

# Entanglement and Sudden Death for a Two-Mode Radiation Field Two Atoms

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**Abstract:** The effect of the field–field interaction on a cavity containing two qubit (TQ) interacting with a two mode of electromagnetic field as parametric amplifier type is investigated. After performing an appropriate transformation, the constants of motion are calculated. Using the Schrödinger differential equation a system of differential equations was obtained, and the general solution was obtained in the case of exact resonance. Some statistical quantities were calculated and discussed in detail to describe the features of this system. The collapses and revivals phenomena have been discussed in details. The Shannon information entropy has been applied for measuring the degree of entanglement (DE) between the qubits and the electromagnetic field. The normal squeezing for some values of the parameter of the field–field interaction is studied. The results showed that the collapses disappeared after the field–field terms were added and the maximum values of normal squeezing decrease when increasing of the field–field interaction parameter. While the revivals and amplitudes of the oscillations increase when the parameter of the field–field interaction increases. Degree of entanglement is partially more entangled with increasing of the field-field interaction parameter. The relationship between revivals, collapses and the degree of entanglement (Shannon information entropy) was monitored and discussed in the presence and absence of the field–field interaction.

**Keywords:** Field–field interaction;  $su(1,1)$  Lie group; degree of entanglement; normal squeezing

## 1 Introduction

The entanglement between the atom-field (AF) interaction for Jaynes–Cummings model (JCM) [1] has been discussed by [2]. This simple model represents the interacting between qubit (Q) and the field placed individually in a high-Q space. It is known that, this model is simple and analytically solvable, which helps to effectively understand quantum optics and information problems. The entanglement between TQ has been studied in one and two JCM and one-photon [3]. The influence of the amplifier terms (two-photon degenerate and non-degenerate case) on the two two-level atoms has been investigated by [4,5]. The



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effect of atom-atom cross interaction has been studied with a maximally entangled state, the occurrence of sudden birth of entanglement has been studied by [6]. The problem of TQ interaction with one mode has been studied by [7–10]. On the other hand, the system of TQ and a system represented by  $su(1,1)$  in existence of classical field has been studied by [6]. However, the generalization of the model from one atom to two atoms with classical field or Stark shift has been investigated by [8,9]. The interaction between the TQ and a one field leads to more entanglement between the subsystem, such that the atoms in general entangled and after adding the field due to the interaction is more increases [10]. The effectiveness of the squeezed state on the TQ system interacting with a field in frame 2-photon has been studied by [11,12]. The degree of entanglement of two atoms with a linear interaction prepared initially in thermal state has been studied by [13].

In this communication, consider a two mode of the electromagnetic field as parametric amplifier interacting with a TQ as well as the field–field interaction as follows:

$$\hat{H} = \frac{1}{2} \sum_{i=1,2} (\Omega_j \hat{S}_z^{(i)}) + \frac{(\omega_1 + \omega_2)}{2} \hat{I} + \omega_1 \hat{a}^* \hat{a} + \omega_2 \hat{b}^* \hat{b} + g(\hat{a} \hat{b} + \hat{a}^* \hat{b}^*) + \sum_{\ell=A,B} \lambda(\hat{a} \hat{b} \hat{S}_\ell^+ + \hat{a}^* \hat{b}^* \hat{S}_\ell^-). \quad (1)$$

where,  $\Omega_j$  represents the TQ frequency. While  $\hat{S}_\ell^\pm$  and  $\hat{S}_\ell^z$  are the Pauli matrices. The  $\lambda$  is the coupling of the TQ-EMF interaction and  $g$  is a coupling of the field-field interaction. If we set  $\omega_1 = \omega_2 = \frac{\omega}{2}$  and introducing the  $su(1,1)$  generators,  $\hat{K}_\pm$  and  $\hat{K}_0$  as follows:

$$\hat{K}_- = \hat{a} \hat{b} = \hat{K}_+^*, \quad \hat{K}_z = \frac{1}{2}(1 + \hat{a}^* \hat{a} + \hat{b}^* \hat{b}), \quad (2)$$

