i=1}^{m}(1-\omega_i)\,t_i^{\alpha_2}\right.$$
$$\left.-\lambda_2\sum_{i=1}^{m-1}(r_i+1)\exp\left\{t_i^{\alpha_2}\right\}-\lambda_2 I_2\exp\left\{t_i^{\alpha_2}\right\}\right\}, \tag{30}$$

$$P_3(\lambda_1|\boldsymbol{t},\alpha_1) \propto \lambda_1^{m_1+a_3-1}\exp\left\{-b_3\lambda_1+\lambda_1\sum_{i=1}^{m-1}(r_i+1)\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)\right.$$
$$\left.+\lambda_1 I_1\left(1-\exp\left\{t_i^{\alpha_1}\right\}\right)\right\}, \tag{31}$$

and

$$P_4(\lambda_2|\boldsymbol{t},\alpha_2) \propto \lambda_2^{m_2+a_4-1}\exp\left\{-b_4\lambda_2+\lambda_2\sum_{i=1}^{m-1}(r_i+1)\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right)\right.$$
$$\left.+\lambda_2 I_2\left(1-\exp\left\{t_i^{\alpha_2}\right\}\right)\right\}. \tag{32}$$

Then the full conditional distributions are reduced to gamma distributions represented by Eqs. (31) and (32), and two distributions similar to normal distributions, shown as Eqs. (29) and (30). The MCMC methods have the forms of Gibbs algorithms, and the more general Metropolis-Hastings (MH) under Gibbs algorithms [17]. The following algorithm describes MCMC methods.

Step 1: Start with an initial vector $\hat{\theta}=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\lambda}_1,\hat{\lambda}_2)$ and indicator $\rho=1$.

Step 2: The values $\lambda_j^{(\rho)}$, $j=1,2$ are generated from conditional distributions presented by Eqs. (31) and (32), respectively.

Step 3: The values $\alpha_j^{(\rho)}$, $j=1,2$ are generated from conditional distributions presented by Eqs. (29) and (30) with the MH algorithm using normal proposal distributions with mean $\alpha_j^{(\rho-1)}$ and variance obtained from approximate information matrix, respectively.

Step 4: The vector $\boldsymbol{\theta}^{(\rho)}=(\alpha_1^{(\rho)},\alpha_2^{(\rho)},\lambda_1^{(\rho)},\lambda_2^{(\rho)})$ is recorded; hence, $\rho=\rho+1$.

Step 5: Steps (2) to (4) are repeated S times.

Table 1: MVs and MSEs of estimators of Chen distributions with $\theta = (1.0, 1.5, 0.2, 0.1)$

(S_1, S_2)	(m, r)	Pa. θ	ML BMCMC prior 0				BMCMC prior 1	
			MVs	MSEs	MVs	MSEs	MVs	MSEs
(30, 25)	$(15, (10, 0^{(13)}))$	α_1	1.311	0.325	1.352	0.321	1.241	0.241
		α_2	1.712	0.521	1.715	0.511	1.669	0.400
		λ_1	0.311	0.100	0.321	0.099	0.217	0.074
		λ_2	0.201	0.081	0.198	0.079	0.147	0.054
	$(15, (0^{(7)}, 10, 0^{(6)}))$	α_1	1.332	0.375	1.372	0.381	1.255	0.262
		α_2	1.732	0.566	1.754	0.559	1.670	0.412
		λ_1	0.325	0.113	0.344	0.110	0.242	0.076
		λ_2	0.231	0.092	0.210	0.088	0.177	0.073
	$(15, (0^{(14)}))$	α_1	1.340	0.382	1.379	0.390	1.266	0.257
		α_2	1.741	0.571	1.762	0.563	1.678	0.417
		λ_1	0.331	0.116	0.340	0.114	0.249	0.081
		λ_2	0.235	0.097	0.213	0.091	0.178	0.075
	$(20, (5, 0^{(18)}))$	α_1	1.209	0.201	1.214	0.199	1.174	0.124
		α_2	1.641	0.410	1.635	0.409	1.611	0.325
		λ_1	0.287	0.082	0.289	0.081	0.216	0.066
		λ_2	0.175	0.055	0.171	0.057	0.144	0.042
	$(20, (0^{(9)}, 5, 0^{(9)}))$	α_1	1.225	0.214	1.227	0.212	1.179	0.131
		α_2	1.652	0.422	1.651	0.417	1.625	0.331
		λ_1	0.292	0.087	0.290	0.089	0.222	0.071
		λ_2	0.181	0.059	0.182	0.058	0.151	0.049
	$(20, (0^{(19)}))$	α_1	1.231	0.217	1.235	0.216	1.181	0.136
		α_2	1.659	0.427	1.656	0.422	1.629	0.335
		λ_1	0.290	0.095	0.291	0.093	0.227	0.076
		λ_2	0.185	0.062	0.181	0.065	0.154	0.053
(50,50)	$(30, (20, 0^{(28)}))$	α_1	1.115	0.125	1.114	0.122	1.113	0.100
		α_2	1.574	0.214	1.569	0.217	1.552	0.158
		λ_1	0.252	0.055	0.249	0.054	0.213	0.036
		λ_2	0.136	0.041	0.129	0.039	0.121	0.018
	$(30, (0^{(29)}))$	α_1	1.126	0.137	1.118	0.141	1.118	0.109
		α_2	1.582	0.221	1.575	0.223	1.561	0.166
		λ_1	0.271	0.059	0.258	0.060	0.218	0.041
		λ_2	0.143	0.048	0.145	0.051	0.127	0.026

Step 6: If we need to the number of iterations to reach convergence in the equilibrium, which called burn-in, say S^*; hence, the Bayes estimators of model parameters are represented by

$$\hat{\theta}_{iB} = \mathrm{E}_P\left(\theta_i|\boldsymbol{t}\right) = \frac{1}{S - S^*} \sum_{i=S^*+1}^{S} \theta^{(i)}, \quad i = 1, 2, 3, 4, \tag{33}$$

Table 2: Two ALs (PCs) of Chen distributions with $\theta = (1.0, 1.5, 0.2, 0.1)$

(S_1, S_2)	(m, r)	Pa. θ	ML		BMCMCprior0		BMCMCprior1	
			ALs	PCs	ALs	PCs	ALs	PCs
(30,25)	$\left(15, \left(10, 0^{(13)}\right)\right)$	α_1	2.854	(0.89)	2.849	(0.90)	2.489	(0.91)
		α_2	3.752	(0.89)	3.762	(0.89)	3.089	(0.90)
		λ_1	0.615	(0.90)	0.619	(0.89)	0.542	(0.91)
		λ_2	0.401	(0.90)	0.409	(0.90)	0.396	(0.90)
	$\left(15, \left(0^{(7)}, 10, 0^{(6)}\right)\right)$	α_1	2.875	(0.89)	2.882	(0.90)	2.521	(0.91)
		α_2	3.791	(0.90)	3.799	(0.89)	3.214	(0.91)
		λ_1	0.651	(0.91)	0.644	(0.89)	0.571	(0.91)
		λ_2	0.434	(0.89)	0.418	(0.90)	0.399	(0.92)
	$\left(15, \left(0^{(14)}\right)\right)$	α_1	2.887	(0.90)	2.891	(0.90)	2.532	(0.91)
		α_2	3.798	(0.90)	3.794	(0.90)	3.218	(0.90)
		λ_1	0.662	(0.90)	0.671	(0.89)	0.580	(0.91)
		λ_2	0.441	(0.90)	0.417	(0.90)	0.400	(0.91)
	$\left(20, \left(5, 0^{(18)}\right)\right)$	α_1	2.624	(0.90)	2.618	(0.91)	2.214	(0.92)
		α_2	3.521	(0.91)	3.524	(0.94)	3.000	(0.92)
		λ_1	0.521	(0.91)	0.518	(0.92)	0.410	(0.91)
		λ_2	0.328	(0.90)	0.333	(0.90)	0.301	(0.91)
	$\left(20, \left(0^{(9)}, 5, 0^{(9)}\right)\right)$	α_1	2.631	(0.91)	2.624	(0.91)	2.217	(0.93)
		α_2	3.528	(0.90)	3.529	(0.93)	3.021	(0.92)
		λ_1	0.528	(0.91)	0.522	(0.92)	0.417	(0.91)
		λ_2	0.341	(0.92)	0.338	(0.91)	0.311	(0.92)
	$\left(20, \left(0^{(19)}\right)\right)$	α_1	2.640	(0.93)	2.639	(0.93)	2.232	(0.93)
		α_2	3.524	(0.90)	3.531	(0.93)	3.024	(0.91)
		λ_1	0.529	(0.91)	0.531	(0.92)	0.422	(0.93)
		λ_2	0.345	(0.92)	0.341	(0.92)	0.310	(0.92)
(50,50)	$\left(30, \left(20, 0^{(28)}\right)\right)$	α_1	2.542	(0.94)	2.555	(0.92)	2.198	(0.95)
		α_2	3.412	(0.931)	3.417	(0.94)	2.900	(0.92)
		λ_1	0.410	(0.91)	0.415	(0.92)	0.397	(0.94)
		λ_2	0.300	(0.93)	0.310	(0.93)	0.289	(0.92)
	$\left(30, \left(0^{(29)}\right)\right)$	α_1	2.551	(0.91)	2.554	(0.92)	2.201	(0.93)
		α_2	3.426	(0.95)	3.425	(0.94)	2.909	(0.92)
		λ_1	0.417	(0.94)	0.421	(0.95)	0.400	(0.94)
		λ_2	0.315	(0.93)	0.318	(0.93)	0.309	(0.93)