which satisfy the relations:  $[\hat{K}_z, \hat{K}_\pm] = \pm \hat{K}_\pm$  and  $[\hat{K}_-, \hat{K}_+] = 2\hat{K}_z$ . Therefore, we find that:  $\hat{K}_+ \hat{K}_- = \hat{K}_z^2 - (\hat{K}_z + \hat{K}^2)$ . The  $\hat{K}^2 = \hat{K}_z^2 - \frac{1}{2}(\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+) = k(k-1)\hat{I}$  is the Casimir operator and  $k$  is the Bargmann number. The Hamiltonian of Eq. (1) can be governed by an  $su(1,1)$  and  $su(2)$  Lie algebra as:

$$\frac{\hat{H}}{\hbar} = \omega \hat{K}_z + \sum_{j=1}^2 \Omega_j \hat{S}_z^{(j)} + i(\hat{K}_+ - \hat{K}_-) \sum_{i=1}^2 \lambda(\hat{S}_+^{(i)} + \hat{S}_-^{(i)}) + g(\hat{K}_+ + \hat{K}_-). \quad (3)$$

So the  $su(1,1)$  operators in the number state representation takes the following form,

$$\begin{aligned} \hat{K}_z |m, k\rangle &= (m+k) |m, k\rangle, \quad \hat{K}_z^2 |m, k\rangle = k(k-1) |m, k\rangle, \\ \hat{K}_- |m, k\rangle &= S_{k,m} |m-1, k\rangle, \quad \hat{K}_+ |m, k\rangle = S_{k,m+1} |m+1, k\rangle, \end{aligned} \quad (4)$$

where  $S_{k,m} = \sqrt{m(m+2k-1)}$ . It is noted that the connection between  $(m, k)$  and the photon number of the field modes is  $\hat{k} = \frac{1}{2}(1 + \hat{a}^* \hat{a} - \hat{b}^* \hat{b})$  and  $\hat{m} + \hat{k} = \frac{1}{2}(1 + \hat{a}^* \hat{a} + \hat{b}^* \hat{b})$ .

The basic aim of this work is to study the behavior of the system Eq. (3) besides to see the influence of the external terms (field–field interaction), which represented the linear combination between  $\hat{K}_+$  and  $\hat{K}_-$  terms on the interaction of the present system. The derivation of the Schrödinger differential equations and their solution will be done in the next Section 2. The relative inversion will be discussed in Section 3 and the degree of entanglement in Section 4. The normal squeezing is considered in Section 5. In Section 6 some brief remarks will be presented.

## 2 Wave Function

To calculate the wave function for the present system Eq. (3), we must solve the Schrödinger differential equations. It is necessary to calculate the constants of the motion, which facilitates the process solution. In the presence of field–field interaction, we cannot obtain the constants of motion. So we apply the following transformations to remove these terms.

$$\begin{aligned} \hat{K}_+ &= \hat{R}_+ \cosh^2 \zeta + \hat{R}_- \sinh^2 \zeta - \hat{R}_z \sinh 2\zeta, \\ \hat{K}_- &= \hat{R}_+ \sinh^2 \zeta + \hat{R}_- \cosh^2 \zeta - \hat{R}_z \sinh 2\zeta, \\ \hat{K}_z &= -\frac{\hat{R}_+}{2} \sinh 2\zeta - \frac{\hat{R}_-}{2} \sinh 2\zeta + \hat{R}_z \cosh 2\zeta, \end{aligned} \tag{5}$$

where the operators  $\hat{R}_\pm, \hat{R}_z$  are generators of the  $su(1,1)$  group with  $\zeta = \frac{1}{2} \tanh^{-1} \left( \frac{2g}{\omega} \right)$ .

Now if we substitute Eq. (5) into the Hamiltonian Eq. (3), the Hamiltonian becomes

$$\frac{\hat{H}}{\hbar} = \Omega_z \hat{R}_z + \sum_{j=1}^2 \left( \frac{\Omega_j}{2} \hat{S}_z^{(j)} + i \lambda (\hat{R}_+ \hat{S}_-^{(j)} - \hat{R}_- \hat{S}_+^{(j)}) \right), \tag{6}$$

With  $\Omega_j$  is the transition frequency for each qubit and  $\Omega_z = \sqrt{\omega^2 - 4g^2}$  indicates to the artificial frequency of the quantum system. The Heisenberg equation of motion is applied to calculate the dynamical operators  $\hat{R}_z, \hat{S}_z^{(1)}$  and  $\hat{S}_z^{(2)}$  as follows:

$$\begin{aligned} \frac{d\hat{R}_z}{dt} &= \sum_{j=1}^2 \lambda (\hat{R}_+ \hat{S}_-^{(j)} + \hat{R}_- \hat{S}_+^{(j)}), \\ \frac{d\hat{S}_z^{(1)}}{dt} &= -2\lambda (\hat{R}_+ \hat{S}_-^{(1)} + \hat{R}_- \hat{S}_+^{(1)}), \quad \frac{d\hat{S}_z^{(2)}}{dt} = -2\lambda (\hat{R}_+ \hat{S}_-^{(2)} + \hat{R}_- \hat{S}_+^{(2)}), \end{aligned} \tag{7}$$

Therefore we introduce the constant operator  $\hat{N}$  as

$$\hat{N} = \hat{R}_z + \sum_{j=1}^2 \frac{\hat{S}_z^{(j)}}{2}, \tag{8}$$

By applying Eq. (8), the Hamiltonian Eq. (6) becomes

$$\frac{\hat{H}}{\hbar} = \omega \hat{N} + \hat{C}, \tag{9}$$

where  $\hat{C}$  is defined as

$$\hat{C} = \sum_{j=1}^2 \left( \frac{\Delta_j}{2} \hat{S}_z^{(j)} + i \lambda (\hat{R}_+ \hat{S}_-^{(j)} - \hat{R}_- \hat{S}_+^{(j)}) \right), \tag{10}$$

where the quantity  $\Delta_j$  refers to the detuning parameter which is represented by

$$\Delta_j = \Omega_z - \Omega_j, \quad j = 1, 2. \tag{11}$$

Consider that the initial condition for the  $su(1,1)$  system is the Barut–Girardello coherent state and TQ are in excited states. Therefore, the initial conditions state for the wave function takes the following equation,

$$|\psi(0)\rangle = |+, +\rangle \otimes |\xi\rangle, \quad (12)$$

where  $|\xi\rangle$  is given by

$$|\xi\rangle = N \sum_{m=0}^{\infty} \frac{\xi^m}{\sqrt{m! \Gamma(2k+m)}} |m, k\rangle. \quad (13)$$

where the quantity  $N$  is the normalized factor. Therefore the time dependence of the state  $|\psi(t)\rangle$  takes the form,

$$|\psi(t)\rangle = \sum_{m=0}^{\infty} (\beta_1(m, t)|+, +\rangle|m, k\rangle + \beta_2(m, t)|+, -\rangle|m+1, k\rangle + \beta_3(m, t)|-, +\rangle|m+1, k\rangle + \beta_4(m, t)|-, -\rangle|m+2, k\rangle) \quad (14)$$

with  $\beta_1(m, t)$ ,  $\beta_2(m, t)$ ,  $\beta_3(m, t)$  and  $\beta_4(m, t)$  are the solutions of Schrodinger equation  $i\hbar \partial|\psi(t)\rangle/\partial t = \hat{H} |\psi(t)\rangle$  which take the following form,

$$\begin{aligned} \frac{d\beta_1}{dt} &= -i v_1(m)(\beta_2 + \beta_3) \\ \frac{d\beta_2}{dt} &= -i \Delta_1 \beta_2 - i v_1(m)\beta_1 - i v_2(m)\beta_4 \\ \frac{d\beta_3}{dt} &= i \Delta_1 \beta_3 - i v_1(m)\beta_1 - i v_2(m)\beta_4 \\ \frac{d\beta_4}{dt} &= -i v_2(m)(\beta_2 + \beta_3) \end{aligned} \quad (15)$$