with posterior variance of Θ,

$$V(\theta_i|\boldsymbol{t}) = \frac{1}{S - S^*} \sum_{i=S^*+1}^{S} (\theta^{(i)} - \hat{\theta}_{iB})^2. \tag{34}$$

Step 7: The $100\,(1-2\gamma)\,\%$ credible intervals can be obtained from the empirical distribution of θ_i after putting the values in ascending order; hence, a credible interval is formed by

$$\left(\hat{\theta}_{i\gamma(S-S^{*})},\hat{\theta}_{i(1-\gamma)(S-S^{*})}\right), \tag{35}$$

where $\boldsymbol{\theta}=(\alpha_1,\alpha_2,\lambda_1,\lambda_2)$.

5 Numerical Computation

5.1 Simulation Studies

Two estimation methods, classical ML and Bayes estimation under Chen lifetime distribution, are discussed and developed in this study. We compare and assess these methods under the MCMC algorithms. We report the results with various sample sizes (S_1, S_2), several sample sizes of failure units m, and censoring procedures r. We fix parameters at $(\alpha_1,\lambda_1)=(0.5,0.5)$ and $(\alpha_1,\lambda_1)=(0.7,0.4)$. The validity of numerical results is determined by the mean value (MV) and mean squared-error (MSE) for point estimators. The probability coverage (PC) and average interval length (AL) are used to measure interval estimators. The results are summarized in Tabs. 1 and 2 for two sets of prior information (non-informative prior 0 and informative prior 1). The simulation study used 1000 balanced progressive type-II samples. For Bayes results, the producer was considered under the rule of the squared-error loss function and 11000 iterations of MCMC, with the first 1000 iterations as burn-in. The results are reported in Tabs. 1 and 2.

5.2 Data Analysis

Let Chen distribution with parameter values $(\alpha_1,\lambda_1)=(1.5,1.1)$ and $(\alpha_2,\lambda_2)=(1.8,0.9)$ and the prior distributions with parameters $(a_1,b_1)=(4,2)$, $(a2,b2)=(3,2.0)$, $(a_3,b_3)=(2.0,1.5)$ and $(a_4,b_4)=(2,2.5)$ are used to apply Bayes approaches.