The coefficients  $\beta_1(m, t)$ ,  $\beta_2(m, t)$ ,  $\beta_3(m, t)$  and  $\beta_4(m, t)$  are given by the following equations,

$$\begin{aligned} \beta_1(m, t) &= Q_m \left[ \frac{(\mu_1(m))^2 - 2v_1^2(m)(1 - \cos \mu_1(m)t)}{(\mu_1(m))^2} \right], \\ \beta_2(m, t) &= -Q_m \left[ \frac{\Delta_1 v_1(m)(1 - \cos \mu_1(m)t)}{(\mu_1(m))^2} + i \frac{v_1(m) \sin \mu_1(m)t}{\mu_1(m)} \right], \\ \beta_3(m, t) &= Q_m \left[ \frac{\Delta_1 v_1(m)(1 - \cos \mu_1(m)t)}{(\mu_1(m))^2} - i \frac{v_1(m) \sin \mu_1(m)t}{\mu_1(m)} \right], \\ \beta_4(m, t) &= -Q_m \left[ \frac{2v_1(m)v_2(m)(1 - \cos \mu_1(m)t)}{(\mu_1(m))^2} \right], \end{aligned} \quad (16)$$

where  $v_1(m) = \lambda\sqrt{(m+1)(m+2k)}$ ,  $v_2(m) = \lambda\sqrt{(m+2)(m+1+2k)}$  with

$$\begin{aligned} \mu_1(m) &= \sqrt{\Delta_1^2 + 2\left((v_1(m))^2 + (v_2(m))^2\right)}, \\ \mu_2(m) &= \sqrt{\Delta_1^2 + \left((v_1(m))^2 + (v_2(m))^2\right)}, \\ \mu_3(m) &= \sqrt{(v_1(m))^2 + (v_2(m))^2} \end{aligned} \quad (17)$$

Now we can measure some physical quantities that help us to understand the behavior of the Hamiltonian Eq. (3). The relative inversion, degree of entanglement and normal squeezing will be discussed in forthcoming.

### 3 Relative Inversion

In quantum optics, the phenomena of collapses and revivals provide us more measures for the system and its connection to the process of entanglement. Therefore, it is known that there are relations between the photon number and the phenomena of collapses and revivals, so we will study the effective of the field-field interaction on the population inversion. Through the two Eqs. (5) and (8), the change in the sum of photons is defined as follows

$$\Delta K_z(t) = \frac{\langle \hat{K}_z(t) \rangle - \langle \hat{K}_z(0) \rangle}{\langle \hat{K}_z(0) \rangle} \tag{18}$$

where  $\hat{K}_z$  is given by Eq. (5). The obtained results are shown in Fig. 1. regarding the population function  $\Delta K_z(t)$  for the fixed parameters  $\xi = 25$ ,  $k = 3$ ,  $\Delta_1 = 0$  and varying the field-field interaction parameter  $\frac{g}{\omega}$ . For example, in Fig. 1a in absence of the field-field interaction  $g = 0$ . In this case, the function of the population since beginning of the interaction revealed a periodic behavior of the collapse followed by revival. This behavior repeated regularly through the time of interaction consideration with period  $\lambda t = n\pi$  ( $n = 1, 2, \dots$ ). The symmetry of the function  $\Delta K_z(t)$  around the horizontal axis is washed after adding the field-field terms into account ( $\frac{g}{\omega} = 0.01$ ). In general, the lower values of the function  $\Delta K_z(t)$  decreased, in contrast the higher values increased with the continuation of the interaction time see Fig. 1b. For increases of the parameter  $\frac{g}{\omega} = 0.1$ , the fluctuations between the lower values and the higher values increased and the collapses phenomena does not occur as seen in Fig. 1c. More increasing the field-field interaction parameter  $\frac{g}{\omega} = 0.35$ , due to more raising in the fluctuation through the periods of revivals with shift upward of the collapses regions.

### 4 Degree of Entanglement

Entanglement is one of the mainstays of many vital applications of quantum information [14–19]. In addition, it forms the support of experiments in quantum information. On the other hand, there are many uses and applications of the disentanglement quantum system [2]. Through dynamic analyzes and conclusions we can describe the behavior of the DF for the system contains the A-F interaction via Shannon information entropies [14] which is defined by

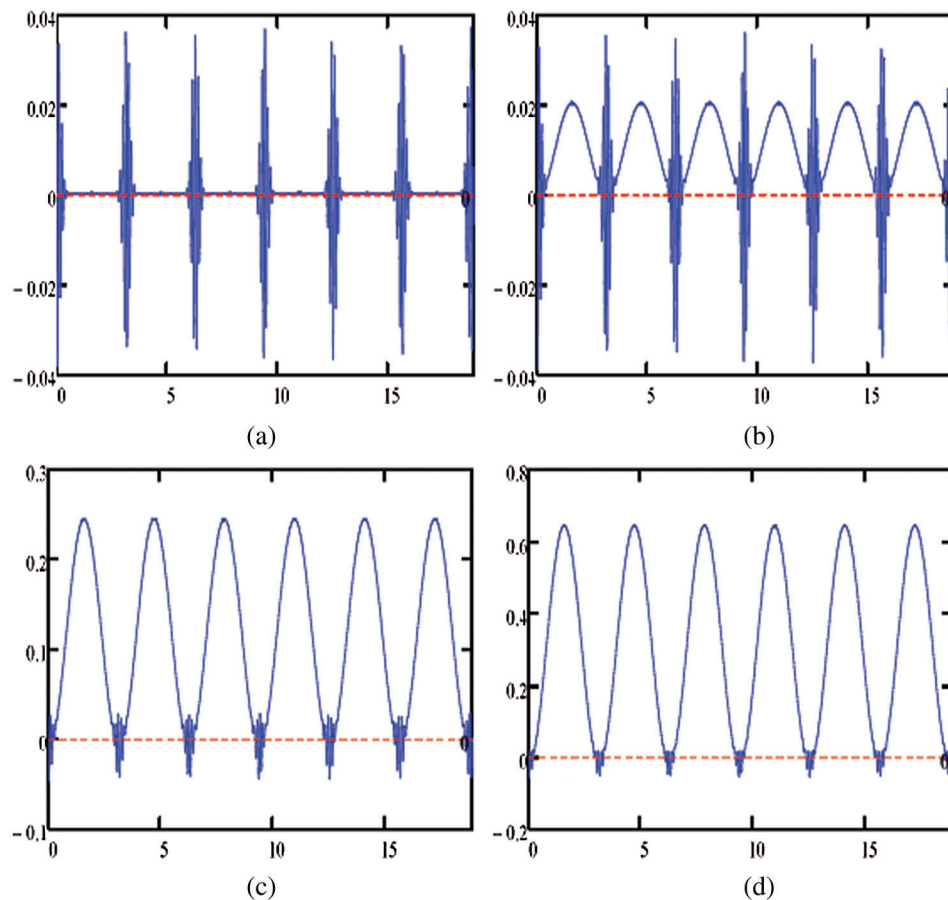
$$H(t) = -\langle \psi(t) | \hat{K}_- \hat{K}_+ | \psi(t) \rangle \ln \langle \psi(t) | \hat{K}_- \hat{K}_+ | \psi(t) \rangle, \tag{19}$$

where  $\hat{K}_+$  and  $\hat{K}_-$  are defined in Eq. (5). From Eqs. (5) and (19) it is easy to calculate the operator  $\langle \psi(t) | \hat{K}_- \hat{K}_+ | \psi(t) \rangle$  and takes the follows form,