Table 3: Balanced joint progressive type-II censoring data

0.0274	0.0435	0.0519	0.0581	0.0740	0.1138	0.1387	0.1839	0.1859	0.1932
0	0	1	1	1	1	1	0	0	1
0.1945	0.2545	0.2613	0.2791	0.2911	0.2973	0.3281	0.3577	0.3955	0.4163
1	0	0	1	0	1	1	1	0	0
0.4671	0.4947	0.5935	0.5990	0.6411	0.6502	0.7318	0.7530	0.9014	1.0391
0	1	1	1	1	1	1	1	0	0

Table 4: Point and 95% confidence and credible intervals (ACIs and CIs)

Parameter θ	ML	BMCMC	95% ACIs	Length	95% CIs	Length
$\alpha_1=1.5$	1.117	1.241	(0.6725,1.5615)	0.889	(0.7206,1.5733)	0.853
$\lambda_1=1.1$	1.147	1.109	(0.5478,1.6028)	1.055	(0.6101,1.6327)	1.022
$\alpha_2=1.8$	1.075	1.321	(0.5176,1.9768)	1.459	(0.6186,1.8626)	1.244
$\lambda_2=0.9$	0.744	0.837	(0.2432,1.2438)	1.001	(0.3389,1.3381)	0.999

Under consideration two sample of size $(S_1,S_2)=(40,40)$, censoring scheme $r=\left\{9,0^{(28)}\right\}$, with the number of failures $m=30$. Then the sample can be generated with sample size $S_1=30$

from a Chen distribution with parameters $(1.5, 1.1)$ and with size S_2 from a Chen distribution with parameters $(1.8, 0.9)$ using the algorithms [18]. The two progressive type-II samples are used to generate balanced joint progressive type-II samples with respect to $r = \{9, 0^{(28)}\}$ and $m = 30$. The joint sample and its type are reported in Tab. 3. The results of point estimation and interval MLEs are reported in Tab. 4. We plot the monitoring of the MCMC and the corresponding histogram in Figs. 3–10, which show the quality of the empirical posterior distribution generated by MCMC methods.

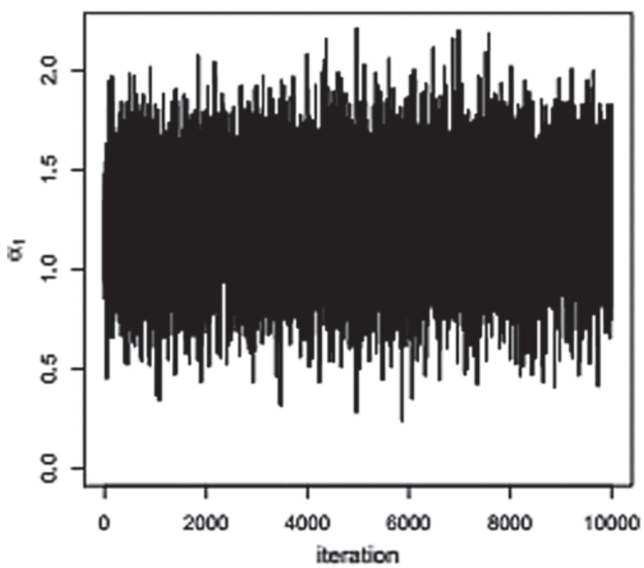

Figure 3: Recording of parameter α_1 generated by the MCMC algorithm

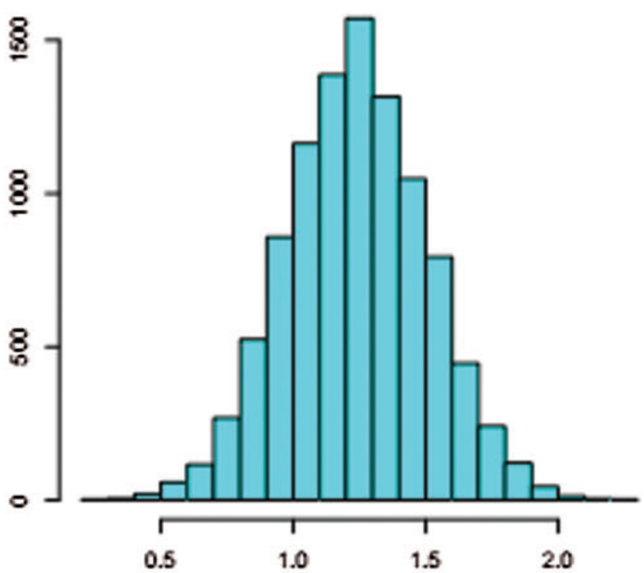

Figure 4: Summary of the analysis for α_1 generated by the MCMC algorithm

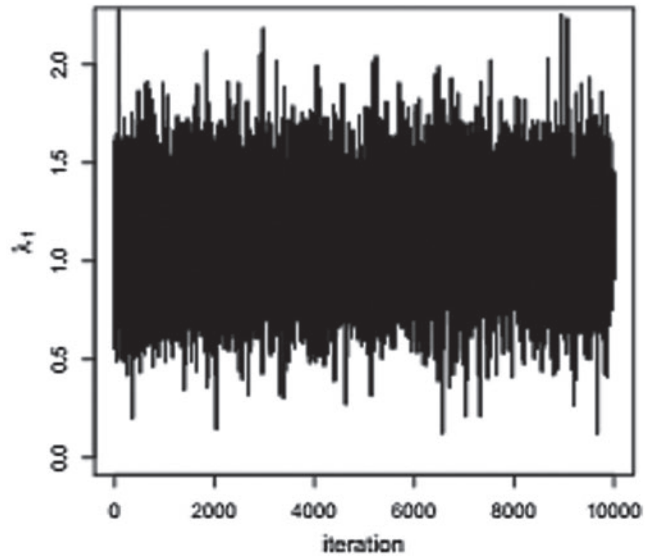

Figure 5: Recording of parameter α_2 generated by the MCMC algorithm

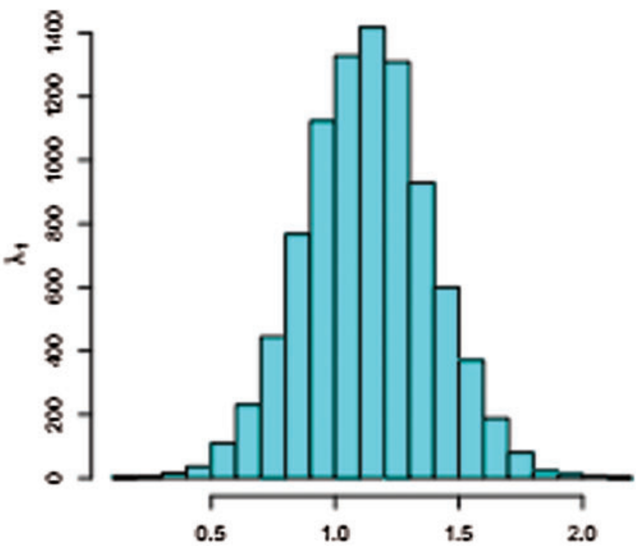

Figure 6: Summary of the analysis for α_2 generated by the MCMC algorithm

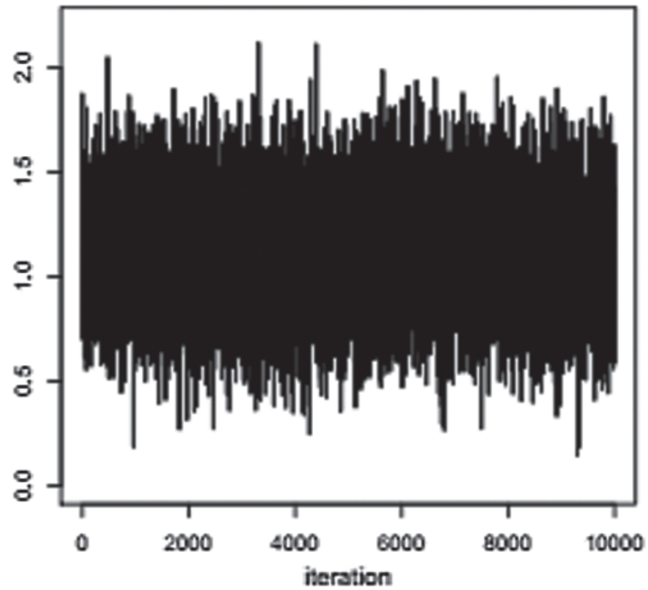

Figure 7: Recording of parameter λ_1 generated by the MCMC algorithm

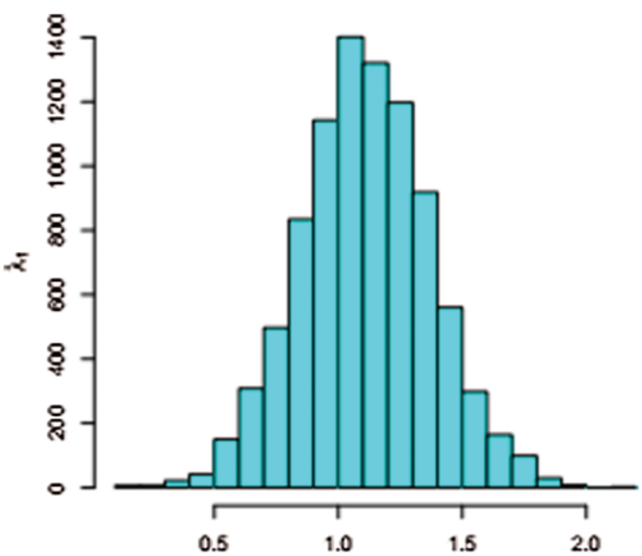

Figure 8: Summary of the analysis for λ_1 generated by the MCMC algorithm

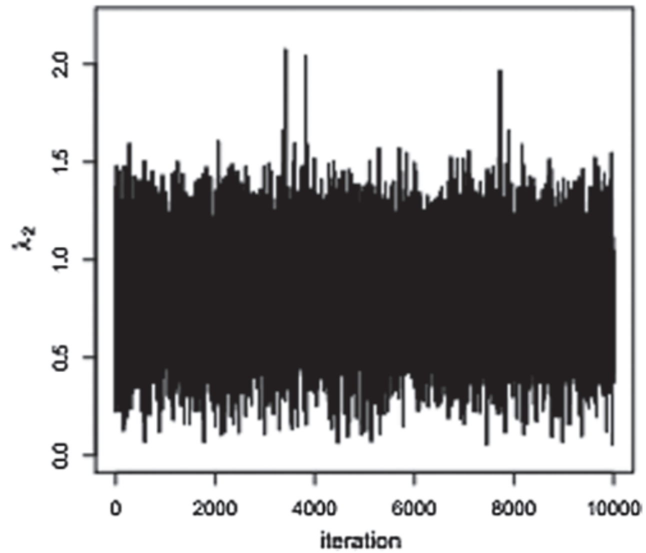

Figure 9: Recording of parameter λ_2 generated by the MCMC algorithm

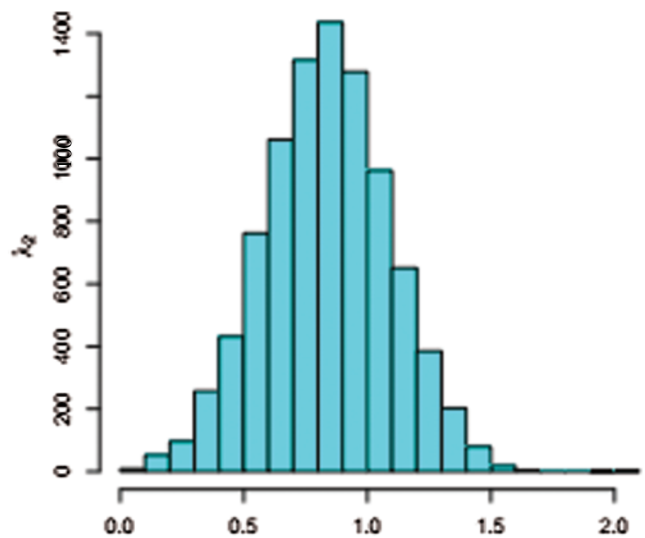

Figure 10: Summary of the analysis for λ_2 generated by the MCMC algorithm

6 Concluding Remarks

Products from different production lines were investigated using a joint censoring procedure under the same conditions. The balanced joint censoring procedure has been shown considerable attention over the last few years. In this study, we discussed products that follow a Chen lifetime distribution. We discussed the ML and Bayes estimates to estimate the underlying parameters of two Chen lifetime distributions. Numerical results were obtained to compare the theoretical performance results. Some points are observed from numerical results, which are summarized as follows.

From the results in Tabs. 1 and 2, show that the balanced joint progressive type-II censoring procedure provides better excellent results for products have Chen lifetime distribution.

Estimation results under the Bayes method and informative prior distribution provide better estimation than ML and non-informative prior methods according to the MSE.