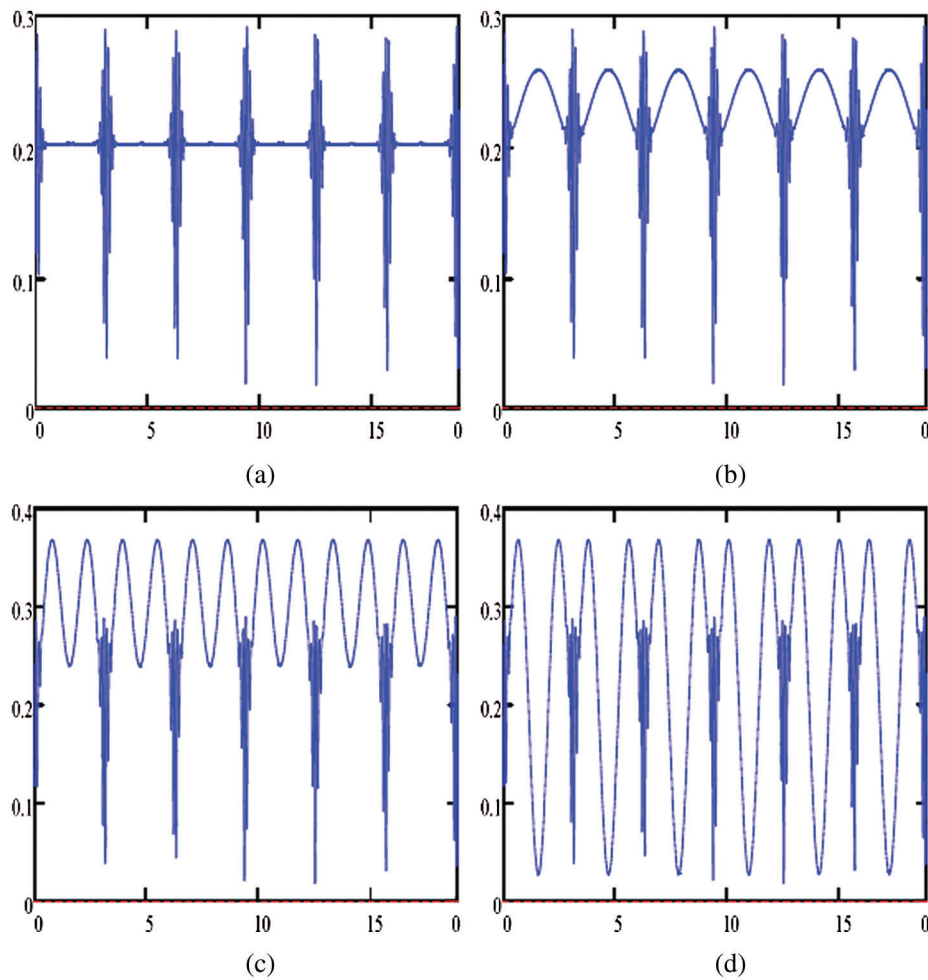
$$\begin{aligned} \langle \psi(t) | \hat{K}_- \hat{K}_+ | \psi(t) \rangle = \langle \psi(t) | \left[ \left( \frac{\sinh 2\zeta}{2} \right)^2 (\hat{R}_+^2 + \hat{R}_-^2 + 4\hat{R}_z^2) + \hat{R}_+ \hat{R}_- (\sinh \zeta)^4 + \hat{R}_- \hat{R}_+ (\cosh \zeta)^4 \right. \\ \left. - (\sinh \zeta)^2 \sinh 2\zeta (\hat{R}_+ \hat{R}_z + \hat{R}_z \hat{R}_-) - (\cosh \zeta)^2 \sinh 2\zeta (\hat{R}_z \hat{R}_+ + \hat{R}_- \hat{R}_z) \right] | \psi(t) \rangle \end{aligned} \tag{20}$$

Now we examine the DF between the interaction of the TQ and the QS with including the field-field interaction which are represented in Eq. (3). Depending on the same condition as mentioned in the

previous section. In the first case with excluding the field-field interaction (i.e.,  $g = 0$ ), the entropy indicates that the system begins to disentanglement followed by partial entanglement and the entropy function  $H(t)$  shows periodic and regular oscillations with a period  $n\pi(n = 0, 1, 2, \dots)$ . The system approaches to zero value (pure state) at the lower values of  $H(t)$  namely  $(n\pi)$  as shown in Fig. 2a. After adding the field-field interaction into account (i.e.,  $\frac{g}{\omega} = 0.01$ ) the lower and higher values of the entropy increases, the system becomes more partial entanglements, the oscillations grow and the maxima holds in centre of the collapses and the revivals regions as comparison between the relative inversion and the degree of entanglement, see Figs. 1b and 2b. It is pointed out the entropy function is symmetric about the value (0.2) in the preceding case, this symmetry vanished after adding the field-field interaction into the interaction cavity, the higher values of the entropy increase and the system still in partial entanglement state. By increasing the parameter field-field interaction adjust  $\frac{g}{\omega} = 0.1$ , the oscillations increase and the extreme values of the entropy function increase too as the time of interaction goes on, see Fig. 2c. With more increase of the parameter  $g$ , we see that the lower values which occurs in the collapses regions decrease and the DF approaches the pure state, see Fig. 2d.



**Figure 1:** The relative inversion with  $\lambda t$ , where  $\zeta = 25$ ,  $k = 3$ ,  $\Delta_1 = 0$  (a)  $\frac{g}{\omega} = 0.0$  (b)  $\frac{g}{\omega} = 0.01$  (c)  $\frac{g}{\omega} = 0.1$  (d)  $\frac{g}{\omega} = 0.35$



**Figure 2:** The DE with  $\lambda t$ , where  $\xi = 25$ ,  $k = 3$ ,  $\Delta_1 = 0$ , (a)  $\frac{g}{\omega} = 0.0$  (b)  $\frac{g}{\omega} = 0.01$  (c)  $\frac{g}{\omega} = 0.1$  (d)  $\frac{g}{\omega} = 0.35$

### 5 Normal Squeezing

One of the fundamentals of quantum mechanics is normal squeezing, which is completely related to the Heisenberg uncertainty principle, which has been suggested by [20–28]. A fundamental assumption in the study of quantum mechanics for any two observable  $\hat{A}$  and  $\hat{B}$  that are not commute (i.e.,  $[\hat{A}, \hat{B}] = i\hat{C}$ ), cannot be determined with the same precision. The uncertainty relation for the observable  $\hat{A}$  and  $\hat{B}$  achieves the following inequality,

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle\hat{C}\rangle|^2, \tag{21}$$

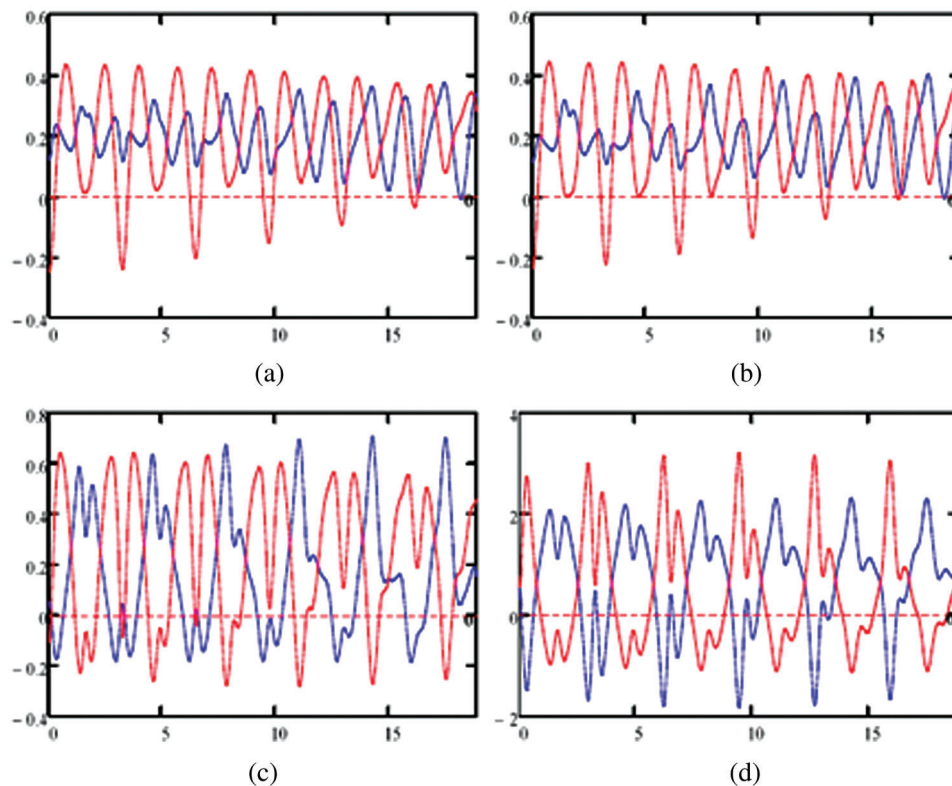
For the case  $\hat{A} = \frac{\hat{K}_+ + \hat{K}_-}{\sqrt{2}}$ ,  $\hat{B} = i\frac{\hat{K}_+ - \hat{K}_-}{\sqrt{2}}$ ,  $\hat{C} = \hat{K}_z$ . Therefore, the uncertainty inequality becomes