For non-informative priors, there are no significant differences between MLEs and Bayes estimates.

The effective sample size m can be increased by reducing the MSEs and interval lengths.

Acknowledgement: The researcher would like to thank LetPub (www.letpub.com) for its linguistic assistance during the preparation of this manuscript. This study was funded by Taif University Researchers Supporting Project number (TURSP-2020/279), Taif University, Taif, Saudi Arabia.

Funding Statement: Taif University.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] N. Balakrishnan and D. Kundu, "Hybrid censoring: Models, inferential results and applications," *Computational Statistics & Data Analysis,* vol. 57, no. 1, pp. 166–209, 2013.

[2] G. Herd, "Estimation of the parameters of a population from a multi-censored sample," Ph.D. dissertation. Iowa State University, USA, 1956.

[3] A. Cohen, "Progressively censored samples in life testing," *Technometrics,* vol. 5, no. 3, pp. 327–339, 1963.

[4] N. L. Johnson, "Life testing and early failure," *Technometrics,* vol. 8, no. 3, pp. 539–545, 1966.

[5] U. V. R. Rao, I. R. Savage and M. Sobel, "Contributions to the theory of rank order statistics: The two-sample censored case," *The Annals of Mathematical Statistics,* vol. 31, no. 2, pp. 415–426, 1960.

[6] K. Mehrotra and G. Bhattacharyya, "Confidence intervals with jointly type-II censored samples from two exponential distributions," *Journal of the American Statistical Association,* vol. 77, no. 378, pp. 441–446, 1982.

[7] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint type-II censoring," *Computational Statistics & Data Analysis,* vol. 52, no. 5, pp. 2725–2738, 2008.

[8] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive type-II censoring," *Communications in Statistics—Theory and Methods,* vol. 39, no. 12, pp. 2172–2191, 2010.

[9] A. Shafay, N. Balakrishnan and Y. Abdel-Aty, "Bayesian inference based on a jointly type-II censored sample from two exponential populations," *Journal of Statistical Computation and Simulation,* vol. 84, no. 11, pp. 2427–2440, 2014.

[10] B. Al-Matrafi and G. A. Abd-Elmougod, “Statistical inferences with jointly type-II censored samples from two rayleigh distributions,” *Global Journal of Pure and Applied Mathematics,* vol. 13*,* no. 12*,* pp. 8361–8372, 2017.

[11] A. Algarni, A. Almarashi, G. A. Abd-Elmougod and Z. A. Abo-Eleneen, “Two compound rayleigh lifetime distributions in analyses the jointly type-II censoring samples,” *Journal of Mathematical Chemistry,* vol. 58*,* no. 1*,* pp. 950–966, 2020.

[12] S. Mondal and D. Kundu, “A new two sample type-II progressive censoring scheme,” *Communications in Statistics-Theory and Methods,* vol. 48*,* no. 10*,* pp. 2602–2618, 2019.

[13] S. Mondal and D. Kundu, “Bayesian inference for Weibull distribution under the balanced joint type-II progressive censoring scheme,” *American Journal of Mathematical and Management Sciences,* vol. 39*,* no. 1*,* pp. 56–74, 2020.

[14] Z. Chen, “A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function,” *Statistics & Probability Letters,* vol. 49*,* no. 2*,* pp. 155–161, 2000.

[15] J. Wu, H. Lu, C. Chen and C. Wu, “Statistical inference about the shape parameter of the new two-parameter bathtub-shaped lifetime distribution,” *Quality and Reliability Engineering International,* vol. 20*,* no. 6*,* pp. 607–616, 2004.

[16] D. Kundu and A. Joarder, “Analysis of type-II progressively hybrid censored data,” *Computational Statistics & Data Analysis,* vol. 50*,* no. 10*,* pp. 2509–2528, 2006.

[17] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller and E. Teller, “Equation of state calculations by fast computing machines,” *Journal of Chemical Physics,* vol. 21*,* no. 6*,* pp. 1087–1092, 1953.

[18] N. Balakrishnan and R. A. Sandhu, “A simple simulational algorithm for generating progressive type-II censored samples,” *American Statistician,* vol. 49*,* no. 2*,* pp. 229–230, 1995.