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle\hat{C}\rangle|. \tag{22}$$

The quantities  $\Delta\hat{A}$  and  $\Delta\hat{B}$  are estimated by the following relationships,

$$\begin{aligned}\Delta\hat{A} &= \langle \hat{A}^2 \rangle - (\langle \hat{A} \rangle)^2 \\ \Delta\hat{B} &= \langle \hat{B}^2 \rangle - (\langle \hat{B} \rangle)^2\end{aligned}\quad (23)$$

To analyze the normal squeezing behavior, consider the same conditions as in the above sections. In the absence of the field–field interaction ( $g = 0$ ), the regular squeezing appears in  $\Delta\hat{A}$  and never occurs in  $\Delta\hat{B}$ . The oscillations of the squeezing repeated periodically with period ( $n\pi$ ,  $n$  is nonnegative integer number) and the oscillations reduced gradually as observed in Fig. 3a. To visualize the field–field interaction by setting  $\frac{g}{\omega} = 0.01$ , we see that the squeezing regions in the quadrature  $\hat{A}$  reduced and the maximum values of the oscillations decreased, see Fig. 3b. By increasing the parameter  $g$ , adjust  $\frac{g}{\omega} = 0.1$ , the squeezing more reduced for the first quadrature  $\hat{A}$  (may takes the minimum values), while for the second quadrature  $\hat{B}$  the squeezing began to grow gradually as the time increasing, Fig. 3c.



**Figure 3:** The normal squeezing with  $\lambda t$ , where  $\xi = 25$ ,  $k = 3$ ,  $\Delta_1 = 0$ , (a)  $\frac{g}{\omega} = 0.0$  (b)  $\frac{g}{\omega} = 0.01$  (c)  $\frac{g}{\omega} = 0.1$  (d)  $\frac{g}{\omega} = 0.35$

Finally, with more increases of the  $g$ , adjust  $\frac{g}{\omega} = 0.35$ , the squeezing regions for  $\hat{B}$  more increases as the time increases, see Fig. 2d. Moreover, there is exchange for squeezing phenomena between the two quadrature  $\hat{A}$  and  $\hat{B}$  which dependence on the field–field interaction parameter  $g$ .



## 6 Conclusion

The effect of a field–field interaction on a cavity containing a pair of qubit interacting with a two-mode field of parametric amplifier is studied. The electromagnetic field transformed into a  $su(1,1)$  Lee group. Appropriate transformations have also been used to determine the constants of motion by calculating the equations of motion for some operators by using the Heisenberg differential formula. The wave function is calculated by solving the Schrödinger differential equation. The relative population, Shannon information entropy as well as the normal squeezing are discussed. The influence of the field-field interaction terms due to reduce of the frequency of the  $su(1,1)$  term in the system Hamiltonian are also presented. The results indicated that the collapses phenomena are reduced by increasing of the ratio  $\frac{g}{\omega}$ . The entanglement between the parties of the system started in separated state and becomes partially entangled, the lower and higher values of the modified Shannon information entropy are related to the periods of collapses and the revivals regions. The periods of squeezing estimated, in the exclude of the field–field interaction terms, the squeezing occurs in quadrature  $\hat{A}$  and after adding the ratio  $\frac{g}{\omega}$  the squeezing grows in the quadrature  $\hat{B}$  and reduced for the quadrature  $\hat{A}$ .

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**Conflicts of Interest:** The authors declare that they have no conflicts of interest to report regarding the present study.

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